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On One-Weight and ACD Codes in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$

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Abstract. In this paper, one-weight and additive complementary dual (ACD) codes in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ are studied. Firstly, it is shown that the image of an equidistant $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code is a binary equidistant code. Then, some properties of the structure and possible weights for one-weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes are described. Finally, it is given the sufficient conditions for a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code to be ACD.

1. Introduction

Let $\mathbb{Z}_m$ be the ring of integers modulo m. Any nonempty subset $\mathcal{C}$ of $\mathbb{Z}_m^n$ is a code and a submodule of a $\mathbb{Z}_m^n$ is called a linear code of length n over $\mathbb{Z}_m$. Specially, for m=2 and $m = 4$ the codes are called binary ($\mathbb{Z}_2$) and quaternary codes ($\mathbb{Z}_4$), respectively.

Constant-weight codes represent an important class of codes within the family of error-correcting codes [12]. A code is called one-weight (one-Lee weight) code if all its nonzero codewords have the same Hamming weight (Lee weight). A code is said to be equidistant if the distance between any two codewords is a constant. A linear equidistant code is necessarily a one weight code.

It is known that, for every positive integer k, there exists a unique (up to equivalence) one-weight binary linear codes of dimension k such that any two columns in its generator matrix are linearly independent [4, 16]. Later, this result has been extended to the ring and to the ring $\mathbb{Z}_{p^m}$, respectively [6, 17]. In [18], Wood determined exactly which modules underlie linear codes of constant-weight.

Additive codes were first defined by Delsarte [8] in terms of association schemes. A $\mathbb{Z}_2\mathbb{Z}_4$-additive code is an additive subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. The structure and properties of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes have been intensely studied, for example in [5, 15] and [1]. Especially, in [9], Dougherty et al. described one-weight $\mathbb{Z}_2\mathbb{Z}_4$-additive codes.

A binary code is said to be linear complementary dual (LCD) if it is linear and $\mathcal{C} \cap \mathcal{C}^\perp = \{\mathbf{0}\}$. Binary LCD codes were defined and characterized in [13]. In that paper, it is shown that these codes are an optimum linear coding solution for the two-user binary adder channel. Complementary dual codes have also been studied in [11] for linear codes over finite chain rings. More recently, in [3] the authors have generalized the notion of LCD codes to additive complementary dual (ACD) codes in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. They constructed infinite families of codes that are ACD and gave conditions for the case when the image of an ACD code is a binary LCD code.

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Recently, in [2] $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes, which are regarded as a generalization of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, have been introduced. The authors in [2] determined the structure of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes and gave the standard forms of generator and parity-check matrices of these codes and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-cyclic codes. Motivated from this work, in this paper, one-weight codes in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ are studied. Some properties of the structure and possible weights for one-weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes are described. Furthermore, it is shown that the image of an equidistant $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$ code is a binary equidistant code. Also the notion of additive complementary dual (ACD) codes in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is generalized to the ACD codes in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$.

2. Preliminaries

2.1. $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-Additive Codes

Definition 2.1. *Let $\mathbb{Z}_2$, $\mathbb{Z}_4$ and $\mathbb{Z}_8$ be the ring of integers modulo 2, 4 and modulo 8, respectively. Then, $\mathcal{C}$ is called a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code if it is a subgroup of $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ which r, s and t are positive integers [2].*

If any subgroup $\mathcal{C}$ of $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ is group isomorphic to the abelian structure

$$\mathbb{Z}_2^{k_0} \times \mathbb{Z}_4^{k_1} \times \mathbb{Z}_2^{k_2} \times \mathbb{Z}_8^{k_3} \times \mathbb{Z}_4^{k_4} \times \mathbb{Z}_2^{k_5}$$

then, $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(r, s, t; k_0, k_1, k_2, k_3, k_4, k_5)$ [2].

Definition 2.2. *Let ϕ_1 and ϕ_2 are the following well-known Gray maps.*

$$\begin{array}{ll} \phi_1 : \mathbb{Z}_4 \to \mathbb{Z}_2^2 & \phi_2 : \mathbb{Z}_8 \to \mathbb{Z}_2^4 \\ 0 \to 00 & 0 \to 0000 \\ 1 \to 01 & 1 \to 0001 \\ 2 \to 11 & 2 \to 0011 \\ 3 \to 10 & 3 \to 0111 \\ & 4 \to 1111 \\ & 5 \to 1110 \\ & 6 \to 1100 \\ & 7 \to 1000 \end{array}$$

For all $\mathbf{u} = (u_0, \ldots, u_{r-1}) \in \mathbb{Z}_2^r$, $\mathbf{v} = (v_0, \ldots, v_{s-1}) \in \mathbb{Z}_4^s$ and $\mathbf{w} = (w_0, \ldots, w_{t-1}) \in \mathbb{Z}_8^t$, a Gray map for codes over $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ can be defined as follows.

$$\Phi : \mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t \to \mathbb{Z}_2^n$$

$$\Phi(\mathbf{u}|\mathbf{v}|\mathbf{w}) = \left(u_0, \ldots, u_{r-1}|\phi_1(v_0), \ldots, \phi_1(v_{s-1})|\phi_2(w_0), \ldots, \phi_2(w_{t-1})\right).$$

Hence, the Gray image $\Phi(\mathcal{C}) = C$ of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{C}$ is a binary code of length $n = r + 2s + 4t$ and called $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code [2].

Let $(\mathbf{u}|\mathbf{v}|\mathbf{w}) \in \mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$. We denote by $w_H(\mathbf{u}|\mathbf{v}|\mathbf{w})$ the Hamming weight of $(\mathbf{u}|\mathbf{v}|\mathbf{w})$. For any two vectors $(\mathbf{u}|\mathbf{v}|\mathbf{w})$, $(\mathbf{u}'|\mathbf{v}'|\mathbf{w}') \in \mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$, the Hamming distance between $(\mathbf{u}|\mathbf{v}|\mathbf{w})$ and $(\mathbf{u}'|\mathbf{v}'|\mathbf{w}')$ is defined to be

$$d_H\left((\mathbf{u}|\mathbf{v}|\mathbf{w}), (\mathbf{u}'|\mathbf{v}'|\mathbf{w}')\right) = w_H\left((\mathbf{u}|\mathbf{v}|\mathbf{w}) - (\mathbf{u}'|\mathbf{v}'|\mathbf{w}')\right) = w_H\left(\mathbf{u} - \mathbf{u}'|\mathbf{v} - \mathbf{v}'|\mathbf{w} - \mathbf{w}'\right).$$

We define the Lee weight of $(\mathbf{u}|\mathbf{v}|\mathbf{w})$ as:

$$wt_L\left(\mathbf{u}|\mathbf{v}|\mathbf{w}\right) = wt_H\left(\Phi(\mathbf{u}|\mathbf{v}|\mathbf{w})\right) = w_H(\mathbf{u}) + w_H\left(\phi_1(\mathbf{v})\right) + w_H\left(\phi_2(\mathbf{w})\right).$$

