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On Geometric Space and its Applications in Topological H_v-Groups

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Abstract. We generalize the concept of topological hypergroup to topological H_v-group and define some topologies on H_v-groups by using the concept of geometric space, which was defined by Freni. By applying these topologies, we have always a topological H_v-group without need any more conditions. Moreover, we state some conditions on a topological H_v-group that make it complete. Our aim is to generalize the concept of topological hypergroup to topological H_v-group via considering the characteristics of the induced geometric space.

1. Introduction and Preliminaries

The concept of hypergroup was first introduced in 1934, by Marty [19]. Then this subject was investigated by many other researchers; for example, see [3, 4, 6, 9, 10, 18, 20–22, 26]. The notion of an H_v-structure is a generalization of algebraic hyperstructures, which was defined by Vougiouklis at the Fourth AHA congress (1990) [27]. He played an important role in extending this concept [28–30]. Davvaz, Spartalis, Dramalidis, Leoreanu-Fotea, S. Hoskova, and others, later joined him in developing this concept in many directions. Finally, Davvaz and Vougiouklis collected all the results until 2018 in [8]. Recall from [8] that a *hypergroupoid* $(H, \circ)$ is a nonempty set H with a map $\circ : H \times H \to \mathcal{P}^*(H)$ called *(binary) hyperoperation*, where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H. The image of the pair (x, y) is denoted by $x \circ y$. Let $A, B \in \mathcal{P}^*(H)$ and let $x \in H$. Then the operation $\circ$ is defined

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b, \quad A \circ x = A \circ \{x\}, \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Consider the hyperoperation "$\circ$" in H:

- It is called *associative* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$. This means that $\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v$.
- It is called *weak associative* if $x \circ (y \circ z) \bigcap (x \circ y) \circ z \neq \emptyset$ for all $x, y, z \in H$.
- It is called *(strong) commutative* if $x \circ y = y \circ x$ for all $x, y \in H$.
- It is called *weak commutative* if $x \circ y \cap y \circ x \neq \emptyset$ for all $x, y \in H$.
- A hypergroupoid $(H, \circ)$ has the *reproduction axiom* if $a \circ H = H \circ a = H$ for all $a \in H$.

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- A hypergroupoid $(H, \circ)$ is *finite* if it has only finite elements.
- A hypergroupoid $(H, \circ)$ is called an *H_v-semigroup* if the hyperoperation "$\circ$" is weak associative.
- A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if it has the reproduction axiom.

A semihypergroup $(H, \circ)$ is called a *hypergroup* if it is a quasihypergroup. An H_v-semigroup $(H, \circ)$ is called an *H_v-group* if it has the reproduction axiom. An element e in hypergroupoid $(H, \circ)$ is called *identity* if $x \in e \circ x \cap x \circ e$ for all $x \in H$. An element $a' \in H$ is called an *inverse* of $a \in H$ if there exists an identity $e \in H$ such that $e \in a \circ a' \cap a' \circ a$. A hypergroup is called *regular* if it has at least one identity and each element has at least one inverse. Each semihypergroup is an H_v-semigroup. If $(H, \circ)$ is a hypergroup such that $| x \circ y |= 1$ for all $x, y \in H$, then $(H, \circ)$ is a group. A nonempty subset K of a semihypergroup $(H, \circ)$ is called a *subsemihypergroup* if it is a semihypergroup. In other words, a nonempty subset K of a semihypergroup $(H, \circ)$ is a subsemihypergroup if $K \circ K \subseteq K$. A nonempty subset K of a hypergroup (H_v-group resp.) $(H, \circ)$ is a *subhypergroup* (*H_v-subgroup* resp.) if $(K, \circ)$ has the reproduction axiom, that is, $a \circ K = K \circ a = K$ for all $a \in K$.