We define the Lee distance between $(\mathbf{u}|\mathbf{v}|\mathbf{w})$ and $(\mathbf{u}'|\mathbf{v}'|\mathbf{w}')$ as

$$d\left(\mathbf{u}|\mathbf{v}|\mathbf{w}), (\mathbf{u}'|\mathbf{v}'|\mathbf{w}')\right) = w_L\left(\mathbf{u} - \mathbf{u}'|\mathbf{v} - \mathbf{v}'|\mathbf{w} - \mathbf{w}'\right) = wt_H(\mathbf{u} - \mathbf{u}') + w_H\left(\phi_1(\mathbf{v} - \mathbf{v}')\right) + w_H\left(\phi_2(\mathbf{w} - \mathbf{w}')\right).$$

Definition 2.3. *Let* $(\mathbf{u}_1|\mathbf{v}_1|\mathbf{w}_1)$, $(\mathbf{u}_2|\mathbf{v}_2|\mathbf{w}_2) \in \mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ *,where* $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}_2^r$, $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{Z}_4^s$ *and* $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{Z}_8^t$. *The inner product between* $(\mathbf{u}_1|\mathbf{v}_1|\mathbf{w}_1)$ *and* $(\mathbf{u}_2|\mathbf{v}_2|\mathbf{w}_2)$ *is defined as follows:*

$$\langle(\mathbf{u}_1 \mid \mathbf{v}_1|\mathbf{w}_1),(\mathbf{u}_2 \mid \mathbf{v}_2|\mathbf{w}_2)\rangle = 4(\sum_{i=1}^{r} u_{1i}u_{2i}) + 2(\sum_{j=r+1}^{r+s} v_{1i}v_{2i}) + (\sum_{l=r+s+1}^{r+s+t} w_{1i}w_{2i}) \in \mathbb{Z}_8.$$

The dual code $\mathcal{C}^{\perp}$ *can be defined in the usual way with respect to this inner product.*

$$\mathcal{C}^{\perp} = \left\{\mathbf{v} \in \mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t | \mathbf{u}\cdot\mathbf{v} = 0 \textit{ for all } \mathbf{u} \in \mathcal{C}\right\}.$$

It is very easy to show that $\mathcal{C}^{\perp}$ *is also a* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-additive code.*

2.2. Generator and Parity-check matrices of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-additive codes*

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(r,s,t;k_0,k_1,k_2,k_3,k_4,k_5)$. Then, it is shown that $\mathcal{C}$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code which has the following standard form generator matrix:

$$G = \left(\begin{array}{cc|ccc|cccc} I_{k_0} & \overline{A}_{01} & 0 & 0 & 2T_1 & 0 & 0 & 0 & 4T_2 \\ \hline 0 & \overline{S}_1 & I_{k_1} & B_{01} & B_{02} & 0 & 0 & 2T_3 & 2T_4 \\ 0 & 0 & 0 & 2I_{k_2} & 2B_{12} & 0 & 0 & 0 & 4T_5 \\ \hline 0 & \overline{S}_2 & 0 & S_{01} & S_{02} & I_{k_3} & A_{01} & A_{02} & A_{03} \\ 0 & \overline{S}_3 & 0 & 0 & 2S_{12} & 0 & 2I_{k_4} & 2A_{12} & 2A_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4I_{k_5} & 4A_{23} \end{array}\right) \quad (1)$$

where $\overline{A}_{01}$, $\overline{S}_1$, $\overline{S}_2$, $\overline{S}_3$ are matrices with all entries from $\mathbb{Z}_2$ and B_{02}, B_{12}, S_{02} and S_{12} are matrices over $\mathbb{Z}_4$. Also, T_4, T_5 and A_{i3} are matrices over $\mathbb{Z}_8$ for $0 \le i \le 2$. Although B_{01}, S_{01} and T_1 are matrices over $\mathbb{Z}_4$, all values of these matrices are from $\mathbb{Z}_2$. Likewise, A_{01} and T_2 are matrices over $\mathbb{Z}_4$ whose all entries are from $\{0,1\}$. T_3, A_{12} and A_{02} are matrices over $\mathbb{Z}_8$, but all values are the elements of the set $\{0,1,2,3\}$. Also, $\mathcal{C}$ has $2^{k_0}2^{2k_1}2^{k_2}2^{3k_3}2^{2k_4}2^{k_5}$ codewords [2].

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code with the generator matrix in (1), then we have the following parity-check matrix:

$$H = \left(\begin{array}{cc|ccc|c} -\overline{A}_{01}^t & I_{\alpha-k_0} & -2\bar{S}_1^t & 0 & 0 & \\ -\overline{T}_1^t & 0 & B_{02}^t - B_{12}^tB_{01}^t & B_{12}^t & I_{\beta-k_1-k_2} & \\ 0 & 0 & -2B_{01}^t & 2I_{k_2} & 0 & P \\ -\overline{T}_1^t & 0 & -T_4^t + T_3^tA_{23}^t + T_5^tB_{01}^t & -T_5^t & 0 & \\ 0 & 0 & -2T_3^t & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \end{array}\right) \quad (2)$$

where P is the matrix as in the (3).

$$P = \begin{pmatrix} 4\overline{S_2}^t - 2\overline{S_3}^t A_{01}^t & -2\overline{S_3}^t & 0 & 0 \\ 2S_{01}^tB_{12}^t - 2S_{02}^t + 2S_{12}^tA_{01}^t & -2S_{12}^t & 0 & 0 \\ -4S_{01}^t & 0 & 0 & 0 \\ -A_{03}^t + A_{13}^tA_{01}^t + A_{23}^tA_{02}^t - A_{23}^tA_{12}^tA_{01}^t + 2S_{01}^tT_5^t & -A_{13}^t + A_{23}^tA_{12}^t & -A_{23}^t & I_{\theta-k_3-k_4-k_5} \\ -2A_{02}^t + 2A_{12}^tA_{01}^t & -2A_{12}^t & 2I_{k_5} & 0 \\ -4A_{01}^t & 4I_{k_4} & 0 & 0 \end{pmatrix} \quad (3)$$

So, the dual code $\mathcal{C}^{\perp}$ is of type $(r,s,t;r-k_0,s-k_1-k_2,k_2,t-k_3-k_4-k_5,k_5,k_4)$ [2].

Let X (respectively Y, Z) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$, $\mathbb{Z}_8$) coordinate positions, so $|X| = r$, $|Y| = s$ and $|Z| = t$. Call $\mathcal{C}_X$ (respectively $\mathcal{C}_Y$, $\mathcal{C}_Z$) the punctured code of $\mathcal{C}$ by deleting the coordinates outside X

(respectively Y, Z). Generator matrix G in (1) is of size $(\sum_{i=0}^{5} k_i) \times (r+s+t)$ for the code $\mathcal{C}$. This generator matrix G can be written as

$$G = \begin{pmatrix} G_X & | & G_Y & | & G_Z \end{pmatrix}$$

where G_X is matrix over $\mathbb{Z}_2$ of size $(\sum_{i=0}^{5} k_i) \times r$, G_Y is matrix over $\mathbb{Z}_4$ of size $(\sum_{i=0}^{5} k_i) \times s$ and G_Z is matrix over $\mathbb{Z}_8$ of size $(\sum_{i=0}^{5} k_i) \times t$. Note that G_X is the generator matrix of C_X, G_Y is the generator matrix of C_Y and G_Z is the generator matrix of Z_X. We define the product

$$G \cdot G^T = \begin{pmatrix} G_X & | & G_Y & | & G_Z \end{pmatrix} \cdot \begin{pmatrix} G_X^T \\ \hline G_Y^T \\ \hline G_Z^T \end{pmatrix}$$

with entries from $\mathbb{Z}_8$, where all entries in G_X are considered as elements in $\{0,1\} \subseteq \mathbb{Z}_8$ and G_Y are considered as elements in $\{0,1,2,3\} \subseteq \mathbb{Z}_8$ the product of a row by a column is computed as the inner product of vectors in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$. Note that in $G_X G_X^T$, $G_Y G_Y^T$ and $G_Z G_Z^T$ the usual matrix multiplication is used, but not in $G \cdot G^T$.

Proposition 2.4. *[13] Let C a binary (n,k) linear code with generator matrix G and parity-check matrix H. The following statements are equivalent:*

1. *C is an LCD code,*
2. *the $k \times k$ matrix $GG^{\perp}$ is non-singular,*
3. *the $(n-k) \times (n-k)$ matrix $HH^{\perp}$ is non-singular.*

Example 2.5. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive one-weight code generated by the matrix*

$$\left(\begin{array}{c|c|c} 110101 & 0022 & 0440 \\ \hline 000000 & 2222 & 0044 \\ \hline 001111 & 0202 & 2266 \end{array}\right)$$

$\mathcal{C}$ is of type $(6,4,4;1;0,1;0,1,0)$ and $\mathcal{C}$ has $2^1 \cdot 2^1 \cdot 4^1 = 16$ codewords. The codewords of $\mathcal{C}$ are:

$$\{(000000|0000|0000), (110101|0022|0440), (000000|2222|0044), (001111|0202|2266),$$
$$(110101|2200|0404), (111010|0220|2626), (001111|2020|2222), (000000|0000|4444),$$
$$(110101|0022|4004), (000000|2222|4400), (111010|2002|2662), (001111|0202|6622),$$
$$(111010|0220|6262), (001111|2020|6666), (110101|2200|4040), (111100|2002|6226)\}.$$

3. One-Weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-Additive codes

In this section, some properties of one-weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(r,s,t;k_0,k_1,k_2,k_3,k_4,k_5)$ are investigated.

Definition 3.1. *Let $\mathcal{C}$ be a nonzero code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$. $\mathcal{C}$ is called a one-weight (Lee-weight) code if all of its nonzero codewords have the same weight. Moreover, if the weight is m, then the code is called a one-weight code with weight m.*

Definition 3.2. *Let $\mathcal{C}$ be a nonzero code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}', \boldsymbol{v}'$ be any four distinct codewords of $\mathcal{C}$. If the distance between $\boldsymbol{u}$ and $\boldsymbol{u}'$ is equal to the distance between $\boldsymbol{v}$ and $\boldsymbol{v}'$ that is, $d(\boldsymbol{u},\boldsymbol{u}') = d(\boldsymbol{v},\boldsymbol{v}')$. then C is called an equidistant code.*

From above definition, we have the following theorem.

Theorem 3.3. *Let $\mathcal{C}$ be a an equidistant code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ with distance m. Then $\Phi(\mathcal{C})$ is a binary equidistant code with the same distance m. Moreover, the binary image $\Phi(\mathcal{C})$ of $\mathcal{C}$ is also a one-weight code with weight m.*

Proof. Let $\Phi(\mathbf{u}),\ \Phi(\mathbf{v}) \in \Phi(C)$ with $\Phi(\mathbf{u}) \neq \Phi(\mathbf{v})$, where $\mathbf{u}, \mathbf{v} \in \mathcal{C}$. Then, since the extended Gray map, defined Definition 2.2, is an isometry, we have

$$d_H(\Phi(\mathbf{u}),\ \Phi(\mathbf{v})) = d(\mathbf{u}, \mathbf{v}) = m.$$

This gives that $\Phi(\mathcal{C})$ is an equidistant code with the same distance m that is the distance of $\mathcal{C}$.

If $\mathbf{0} \in \mathcal{C}$, then for any nonzero codeword $\mathbf{u} \in \mathcal{C}$, we have $wt(\mathbf{u}) = d(\mathbf{u}, \mathbf{0}) = m$. Note that $\Phi(\mathbf{0}) = \mathbf{0}$, then the equality above gives that

$$w_H(\Phi(\mathbf{u})) = d_H(\Phi(\mathbf{u})), \Phi(\mathbf{0})) = d(\mathbf{u}, \mathbf{0}) = wt(\mathbf{u}) = m.$$

This finishes the proof. □

It is easy to see that if $\mathcal{C}$ is an additive code, then $\mathcal{C}$ is a one-weight code if and only if $\mathcal{C}$ is an equidistant code.

Example 3.4. *The code $\mathcal{C} = \langle(1|1|246)\rangle$ is an additive code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$, which is a one-weight code with weight 10. It can be verified that the image $\Phi(\mathcal{C})$ of the code $\mathcal{C}$ is a binary simplex code of length 15. $\mathcal{C}$ is of type $(1, 1, 3; 0; 1, 0; 0, 0, 0)$ and $\mathcal{C}$ has $2^2 = 4$ codewords. The codewords of $\mathcal{C}$ are:*

$$\{(0|0|000), (1|1|246), (0|2|404), (1|3|642)\}.$$

The codewords of $\Phi(\mathcal{C})$ are:

$$\{(0|00|000000000000), (1|10|001111111100), (0|11|111100001111), (1|01|110011110011)\}.$$

It is worth to note that the dual of a one-weight code is not necessarily a one-weight code. The dual code of $\mathcal{C}$ given in above example is generated by the following matrix

$$\left(\begin{array}{c|c|c} 1 & 2 & 000 \\ \hline 0 & 3 & 100 \\ 0 & 2 & 010 \\ 0 & 1 & 001 \end{array}\right)$$

and is of type $(1, 1, 3; 1; 0, 0; 3, 0, 0)$. But it is not a one-weight code.

The following result gives a construction of one-weight additive codes in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$.