Until now, limited papers have noticed the concept of topological hyperstructures; for example, see [2, 13–15, 17, 24]. Heidari, Davvaz, and Modarres [13] introduced the notion of topological hypergroup. Singha, Das, and Davvaz [25] defined the concept of topological complete hypergroups and investigated some of their properties. In this paper, we generalize the concept of topological hypergroup to topological H_v-group and introduce some topologies on H_v-groups by the concept of geometric space, which was defined by Freni [11]. Recall from [11] that, a *geometric space* is a pair $(S, \mathcal{B})$ such that S is a nonempty set in which its elements are called *points* and $\mathcal{B}$ is a nonempty family of subsets of S, which its elements are called *blocks*. If C is a subset of S, then it is called a *$\mathcal{B}$-part* of S if $B \cap C \neq \emptyset$ implies $B \subseteq C$ for every $B \in \mathcal{B}$. For a subset $X \subseteq S$, the intersection of all $\mathcal{B}$-parts of S containing X is denoted by $\Gamma(X)$. For each subsets X and Y of a geometric space $(S, \mathcal{B})$, the following properties are true:

(P1) $X \subseteq \Gamma(X)$,

(P2) $X \subseteq Y \Rightarrow \Gamma(X) \subseteq \Gamma(Y)$,

(P3) $\Gamma(\Gamma(X)) = \Gamma(X)$,

(P4) $\Gamma(X) = \bigcup_{x \in X} \Gamma(x)$, where $\Gamma(x) = \Gamma(\{x\})$.

For all subsets X of S, Freni [11] described an ascending chain of subsets $(\Gamma_n(X))_n$, called *cone* of X, which has the following conditions: $\Gamma_0(X) = X$ and$\Gamma_{n+1}(X) = \Gamma_n(X) \cup (\cup\{B \in \mathcal{B} \mid B \cap \Gamma_n(X) \neq \emptyset\})$ for every integer $n \geq 0$. Using the notion of cone of X, we can obtain the closure of X, as shown in the next result.

Proposition 1.1. ([11]) *Let $(S, \mathcal{B})$ be a geometric space. For every $n \in \mathbb{N}$ and for every subsets X, Y of S, the following properties are true:*

1. $X \subseteq Y \Rightarrow \Gamma_n(X) \subseteq \Gamma_n(Y)$.
2. $\Gamma_n(X) = \cup_{x \in X} \Gamma_n(x)$ *where* $\Gamma_n(x) = \Gamma_n(\{x\})$.
3. $\Gamma_n(\Gamma_m(X)) = \Gamma_{n+m}(X)$.
4. $\Gamma_n(X) = \cup_{n \in \mathbb{N}} \Gamma_n(X)$.
5. *If the family $\mathcal{B}$ is a covering of S (i.e., $S = \cup_{B \in \mathcal{B}} B$), then* $\Gamma_{n+1}(X) = \cup\{B \in \mathcal{B} \mid B \cap \Gamma_n(X) \neq \emptyset\}$.

Corollary 1.2. ([11]) *Let $(S, \mathcal{B})$ be a geometric space and let B be an element of $\mathcal{B}$. Then*

1. $\Gamma(x) = \Gamma(y)$ *for each* $x, y \in B$.
2. $\Gamma(B) = \Gamma(x)$ *for all* $x \in B$.

Corollary 1.3. ([11]) *Let $(S, \mathcal{B})$ be a geometric space and let X be a subset of S. If there exists $n \in \mathbb{N}$ such that $\Gamma_{n+1}(X) \subseteq \Gamma_n(X)$, then $\Gamma_k(X) = \Gamma_n(X)$ for every integer $k > n$. Moreover $\Gamma(X) = \Gamma_n(X)$.*

2. Geometric Spaces

It is interesting to find the relation between the concept of basis of a topology and blocks of a geometric space. In fact, we are willing to know that, when the blocks of a geometric space form a basis for a topology. By the definition of geometric space, the family $\mathcal{F}_{\mathcal{B}}(S)$ of all $\mathcal{B}$-parts of a geometric space S has following properties:

1. $\emptyset, S \in \mathcal{F}_{\mathcal{B}}(S)$.
2. $\mathcal{F}_{\mathcal{B}}(S)$ is closed under the intersection.
3. $\mathcal{F}_{\mathcal{B}}(S)$ is closed under the union.