Theorem 3.5. *Let $\mathcal{C}$ be a one-weight additive code with weight m in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$. Then for any positive integer δ, there exists a one-weight additive code of weight δm in $\mathbb{Z}_2^{\delta r} \times \mathbb{Z}_4^{\delta s} \times \mathbb{Z}_8^{\delta t}$.*

Proof. Let G be a generator matrix of the one-weight code $\mathcal{C}$ with weight m. We can write G as $G = (G_1|G_2|G_3)$, where G_1, G_2 and G_3 are the binary, quaternary and $\mathbb{Z}_8$ parts of the generator matrix G respectively. Let $\mathcal{C}'$ be an additive code generated by the following matrix

$$G' = \left(\overbrace{G_1, \ldots, G_1}^{\delta} \,|\, \overbrace{G_2, \ldots, G_2}^{\delta} \,|\, \overbrace{G_3, \ldots, G_3}^{\delta} \right).$$

Then for any nonzero codeword $\mathbf{c}' \in \mathcal{C}'$, there exists a nonzero vector $\mathbf{u}$ such that $\mathbf{c}' = \mathbf{u}G'$. Note here that the vector $\mathbf{u}$ is an integer vector and multiplication of a row by an integer is simply the row added to itself that many times. Hence

$$wt(\mathbf{c}') = wt(\mathbf{u}G') = wt \left(\overbrace{\mathbf{u}G_1, \ldots, \mathbf{u}G_1}^{\delta} \,|\, \overbrace{\mathbf{u}G_2, \ldots, \mathbf{u}G_2}^{\delta} \,|\, \overbrace{\mathbf{u}G_3, \ldots, \mathbf{u}G_3}^{\delta} \right)$$

$$= \delta \cdot wt_H(\mathbf{u}G_1) + \delta \cdot wt_H(\mathbf{u}G_2) + \delta \cdot wt_H(\mathbf{u}G_3)$$

$$= \delta \cdot (wt_H(\mathbf{u}G_1) + wt_H(\mathbf{u}G_2) + wt_H(\mathbf{u}G_3))$$

$$= \delta \cdot wt_H(\mathbf{u}G)$$

Note that $\mathcal{C}$ is a one-weight code with weight m, which implies that $wt_H(\mathbf{u}G) = m$ for all nonzero $\mathbf{u}$. Then we get that $wt(\mathbf{c}') = \delta m$. □

Definition 3.6. *Let $\mathcal{C}$ a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code. Let $\mathcal{C}_r$ (respectively $\mathcal{C}_s$, $\mathcal{C}_t$) be the punctured code of $\mathcal{C}$ by deleting the coordinates outside r (respectively s, t). If $\mathcal{C} = \mathcal{C}_r \times \mathcal{C}_s \times \mathcal{C}_t$ then $\mathcal{C}$ is called separable. Otherwise it is called non-separable.*

Proposition 3.7. *There do not exist separable one-weight additive codes in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ with $\mathcal{C}_r \neq 0$, $\mathcal{C}_s \neq 0$ and $\mathcal{C}_t \neq 0$.*

Proof. Suppose $\mathcal{C}_r \times \mathcal{C}_s \times \mathcal{C}_t$ is a separable one-weight additive code with $\mathcal{C}_r \neq \mathbf{0}$, $\mathcal{C}_s \neq \mathbf{0}$ and $\mathcal{C}_t \neq \mathbf{0}$. Consider the codeword $\mathbf{0} \neq (\mathbf{u}|\mathbf{v}|\mathbf{w}) \in \mathcal{C}_r \times \mathcal{C}_s \times \mathcal{C}_t$ with weight m. Note that $\mathcal{C}_r \times \mathcal{C}_s \times \mathcal{C}_t$ is an additive code, we get that $\mathbf{0} \in \mathcal{C}_r$, $\mathbf{0} \in \mathcal{C}_s$ and $\mathbf{0} \in \mathcal{C}_s$. Therefore, $(\mathbf{u}|\mathbf{0}|\mathbf{0})$, $(\mathbf{0}|\mathbf{v}|\mathbf{0})$ and $(\mathbf{0}|\mathbf{0}|\mathbf{w})$ are elements of $\mathcal{C}_r \times \mathcal{C}_s \times \mathcal{C}_t$. If $\mathbf{u} \neq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$, then $wt(\mathbf{u}|\mathbf{v}|\mathbf{w}) \neq wt(\mathbf{u})$, $wt(\mathbf{u}|\mathbf{v}|\mathbf{w}) \neq wt(\mathbf{v})$ and $wt(\mathbf{u}|\mathbf{v}|\mathbf{w}) \neq wt(\mathbf{w})$. Hence, we have a contradiction. So, there is no separable one-weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code. □

Lemma 3.8. *Let $\mathcal{C}$ be $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(r, s, t; k_0, k_1, k_2, k_3, k_4, k_5)$ with no all zero columns in the generator matrix of $\mathcal{C}$. Then the sum of the weights of all the codewords of $\mathcal{C}$ is equal to $\frac{|\mathcal{C}|}{2}(r + 2s + 4t)$.*

Proof. Let G be a matrix whose rows are all codewords of $\mathcal{C}$. Since $\mathcal{C}$ is an additive code, in the first r columns of G, the number of coordinates containing 0 is equal to the number of coordinates containing 1.

Now consider the second s columns of G. Any column in this part either contains an equal number of 0, 1, 2 and 3 or it contains an equal number of 0 and 2 and does not contain any coordinates with a 1 or 3 in it. Assume there are ρ of these columns containing only 0 and 2 in the last s columns. Then there are $s - \rho$ columns containing an equal number of 0, 1, 2 and 3.

Finally, consider the last t columns of G, namely those containing the $\mathbb{Z}_8$ part of the codewords. Any column in this part contains an equal number of 0, 1, 2, 3, 4, 5, 6 and 7 or it contains an equal number of 0, 2, 4 and 6 and does not contain any coordinates with a 1, 3, 5 or 7 in it or it contains an equal number of 0 and 4 and does not contain any coordinates with a 1, 2, 3, 5, 6 or 7 in it. Suppose there are σ of these columns containing only 0, 2, 4 and 6 and γ of these columns containing only 0 and 4 in the last t columns. Then there are $t - \sigma - \gamma$ columns containing an equal number of 0, 1, 2, 3, 4, 5, 6 and 7.