Corollary 2.1. *Let $(S, \mathcal{B})$ be a geometric space; then the following properties hold:*

1. *The family of $\mathcal{B}$-parts of S is a topology on S. The open subsets of this topology are $\mathcal{B}$-parts.*
2. *The family of the complement of $\mathcal{B}$-parts of S is a topology on S. The closed subsets of this topology are $\mathcal{B}$-parts.*

Proof. 1. Since $\emptyset, S \in \mathcal{F}_{\mathcal{B}}(S)$ and $\mathcal{F}_{\mathcal{B}}(S)$ is closed under union and (finite) intersection, $\mathcal{F}_{\mathcal{B}}(S)$ is a topology on S. Therefore, the open subsets of this topology are $\mathcal{B}$-parts.

2. Since $\mathcal{F}_{\mathcal{B}}(S)$ is closed under intersection and (finite) union, the family of the complement of $\mathcal{B}$-parts of S is closed under union and (finite) intersection. Therefore, the family of the complement of $\mathcal{B}$-parts of S is a topology on S and the closed subsets of this topology are $\mathcal{B}$-parts. □

Definition 2.2. Let $(S, \mathcal{B})$ be a geometric space. The *open topology corresponding to* $\mathcal{B}$ is $\mathcal{T}^o_{\mathcal{B}}(S) = \mathcal{F}_{\mathcal{B}}(S)$ and the *closed topology corresponding to* $\mathcal{B}$ is $\mathcal{T}^c_{\mathcal{B}}(S) = \left\{S \setminus B \,\middle|\, B \in \mathcal{F}_{\mathcal{B}}(S)\right\}$.

The open sets of topology $\mathcal{T}^o_{\mathcal{B}}(S)$ are the closed sets of topology $\mathcal{T}^c_{\mathcal{B}}(S)$, and vice versa. By Corollary 2.1, $\Gamma(X)$ is the *closure* of X in topology $\mathcal{T}^c_{\mathcal{B}}(S)$. Throughout of the paper, whenever the type of these topologies is not important or the properties are common for them, we call it by *corresponding topology* and notify it by $\mathcal{T}_{\mathcal{B}}(S)$.

Example 2.3. 1. If $(S, \mathcal{B})$ is a geometric space and $\mathcal{B}$ is the family of singletons, then $\mathcal{F}_{\mathcal{B}}(S) = \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the set of all subsets of S. Thus $\mathcal{T}^o_{\mathcal{B}}(S)$ and $\mathcal{T}^c_{\mathcal{B}}(S)$ are the discrete topology.

2. If (S, τ) is a topological space, then (S, τ) is a geometric space and $\mathcal{F}_\tau(S) = \{\emptyset, S\}$. Therefore, $\mathcal{T}^o_\tau(S)$ is the trivial topology.

3. Let $S = \{a, b, c, d\}$, let $\mathcal{B}_1 = \{\{a\}, \{b\}, \{a, c\}, \{c, d\}\}$, let $\mathcal{B}_2 = \{\{a\}, \{c, d\}\}$, and let $\mathcal{B}_3 = \{\{a\}, \{a, c\}\}$. Then

$$\mathcal{T}^o_{\mathcal{B}_1}(S) = \mathcal{F}_{\mathcal{B}_1}(S) = \{\emptyset, \{b\}, \{a, c, d\}, S\};$$

$$\mathcal{T}^o_{\mathcal{B}_2}(S) = \mathcal{F}_{\mathcal{B}_2}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, S\};$$

$$\mathcal{T}^o_{\mathcal{B}_3}(S) = \mathcal{F}_{\mathcal{B}_3}(S) = \{\emptyset, \{b\}, \{d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, S\};$$

$$\mathcal{T}^c_{\mathcal{B}_1}(S) = \{\emptyset, \{b\}, \{a, c, d\}, S\};$$

$$\mathcal{T}^c_{\mathcal{B}_2}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, S\};$$

$$\mathcal{T}^c_{\mathcal{B}_3}(S) = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}, S\}.$$