Therefore, the sum of the weights in $\mathcal{C}$ is

$$\sum_{c \in \mathcal{C}} wt(c) = r \cdot \left(\frac{|\mathcal{C}|}{2}\right) + \left(\frac{|\mathcal{C}|}{2} \cdot 2\right) \cdot \rho + \left(\frac{|\mathcal{C}|}{4} + \frac{|\mathcal{C}|}{4} \cdot 2 + \frac{|\mathcal{C}|}{4}\right) \cdot (s - \rho)$$

$$+ \left(\frac{|\mathcal{C}|}{4} + \frac{|\mathcal{C}|}{4} \cdot 2 + \frac{|\mathcal{C}|}{4} \cdot 3 + \frac{|\mathcal{C}|}{4} \cdot 4\right) \cdot (t - \sigma - \gamma)$$

$$+ \left(\frac{|\mathcal{C}|}{2} \cdot 2 + \frac{|\mathcal{C}|}{4} \cdot 4\right) \cdot \sigma + \left(\frac{|\mathcal{C}|}{2} \cdot 4\right) \cdot \gamma$$

Then we have

$$\sum_{c \in \mathcal{C}} wt(c) = r \cdot \left(\frac{|\mathcal{C}|}{2}\right) + s \cdot |\mathcal{C}| + 2|\mathcal{C}| \cdot t = \frac{|\mathcal{C}|}{2}(r + 2s + 4t)$$

□

Theorem 3.9. *Let $\mathcal{C}$ be a one-weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(r, s, t; k_0, k_1, k_2, k_3, k_4, k_5)$ with weight m in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ such that there exists no zero columns in the generator matrix of $\mathcal{C}$, and suppose $\mathcal{C} \cong \mathbb{Z}_2^{k_0} \times \mathbb{Z}_2^{2k_1} \times \mathbb{Z}_2^{k_2} \times \mathbb{Z}_2^{3k_3} \times \mathbb{Z}_2^{2k_4} \times \mathbb{Z}_2^{k_5}$ as an additive group. Then there exists a unique positive integer μ such that $m = \mu \cdot 2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3-1}$, where r, s and t satisfy $r + 2s + 4t = \mu \cdot 2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3-1}$. Furthermore, if m is an odd integer, then r is odd and $\mathcal{C} = \{(\mathbf{0}_r|\mathbf{0}_s|\mathbf{0}_t), (\mathbf{1}_r|\mathbf{2}_s|\mathbf{4}_t)\}$, where $\mathbf{1}_r = (1, \ldots, 1) \in \mathbb{Z}_2^r$, $\mathbf{2}_s = (2, \ldots, 2) \in \mathbb{Z}_4^s$ and $\mathbf{4}_t = (4, \ldots, 4) \in \mathbb{Z}_8^t$.*

Proof. Since $\mathcal{C}$ is a one-weight code of weight m, the sum of the weights of all codewords in $\mathcal{C}$ is $(|\mathcal{C}| - 1)\, m$. And also we know from Lemma 3.8, we have that

$$\sum_{c \in \mathcal{C}} wt(c) = \frac{|\mathcal{C}|}{2} (r + 2s + 4t)\,.$$

So we have

$$(|\mathcal{C}| - 1)\, m = \frac{|\mathcal{C}|}{2} (r + 2s + 4t)\,.$$

Note that the cardinality of $\mathcal{C}$ is $|\mathcal{C}| = 2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3}$, and

$$gcd\left((|\mathcal{C}| - 1)\,, \frac{|\mathcal{C}|}{2}\right) = gcd\left(2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3} - 1, 2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3-1}\right) = 1.$$

Hence there exists a positive integer μ such that

$$m = \mu \cdot \frac{|\mathcal{C}|}{2} = \mu \cdot 2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3-1}$$

and

$$r + 2s + 4t = \mu \cdot (|\mathcal{C}| - 1) = \mu \cdot 2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3} - 1.$$

Furthermore, if m is odd, then $\mu \cdot 2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3-1}$ is odd. This implies that μ is odd and $2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3-1} = 1$, which gives that any of k_0, k_2, k_5 is 1, others are 0 and $k_1 = k_3 = k_4 = 0$, so $m = \mu$ is obtained. In this case $r + 2s + 4t = \mu = m$ is odd, hence r is odd. Recall that $\mathcal{C}$ is a one-weight additive code with weight $m = r + 2s + 4t$. Since $(\mathbf{1}_r|\mathbf{2}_s|\mathbf{4}_t)$ is the only word with weight $r + 2s + 4t$, we get

$$\mathcal{C} = \{(\mathbf{0}_r|\mathbf{0}_s|\mathbf{0}_t), (\mathbf{1}_r|\mathbf{2}_s|\mathbf{4}_t)\}\,.$$

□

Theorem 3.10. *Let $\mathcal{C}$ be an additive code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$. Then the weights of all codewords of $\mathcal{C}$ are even if and only if $(\mathbf{1}_r|\mathbf{2}_s|\mathbf{4}_t) \in \mathcal{C}^\perp$.*

Proof. Let $(\mathbf{u}|\mathbf{v}|\mathbf{w}) = (u_1, \ldots, u_r|v_1, \ldots, v_s|w_1, \ldots, w_t) \in \mathcal{C}$, where $(u_1, \ldots, u_r) \in \mathbb{Z}_2^r$, $(v_1, \ldots, v_s) \in \mathbb{Z}_4^s$ and $(w_1, \ldots, w_t) \in \mathbb{Z}_8^t$. We consider the following equality

$$\langle (\mathbf{u}|\mathbf{v}|\mathbf{w}), (\mathbf{1}_r|\mathbf{2}_s|\mathbf{4}_t) \rangle = 4\left(\sum_{i=1}^{r} u_i\right) + 2\left(\sum_{j=r+1}^{r+s} 2v_j\right) + \sum_{t=r+s+1}^{r+s+t} 4w_l.$$

It is easy to see that $\langle (\mathbf{u}|\mathbf{v}|\mathbf{w}), (\mathbf{1}_r|\mathbf{2}_s|\mathbf{4}_t) \rangle = 0$ if and only if $wt(\mathbf{u}|\mathbf{v}|\mathbf{w})$ is even, which gives the result. □

Corollary 3.11. *Let $\mathcal{C}$ be an one-weight additive code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$. Then the weight of $\mathcal{C}$ is even if and only if $(\mathbf{1}_r|\mathbf{2}_s|\mathbf{4}_t) \in \mathcal{C}^\perp$.*

4. The Structure of One-Weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-Additive codes

In this section, we give some properties to construct one-weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes of type $(r, s, t; k_0, k_1, k_2, k_3, k_4, k_5)$.

We begin by noting that if $\mathcal{C}$ is a nontrivial one-weight code with weight m, with a generator matrix that has no zero columns, then $m = \mu \cdot 2^{k_0+k_2+k_5+2(k_1+k_4)+3k_3-1}$ is even, where μ is a positive integer, and $\mathcal{C} \cong \mathbb{Z}_2^{k_0} \times \mathbb{Z}_2^{2k_1} \times \mathbb{Z}_2^{k_2} \times \mathbb{Z}_2^{3k_3} \times \mathbb{Z}_2^{2k_4} \times \mathbb{Z}_2^{k_5}$ as a group. Also, we assume that the quaternary part of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code satisfies the conditions Lemma 5.1, Lemma 5.2 and Lemma 5.5 in [9].