Lemma 2.4. *Let $(S, \mathcal{B})$ be a geometric space and let $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 \neq \emptyset$. Every $\mathcal{B}$-part contains B_1, and also contains B_2.*

Proof. Let $C \subseteq S$ be a $\mathcal{B}$-part containing B_1; then $\emptyset \neq B_2 \cap B_1 \subseteq B_2 \cap C$, so $B_2 \subseteq C$. □

Proposition 2.5. *Let $(S, \mathcal{B}_1)$ and $(S, \mathcal{B}_2)$ be geometric spaces and let $\mathcal{B}_2 \subseteq \mathcal{B}_1$. Then $\mathcal{T}_{\mathcal{B}_1}(S) \subseteq \mathcal{T}_{\mathcal{B}_2}(S)$.*

Proof. By Lemma 2.4, $\mathcal{F}_{\mathcal{B}_1}(S) \subseteq \mathcal{F}_{\mathcal{B}_2}(S)$, thus $\mathcal{T}_{\mathcal{B}_1}(S) \subseteq \mathcal{T}_{\mathcal{B}_2}(S)$. □

The n-tuple $(B_1, B_2, \ldots, B_n)$ of blocks of geometric space $(S, \mathcal{B})$ is called a *polygonal* if $B_i \cap B_{i+1} \neq \emptyset$ for each $1 \leq i < n$. By the concept of polygonal, Freni [11], defined the relation $\approx$ as follows:

$$x \approx y \iff x = y \text{ or there exists a polygonal } (B_1, B_2, \ldots, B_n) \text{ such that } x \in B_1 \text{ and } y \in B_n.$$

The relation $\approx$ is an equivalence and coincides with the transitive closure of the following relation:

$$x \sim y \iff x = y \text{ or there exists } B \in \mathcal{B} \text{ such that } \{x, y\} \subseteq B.$$

Hence $\approx$ is equal to $\bigcup_{n\geq 1} \sim^n$, where $\sim^n = \underbrace{\sim \circ \sim \circ \cdots \circ \sim}_{n \text{ times}}$. Freni [11] proved that the $\approx$-class of x in S identified by $[x]$, coincides with $\Gamma(x)$ (see the following proposition).

Proposition 2.6. ([11]) *Let $(S, \mathcal{B})$ be a geometric space. Then for each integer $n \geq 1$ and for each $x, y \in S$,*

1. $y \sim^n x \iff y \in \Gamma_n(x)$,
2. $[x] = \Gamma(x)$.

Corollary 2.7. ([11]) *Let $(S, \mathcal{B})$ be a geometric space. Then for each integer $n \geq 1$,*

1. $\sim^n$ *is transitive* $\iff \Gamma(x) = \Gamma_n(x)$ *for all* $x \in S$,
2. $\sim$ *is transitive* $\iff \Gamma(x) = \Gamma_1(x)$ *for all* $x \in S$.

Proposition 2.8. ([11]) *Let $(S, \mathcal{B})$ be a geometric space. Then for all blocks $A, B \in \mathcal{B}$ and for each integer $n \geq 1$, the following conditions are equivalent:*

1. *If $A \cap B \neq \emptyset$ and $x \in B$, then there exists $C \in \mathcal{B}$ such that $(A \cup \{x\}) \subseteq C$.*
2. *If $A \cap B \neq \emptyset$ and $x \in \Gamma_n(B)$, then there exists $C \in \mathcal{B}$ such that $(A \cup \{x\}) \subseteq C$.*
3. *If $A \cap \Gamma_n(B) \neq \emptyset$ and $x \in \Gamma_n(B)$, then there exists $C \in \mathcal{B}$ such that $(A \cup \{x\}) \subseteq C$.*

If one of the equivalent conditions of the previous proposition is satisfied and $\mathcal{B}$ is a covering of S, then the geometric space $(S, \mathcal{B})$ was called *strongly transitive* by Freni [11]. If $(S, \mathcal{B})$ is a strongly transitive geometric space, then the relation $\sim$ on S is transitive, that is, $\sim = \approx$ (see [11]).