Proposition 4.1. *Let $\mathcal{C}$ be a one-weight code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ with weight m, then for any $\mathbf{c} = (\mathbf{u}|\mathbf{v}|\mathbf{w})$, order 8 codeword of $\mathcal{C}$, the number of units $\{1$ or $7\}$ in w is equal to the number of units $\{3$ or $5\}$ in w.*

Proof. Let $\mathcal{C}$ be a one-weight code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ with weight m and $\mathbf{c} = (\mathbf{u}|\mathbf{v}|\mathbf{w}) \in \mathcal{C}$ be a codeword of order 8. Then we have

$$wt(\mathbf{c}) = wt_H(\mathbf{u}) + wt_L(\mathbf{v}) + wt_L(\mathbf{w}) = m.$$

Assume that the number of units $\{1$ or $7\}$ in $\mathbf{w}$ is k_1 and the number of units $\{3$ or $5\}$ in w is k_2. Also suppose that $k_1 \neq k_2$, so we can take $k_1 > k_2$. Since $\mathcal{C}$ is an additive code, $5\mathbf{c}$ must be in $\mathcal{C}$. Since $5\mathbf{c} = (5\mathbf{u}|5\mathbf{v}|5\mathbf{w}) = (\mathbf{u}|\mathbf{v}|5\mathbf{w})$, then we have $wt(5\mathbf{c}) = wt_H(\mathbf{u}) + wt_L(\mathbf{v}) + wt_L(5\mathbf{w}) = m$. If $\mathbf{c}$ has $\{0, 2, 4, 6\}$, then we know that $\mathbf{c}$ and $5\mathbf{c}$ have same number of $\{0, 2, 4, 6\}$. So, we have $wt_L(\mathbf{w}) < wt_L(5\mathbf{w})$. Therefore $wt(5\mathbf{c}) \neq m$. It is a contradiction. □

Proposition 4.2. *Let G be the generator matrix of a one-weight code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ with weight m. Then if $\boldsymbol{c} = (\boldsymbol{u}|v|w)$ is a row of G, then the number of units $\{1, 3, 5, 7\}$ in w is either 0 or $\frac{m}{4}$.*

Proof. Let $\mathbf{c} = (\mathbf{u}|\mathbf{v}|\mathbf{w})$ be a row of G. Then, the weight of $\mathbf{c}$ is $wt(\mathbf{c}) = wt_H(\mathbf{u}) + wt_L(\mathbf{v}) + wt_L(\mathbf{w}) = m$. Since $\mathcal{C}$ is an additive code $4\mathbf{c} = (0|0|4\mathbf{w})$ is also in $\mathcal{C}$. Then, if $4\mathbf{c} = (0|0|4\mathbf{w}) = \mathbf{0}$ then $\mathbf{w}$ does not contain units.

If $4\mathbf{c} \neq \mathbf{0}$, then $4\mathbf{u} = 4\mathbf{v} = \mathbf{0}$ and $4\mathbf{c} = (0|0|4\mathbf{w})$. This means that $wt_L(4\mathbf{w}) = m$. Let the number of units in $\mathbf{w}$ be 2k and assume that the coordinate positions where $\mathbf{w}$ has units $\{1$ or $7\}$ and $\{3$ or $5\}$ are k. So, $wt_L(4\mathbf{w}) = 8k = m$. Hence, the number of units in $\mathbf{w}$ is $\frac{m}{4}$. □

Corollary 4.3. *Let $\mathcal{C}$ be a one-weight code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ with weight m. For any $\boldsymbol{c} = (\mathbf{u}|\mathbf{v}|\mathbf{w})$, order 8 codeword of $\mathcal{C}$, both the number of units $\{1$ or $7\}$ and the number of units $\{3$ or $5\}$ in $\boldsymbol{w}$ are $\frac{m}{8}$.*

Example 4.4. *Consider the vector* $(11|132|75132)$. *This generates the code*

$$\{(00|000|00000), (11|132|75132), (00|220|62264), (11|312|57316),$$
$$(00|000|44440), (11|132|31572), (00|220|26624), (11|312|13756)\}.$$

This code is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code type $(2, 3, 5; 0, 0, 0, 1, 0, 0)$ *and its weight m is 16.*

Example 4.5.

$$G_{k_0=k_3=1} = \left(\begin{array}{c|c|c} 111100 & 1331 & 1133 \\ \hline 110011 & 1133 & 1537 \end{array} \right)$$

$\mathcal{C}$ is of type $(6, 4, 4; 1; 0, 0; 1, 0, 0)$ *and C has* $2^1 \cdot 8^1 = 16$ *codewords. The codewords of C are:*

$$\{(111100|1331|1133), (110011|1133|1537), (000000|2222|2266), (111100|3113|3311),$$
$$(000000|000000|4444), (111100|1331|5577), (00000|2222|6622), (111100|3113|7755),$$
$$(110011|3311|3715), (110011|1133|5173), (110011|3311|7351), (001111|0202|4040),$$
$$(001111|2020|6226), (001111|0202|0404), (001111|2020|2662), (000000|0000|0000)\}.$$

So, we have

$$\left(2^{1+3} - 1\right) \cdot m = 2^{1+3-1} (6 + 2 \cdot 4 + 4 \cdot 4) \Rightarrow 15m = 240 \Rightarrow m = 16.$$

Standard matrix,

$$G_S = \left(\begin{array}{c|c|c|c|c} I_{k_0} & \overline{A}_{01} & 2T_1 & 0 & 4T_2 \\ \hline 0 & \overline{S}_2 & S_{02} & I_{k_3} & A_{03} \end{array}\right)$$

That's,

$$G_S = \left(\begin{array}{c|c|c} 100111 & 0202 & 0404 \\ \hline 011011 & 1133 & 1537 \end{array}\right)$$

Proposition 4.6. *Let C be a one-weight code in $\mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ with weight m. If $\mathbf{c}_1 = (\mathbf{u}_1|\mathbf{v}_1|\mathbf{w}_1)$ and $\mathbf{c}_2 = (\mathbf{u}_2|\mathbf{v}_2|\mathbf{w}_2)$ are two distinct order 8 codewords of C, then $\{i|\,(\mathbf{w}_1)_i = \pm1, \pm3\} = \{i|\,(\mathbf{w}_2)_i = \pm1, \pm3\}$*