Definition 2.9. An element x of the geometric space $(S, \mathcal{B})$ is called *single* if $\{x\}$ is a $\mathcal{B}$-part. It means that x is a single if $\Gamma(x) = \{x\}$.

By the definition of $\Gamma(x)$, it is clear that x is a single element of geometric space $(S, \mathcal{B})$ if and only if $\{x\} \cap B = \emptyset$ for all $B \in \mathcal{B} \setminus \{x\}$.

In the corresponding topologies of a geometric space, a subset U is open (closed resp.) if and only if U is a $\mathcal{B}$-part. In general, we have the following definition.

Definition 2.10. A geometric space $(S, \mathcal{B})$ with topology τ on S is called *τ-open* (*τ-closed* resp.) if every block $B \in \mathcal{B}$ is an open (closed resp.) subset of (S, τ).

A geometric space $(S, \mathcal{B})$ is $\mathcal{T}^o_{\mathcal{B}}(S)$-open and $\mathcal{T}^c_{\mathcal{B}}(S)$-closed.

Proposition 2.11. *Let $(S, \mathcal{B})$ be a geometric space with topology τ on S and let $U \subseteq S$. Then we have the following properties:*

1. *If $(S, \mathcal{B})$ is a τ-open geometric space and U is open, then $\Gamma(U)$ is open.*
2. *If $(S, \mathcal{B})$ is a τ-closed geometric space such that $\mathcal{B}$ is finite and U is closed, then $\Gamma(U)$ is closed.*

Proof. 1. Since $\Gamma_1(U) = U \cup (\cup\{B \in \mathcal{B} \mid B \cap U \neq \emptyset\})$ is a union of open subsets of S, it is open. By induction, $\Gamma_n(X)$ is open, Hence $\Gamma(U)$ is open.

2. Since $\Gamma_1(U) = U \cup (\cup\{B \in \mathcal{B} \mid B \cap U \neq \emptyset\})$ is a finite union of closed subsets of S, it is closed; like the open case, $\Gamma(U)$ is closed. □

The first part of Proposition 2.11, means that the natural map $\Gamma : S \to \mathcal{F}_{\mathcal{B}}(S)$ is an open map. In the special case, when $\mathcal{B}$ is a covering of S, we have the following corollary.

Corollary 2.12. *Let $(S, \mathcal{B})$ be a geometric space with topology τ on S, such that $S = \bigcup_{B \in \mathcal{B}} B$ and $U \subseteq S$.*

1. *If $(S, \mathcal{B})$ is a τ-open geometric space, then $\Gamma(U)$ is open.*
2. *If $(S, \mathcal{B})$ is a τ-closed geometric space such that $\mathcal{B}$ is finite, then $\Gamma(U)$ is closed.*

Recall from the topological theory (see [23, 31]) that, if (X, τ) is a topological space, a *basis (open basis)* for τ, is a collection $\mathfrak{B} \subseteq \tau$ such that every open set is a union of subfamily of $\mathfrak{B}$. In other words, $\mathfrak{B}$ is a basis for a topological space (X, τ) if and only if

1. $X = \bigcup_{B \in \mathfrak{B}} B$,
2. for each $B_1, B_2 \in \mathfrak{B}$ with $x \in B_1 \cap B_2$, there is some $B_3 \in \mathfrak{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Closed sets can equally describe the topology of a space. Thus there is a dual notion of a basis for the closed sets of a topological space. A *closed basis* or a *basis for the closed sets* of topological space (X, τ) is any family of closed sets in X such that every closed set is an intersection of the subfamily. In other words, $\mathfrak{F}$ is a closed basis for a topological space (X, τ) if and only if

1. $\bigcap_{F \in \mathfrak{F}} F = \emptyset$,
2. for each $F_1, F_2 \in \mathfrak{F}$, the union $F_1 \cup F_2$ is the intersection of some subfamily of $\mathfrak{F}$ (i.e., for any x not in F_1 or F_2, there is $F_3 \in \mathfrak{F}$ containing $F_1 \cup F_2$ and not containing x).

It is easy to check that $\mathfrak{F}$ is a closed basis of X if and only if the family of complements of members of $\mathfrak{F}$ is an open basis of X.