Proof. Assume $\mathbf{c}_1 = (\mathbf{u}_1|\mathbf{v}_1|\mathbf{w}_1)$ and $\mathbf{c}_2 = (\mathbf{u}_2|\mathbf{v}_2|\mathbf{w}_2)$ are two distinct order 8 vectors in a one-weight code C with weight m. From Proposition 4.2, we have that the number of units in $\mathbf{w}_1$ and $\mathbf{w}_2$ is $\frac{m}{4}$. Consider the vector $\mathbf{c} = 4\mathbf{c}_1 + 4\mathbf{c}_2$. Binary and quaternary part of this vector are 0 and the $\mathbb{Z}_8$ part consists of elements that are either 0 or 4. If $(\mathbf{w}_1)_i = \pm1, \pm3$ whenever $(\mathbf{w}_2)_i = \pm1, \pm3$, then the number of coordinates in $\mathbf{c}$ that are 2,4,6 is 0. That is $\mathbf{c} = 0$. So, assume that they have some units in different coordinates. Since C is a one- weight code with weight m, if $\mathbf{c} \neq 0$. then the number of coordinates where $\mathbf{w}_1$ and $\mathbf{w}_2$ have units in different places must be $\frac{m}{4}$. To obtain this we need

$$\{i|(\mathbf{w}_1)_i = \pm1, \pm3 = -(\mathbf{w}_2)_i = \pm1, \pm3\} = \frac{m}{8}$$

and in all other coordinates where $(\mathbf{w}_1)_i = \pm1, \pm3$ we need that $(\mathbf{w}_2)_i = 0, 2, 4$ or 6, and in all other coordinates where $(\mathbf{w}_2)_i = \pm1, \pm3$ we need that $(\mathbf{w}_1)_i = 0, 2, 4$ or 6. Then consider $\mathbf{d} = \mathbf{c}_1 + 5\mathbf{c}_2$. This vector has the same binary part as $\mathbf{c}_1 + 5\mathbf{c}_2$ but it has a 2, 4 or 6 in the coordinates where $(\mathbf{w}_1)_i = \pm1, \pm3 = -(\mathbf{w}_2)_i$ and has coordinates of equal Lee weight in all other coordinates. Therefore $\mathbf{d}$ has weight greater than m. This gives our result. □

We can give an example related to the above theorem, consider the two $\mathbb{Z}_8$ vectors $\mathbf{x} = (135700)$ and $\mathbf{y} = (003175)$. They both have weight 8 with 4 units, and their sum has 8 weight 4 units. But $\mathbf{x}+5\mathbf{y} = (134431)$ has weight 16.

5. $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-Additive complementary dual codes

In this section, we generalize the concept of additive complementary duality to $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes

Definition 5.1. *A code $C \subseteq \mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ is additive complementary dual (briefly ACD) if it is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code such that $C \cap C^\perp = 0$.*

For $\mathbb{Z}_2\mathbb{Z}_4$-additive complementary dual codes we have the following property.

Lemma 5.2. *([3]) Let $C \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ an ACD code. Then any vector $\mathbf{w} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ can be written uniquely as $\mathbf{w}_1 + \mathbf{w}_2$, for $\mathbf{w}_1 \in C$ and $\mathbf{w}_2 \in C^\perp$.*

Lemma 5.3. *Let $C \subseteq \mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ an ACD code. Then any vector $\mathbf{c} \in \mathbb{Z}_2^r \times \mathbb{Z}_4^s \times \mathbb{Z}_8^t$ can be written uniquely as $\mathbf{c}_1 + \mathbf{c}_2$, for $\mathbf{c}_1 \in C$ and $\mathbf{c}_2 \in C^\perp$.*

Now, we state sufficient conditions for a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code to be ACD.

Theorem 5.4. *Let C be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code and G be a generator matrix for C, which has the rows $g_1, \ldots, g_r$. If $\langle g_i, g_i \rangle \in \{0, 2, 4, 6\}$ and $\langle g_j, g_j \rangle \notin \{0, 2, 4, 6\}$ for all $i, j = 1, \ldots, r$ such that $i \neq j$, then C is an ACD code and C_Z is an octal LCD code.*

Proof. Let $\mathbf{c} \in \mathcal{C} \setminus \{0\}$ be any nonzero codeword. We want to show that $\mathbf{c} \notin \mathcal{C}^{\perp}$. Since $\mathbf{c} \in \mathcal{C}$, $\mathbf{c}$ can be written as $\mathbf{c} = \sum_{i \in J} \epsilon_i \mathbf{g}_i$, where $J = \{1, \ldots, r\}$ and $\epsilon_i \in \mathbb{Z}_8$.

First, assume there exists $j \in J$ such that $\epsilon_j \in \{1, 3, 5, 7\}$. Thus,

$$\left\langle \mathbf{c}, \mathbf{g}_i \right\rangle = \sum_{i \in J} \epsilon_i \left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle = \sum_{i \in J \setminus \{j\}} \epsilon_i \left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle + \epsilon_j \left\langle \mathbf{g}_j, \mathbf{g}_j \right\rangle .$$

Since $\epsilon_i \left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle \in \{0, 2, 4, 6\}$ for $i \neq j$, we have that $\left\langle \mathbf{c}, \mathbf{g}_i \right\rangle \neq 0$ and $\mathbf{c} \notin \mathcal{C}^{\perp}$.

Finally, if $\epsilon_i \in \{0, 2, 4, 6\}$ for all $i \in J$, let $j \in J$ such that $\epsilon_j \in \{2, 4, 6\}$. Note that $\sum_{i \in J} \epsilon_i \left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle \in \{0, 4\}$ and $\epsilon_j \left\langle \mathbf{g}_j, \mathbf{g}_j \right\rangle \in \{2, 6\}$. Hence

$$\left\langle \mathbf{c}, \mathbf{g}_i \right\rangle = \sum_{i \in J} \epsilon_i \left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle = \sum_{i \in J \setminus \{j\}} \epsilon_i \left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle + \epsilon_j \left\langle \mathbf{g}_j, \mathbf{g}_j \right\rangle \in \{2, 6\} .$$

So, $\mathbf{c} \notin \mathcal{C}^{\perp}$. □

Corollary 5.5. *Let G be generator matrix for a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{C}$ and consider the matrix $G \cdot G^T = (a_{ij})_{1 \leq i, j \leq r}$ with entries from $\mathbb{Z}_8$. If $a_{ij} \in \{0, 2, 4, 6\}$ and $a_{ii} \notin \{0, 2, 4, 6\}$ for all $i, j = 1, \ldots, k$ such that $i \neq j$, then $\mathcal{C}$ is an ACD code and $\mathcal{C}_Z$ is an octal LCD code.*

Proof. It is clear from the Theorem 5.4. □

Remark 5.6. *The reverse statements of Theorem 5.4 and Corollary 5.5 are not true in general. Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code generated by*

$$\left(\begin{array}{c|c|cc} 1 & 2 & 0 & 4 \\ 0 & 3 & 4 & 2 \end{array} \right)$$

We have that $\mathcal{C}$ is ACD, but in this case

$$G \cdot G^T = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$$

Corollary 5.7. *If $G \cdot G^T$ is invertible (over $\mathbb{Z}_8$), then $\mathcal{C}$ is an ACD code.*

Remark 5.8. *Again, the reverse statements of Corollary 5.7 is not true in general. Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code generated by*

$$\left(\begin{array}{c|c|cc} 1 & 2 & 0 & 4 \\ 0 & 3 & 4 & 2 \end{array} \right)$$

We have that $\mathcal{C}$ is ACD, but in this case

$$G \cdot G^T = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$$

that is not invertible (over $\mathbb{Z}_8$).

Theorem 5.9. *Let C_1 be a binary (r, k) code and let $\{u_1, \ldots, u_k\}$ be a basis for C_1. Let C_2 be a quaternary (s, l) code and let $\{v_1, \ldots, v_l\}$ be a minimum generating set for C_2. Let $\eta \geq k + l$ and let G_X and G_Y be the $\eta \times r$ and $\eta \times s$ matrices whose non-zero row vectors are $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_l\}$, respectively. Then, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{C}$ of type $(r, s, \eta; 0, 0, 0, \eta, 0, 0)$ generated by*

$$G = \left(\begin{array}{c|c|c} G_X & G_Y & I_\eta \end{array} \right)$$

is an ACD code.

Proof. Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code with generator matrix

$$G = \left(\begin{array}{c|c|c} G_X & G_Y & I_\eta \end{array} \right).$$

Note that $G \cdot G^T = 4G_XG_X^T + 2G_YG_Y^T + I_\eta = (w_{ij})_{1\leq i,j\leq \eta}$ where all entries in G_X and G_Y are considered as elements in $\{0,1\} \subseteq \mathbb{Z}_8$ and $\{0,1,2,3\} \subseteq \mathbb{Z}_8$, respectively. Hence $4G_XG_X^T$ has all entries in $\{0,4\}$ and $2G_YG_Y^T$ has all entries in $\{0,2,4,6\}$. Therefore, $a_{ij} \in \{0,2,4,6\}$ and $a_{ii} \notin \{0,2,4,6\}$ for all $i, j = 1,\ldots,\eta$ such that $i \neq j$, and $\mathcal{C}$ is ACD by Corollary 5.5. The generator matrix G is in standard form and it is easy to see that $\mathcal{C}$ is of type $(r,s,\eta;0,0,0,\eta,0,0)$. □

5.1. Complementary duality of $\mathcal{C}$, $\mathcal{C}_X$, $\mathcal{C}_Y$, $\mathcal{C}_Z$

In this section, the complementary duality of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code in terms of the complementary duality of $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$ is studied.

Example 5.10. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(r,s,t;k_0,k_1,k_2,k_3,k_4,k_5)$ such that $r = t = k_0 = r$ and $k_1 = k_2 = k_3 = k_4 = 0$ and $k_5 = 1$ and generated by*

$$\left(\begin{array}{c|c|c} I_r & I_r & I_r \\ \mathbf{1} & \mathbf{2} & \mathbf{4} \end{array} \right).$$

So, we have $\mathcal{C}_X = \mathbb{Z}_2^r$ is an LCD, $\mathcal{C}_Y = \mathbb{Z}_4^r$ is an LCD and $\mathcal{C}_Z = \mathbb{Z}_8^r$ is also an LCD. But, the last row of the generator matrix $g_{r+1} = (\mathbf{1}|\mathbf{2}|\mathbf{4})$ is orthogonal to any row in the generator matrix. Therefore, $g_{r+1} \in \mathcal{C} \cap \mathcal{C}^\perp$ and $\mathcal{C}$ is not complementary dual.

Example 5.11. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code generated by*

$$G = \left(\begin{array}{cc|cc|cc} 1 & 1 & 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 2 & 5 & 2 \end{array} \right).$$

So, we have the parity-check matrix of $\mathcal{C}$ is

$$H = \left(\begin{array}{cc|cc|cc} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 & 4 & 0 \\ 0 & 0 & 2 & 0 & 6 & 1 \end{array} \right).$$

Note that $(1|1) \in \mathcal{C}_X \cap \mathcal{C}_X^\perp$, $(2|0) \in \mathcal{C}_Y \cap \mathcal{C}_Y^\perp$ and $(4|0) \in \mathcal{C}_Z \cap \mathcal{C}_Z^\perp$. Hence, $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$ are not a binary LCD, a quaternary LCD and an octal LCD, respectively. But, we have that $\mathcal{C}$ is an ACD code since $\mathcal{C} \cap \mathcal{C}^\perp = \{\mathbf{0}\}$

Proposition 5.12. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code. If $\mathcal{C}$ is separable, then $\mathcal{C}$ is an ACD code if and only if $\mathcal{C}_X$ is a binary LCD code, $\mathcal{C}_Y$ is a quaternary LCD code and $\mathcal{C}_Z$ is an octal LCD code.*

Proof. Since $\mathcal{C}$ is a separable $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code. So, we have $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y \times \mathcal{C}_Z$. Assume that $\mathcal{C}$ is an ACD code. Any codeword $\mathbf{c} = (\mathbf{u}|\mathbf{v}|\mathbf{w}) \in \mathcal{C} \cap \mathcal{C}^\perp$ is the zero codeword. Let $\mathbf{u} \in \mathcal{C}_X \cap \mathcal{C}_X^\perp$. The codeword $(\mathbf{u}|\mathbf{0}|\mathbf{0}) \in \mathcal{C} \cap \mathcal{C}^\perp$ and this implies that $\mathbf{u} = \mathbf{0}$. Therefore, $\mathcal{C}_X$ is a binary LCD code. Similarly, it easily shown that $\mathcal{C}_Y$ and $\mathcal{C}_Z$ are quaternary LCD code and octal LCD code, respectively. Suppose that $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$ are binary, quaternary and octal LCD code, respectively. Let $\mathbf{c} = (\mathbf{u}|\mathbf{v}|\mathbf{w}) \in \mathcal{C} \cap \mathcal{C}^\perp$. This implies that $\mathbf{u} \in \mathcal{C}_X \cap \mathcal{C}_X^\perp = \{\mathbf{0}\}$, $\mathbf{v} \in \mathcal{C}_Y \cap \mathcal{C}_Y^\perp = \{\mathbf{0}\}$ and $\mathbf{w} \in \mathcal{C}_Z \cap \mathcal{C}_Z^\perp = \{\mathbf{0}\}$. Then, $\mathbf{c} = (\mathbf{u}|\mathbf{v}|\mathbf{w})$ is the zero codeword and $\mathcal{C}$ is an ACD code. □

Example 5.13. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code generated by*

$$G = \left(\begin{array}{cc|cc|cc} 1 & 0 & 2 & 1 & 7 & 0 \\ 0 & 1 & 1 & 2 & 0 & 5 \end{array} \right).$$

Let g_1 and g_2 be the row vectors of G. Clearly, $\langle g_1, g_2 \rangle = 0$, $\langle g_1, g_1 \rangle \notin \{0,2,4,6\}$ and $\langle g_2, g_2 \rangle \notin \{0,2,4,6\}$. Hence, from Theorem 5.4 $\mathcal{C}$ is an ACD code. Furthermore, $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$ are LCD codes.

6. Conclusion

In this paper, it is studied that some properties of one-weight $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes and the concept of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive complementary dual codes (or ACD). When $\mathcal{C}$ is an ACD code, it is considered complementary duality of $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$. Also, it is presented some illustrative examples for one-weight and ACD $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes.

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