A *subbasis* of topological space (X, τ) is a subcollection $\mathfrak{B}$ of τ that generates the topology τ. This means that τ is the smallest topology containing $\mathfrak{B}$. In other words, the collection of open sets consisting of all finite intersections of elements of $\mathfrak{B}$, together with the set X, forms a basis for τ. This means that every proper open set in τ can be written as a union of finite intersections of elements of $\mathfrak{B}$.

Equally, a *closed subbasis* of topological space (X, τ) is defined as a subcollection $\mathfrak{F}$ of closed subsets X, such that every closed set of X can be written as a union of elements of $\mathfrak{F}$.

Definition 2.13. Let $(S, \mathcal{B})$ be a geometric space. A block $B \in \mathcal{B}$ is *complete* if B is a $\mathcal{B}$-part. A geometric space is *complete* if every block of it is complete.

By the definition of Γ, the following corollary is immediately obtained.

Corollary 2.14. *A geometric space $(S, \mathcal{B})$ is complete, if and only if $\Gamma(B) = B$ for each block $B \in \mathcal{B}$.*

Proposition 2.15. *The geometric space $(S, \mathcal{B})$ is complete if and only if $\mathcal{B}$ is pairwise disjoint.*

Proof. If $\mathcal{B}$ is pairwise disjoint, then clearly the geometric space $(S, \mathcal{B})$ is complete. Conversely let $(S, \mathcal{B})$ be complete, and there exist $B_i, B_j \in \mathcal{B}$ such that $B_i \neq B_j$ and $B_i \cap B_j \neq \emptyset$. By Corollary 1.2, $\Gamma(B_i) = \Gamma(B_j)$, so by (P1), we have $B_i \cup B_j \subseteq \Gamma(B_i)$, thus $B_i \neq \Gamma(B_i)$, which contradicts with completeness of the geometric space $(S, \mathcal{B})$. □

Theorem 2.16. *If $(S, \mathcal{B})$ is a complete geometric space such that $S = \bigcup_{B \in \mathcal{B}} B$, then $\mathcal{B}$ is an open basis of topology $\mathcal{T}^o_{\mathcal{B}}(S)$, and a closed subbasis of topology $\mathcal{T}^c_{\mathcal{B}}(S)$. Moreover $(S, \mathcal{B})$ is transitive.*

Proof. Let $X \subseteq S$ be open (closed resp.); then $\Gamma(X) = X$. By (P4) and Corollary 1.2, we have

$$X = \bigcup_{x \in X} \Gamma(x) = \bigcup_{\substack{B \in \mathcal{B} \\ B \cap X \neq \emptyset}} \Gamma(B) = \bigcup_{\substack{B \in \mathcal{B} \\ B \cap X \neq \emptyset}} B.$$

Thus X is the union of elements of $\mathcal{B}$, which means that $\mathcal{B}$ is an open basis of topology $\mathcal{T}^o_{\mathcal{B}}(S)$ and a closed subbasis of topology $\mathcal{T}^c_{\mathcal{B}}(S)$.

By Proposition 1.1(5), $\Gamma_1(X) = \bigcup_{\substack{B \in \mathcal{B} \\ B \cap X \neq \emptyset}} B = \Gamma(X)$, so $\sim = \approx$, which means that the geometric space is transitive. □

In general, if $(S,\mathcal{B})$ is a τ-open geometric space such that $S = \bigcup_{B\in\mathcal{B}} B$, then $\mathcal{B}$ is an open basis of topology τ. Conversely if (S,τ) is a topological space and $\mathfrak{B}$ is a basis of τ, then $(S,\mathfrak{B})$ is a τ-open geometric space.

By the definition of open corresponding topology of a geometric space, every open set is a $\mathcal{B}$-part, or in general we have the following definition.

Definition 2.17. A geometric space $(S,\mathcal{B})$ with a topology τ on S is called *τ-complete* if every open set of S is a $\mathcal{B}$-part. In other words, the geometric space $(S,\mathcal{B})$ with a topology τ on S is τ-complete if $\Gamma(U) = U$ for every $U \in \tau$.

Immediately, we have the following corollary.

Corollary 2.18. *Let $(S,\mathcal{B})$ be a geometric space and let τ be a topology on S. If $(S,\mathcal{B})$ is τ-complete, then the natural map $\Gamma : S \to \mathcal{F}_{\mathcal{B}}(S)$ is an open map.*

Clearly, the $\mathcal{T}^o_{\mathcal{B}}(S)$-complete geometric space $(S,\mathcal{B})$, is complete.

If $(S,\mathcal{B})$ is a τ-complete geometric space and $\mathfrak{B}$ is a basis of τ, then by Corollary 2.14, $(S,\mathfrak{B})$ is a complete geometric space, and so by Proposition 2.15, $\mathfrak{B}$ is a family of pairwise disjoint subsets of S.

Proposition 2.19. *If a geometric space $(S,\mathcal{B})$ is τ-open and τ-complete for a topology τ on S, then it is complete.*

Proof. Let $B \in \mathcal{B}$. Since $(S,\mathcal{B})$ is τ-open, then $B \in \tau$, so by τ-completeness, $\Gamma(B) = B$. Hence Corollary 2.14 yields that $(S,\mathcal{B})$ is complete. □

Let $(S_i,\mathcal{B}_i)$ be a geometric space with topology τ_i on S_i for $i = 1,2$. Then $(S_1 \times S_2, \mathcal{B}_1 \times \mathcal{B}_2)$ is a geometric space, where $\mathcal{B}_1 \times \mathcal{B}_2 = \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$. Let $X \subseteq S_1 \times S_2$. Proposition 1.1 implies

$$\Gamma_i(X) = \bigcup_{x\in X}\Gamma_i(x) = \bigcup_{(x_1,x_2)\in X}\Gamma_i(x_1,x_2) = \bigcup_{(x_1,x_2)\in X}\Gamma_i(x_1)\times\Gamma_i(x_2).$$

Thus $\Gamma(X) = \bigcup_{(x_1,x_2)\in X}\Gamma(x_1)\times\Gamma(x_2)$. If $X = X_1 \times X_2$, then $\Gamma_i(X) = \Gamma_i(X_1 \times X_2) = \Gamma_i(X_1) \times \Gamma_i(X_2)$ for each $i \in \mathbb{N}$. Clearly if $(S_i,\mathcal{B}_i)$ is (strongly) transitive (τ_i-open, τ_i-closed, τ_i-complete, or complete resp.) for $i = 1,2$, then $(S_1 \times S_2, \mathcal{B}_1 \times \mathcal{B}_2)$ is (strongly) transitive $\big((\tau_1 \times \tau_2)$-open, $(\tau_1 \times \tau_2)$-closed, $(\tau_1 \times \tau_2)$-complete, or complete resp.$\big)$. By induction, we have the following proposition.

Proposition 2.20. *Let $(S_i,\mathcal{B}_i)$ be geometric spaces with topology τ_i on S_i for $i = 1,2,\ldots,n$. Then $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathcal{B}_i)$ is a geometric space and we have the following properties:*

1. *If $X \subseteq \prod_{i=1}^n S_i$, then $\Gamma_t(X) = \prod_{(x_1,\ldots,x_n)\in X}\Gamma_t(x_i)$ for each $t \in \mathbb{N}$ and thus $\Gamma(X) = \prod_{(x_1,\ldots,x_n)\in X}\Gamma(x_i)$.*
2. *If $(S_i,\mathcal{B}_i)$ is (strongly) transitive for $i = 1,2,\ldots,n$, then $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathcal{B}_i)$ is (strongly) transitive.*
3. *If $(S_i,\mathcal{B}_i)$ is τ_i-open (τ_i-closed resp.) for $i = 1,2,\ldots,n$, then $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathcal{B}_i)$ is $(\prod_{i=1}^n \tau_i)$-open ($(\prod_{i=1}^n \tau_i)$-closed resp.).*
4. *If $(S_i,\mathcal{B}_i)$ is τ_i-complete (complete resp.) for $i = 1,2,\ldots,n$, then $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathcal{B}_i)$ is $(\prod_{i=1}^n \tau_i)$-complete (complete resp.).*

Let (S,τ) be a topological space. Then, the family $\boldsymbol{\mathcal{U}}$ consisting of all sets

$$S_U = \{V \in \mathcal{P}^*(S) \mid V \subseteq U\}, \quad U \in \tau,$$

is a base for a topology on $\mathcal{P}^*(S)$, which is denoted by τ^*; see [15]. For more information and some example see [1, 5, 16].

Definition 2.21. Let $(S,\mathcal{B})$ be a geometric space. The *induced geometric space* on $\mathcal{P}^*(S)$ is $(\mathcal{P}^*(S),\mathcal{B}^*)$, where

$$\mathcal{B}^* = \{B^* \mid B \in \mathcal{B}\}, \text{ where } B^* = \{V \in \mathcal{P}^*(S) \mid V \subseteq B\}.$$

Proposition 2.22. *Let $(S,\mathcal{B})$ be a geometric space with topology τ on S and let $(\mathcal{P}^*(S),\mathcal{B}^*)$ be the induced geometric space with topology τ^* on $\mathcal{P}^*(S)$. then*

1. *$(S,\mathcal{B})$ is complete $\iff$ $(\mathcal{P}^*(S),\mathcal{B}^*)$ is complete.*
2. *$(S,\mathcal{B})$ is τ-open $\Longrightarrow$ $(\mathcal{P}^*(S),\mathcal{B}^*)$ is τ^*-open.*
3. *$(S,\mathcal{B})$ is τ-complete $\Rightarrow$ $(\mathcal{P}^*(S),\mathcal{B}^*)$ is τ^*-complete.*

Proof. 1. Let $(S,\mathcal{B})$ be complete; then by Proposition 2.15, $\mathcal{B}$ is pairwise disjoint. Let $\mathcal{B}^*$ not be pairwise disjoint, that is, there are $B_1^*, B_2^* \in \mathcal{B}*$ such that $B_1^* \cap B_2^* \neq \emptyset$. Then there is $V \in \mathcal{P}^*(S)$ such that $V \in B_1^* \cap B_2^*$. Therefore $V \subseteq B_1 \cap B_2 = \emptyset$, which contradicts with completeness of $(S,\mathcal{B})$, Hence by Proposition 2.15, $(\mathcal{P}^*(S),\mathcal{B}^*)$ is complete.

Conversely, if $\{x\} \subseteq B_1 \cap B_2 \neq \emptyset$for $B_1, B_2 \in \mathcal{B}$, then $\{x\} \in B_1^* \cap B_2^*$, which contradicts with completeness of $(\mathcal{P}^*(S),\mathcal{B}^*)$. Hence Proposition 2.15 completes the proof.

2. Let $(S,\mathcal{B})$ be a τ-open geometric space and let $B^* \in \mathcal{B}^*$. Since $(S,\mathcal{B})$ is τ-open, we have $B^* \in \tau^*$. Thus $(\mathcal{P}^*(S),\mathcal{B}^*)$ is τ^*-open.

3. Let $(S,\mathcal{B})$ be a τ-complete geometric space and let $S_U = \{V \in \mathcal{P}^*(S) \mid V \subseteq U\}$ be an element of the basis of τ^*. Since every open set is a union of subfamily of the basis, by (P4), it is sufficient to prove that $\Gamma(S_U) = S_U$. By the definition, $\Gamma_1(S_U) = S_U \cup \bigcup\{B^* \in \mathcal{B}^* \mid S_U \cap B^* \neq \emptyset\}$, but $S_U \cap B^* = \{V \in \mathcal{P}^*(S) \mid V \subseteq U \cap B\}$. Therefore $B^* \subseteq \Gamma_1(S_U)$ if and only if $U \cap B \neq \emptyset$ if and only if $B \subseteq \Gamma_1(U) \subseteq \Gamma(U)$, but $\Gamma(U) = U$ since $(S,\mathcal{B})$ is τ-complete. Hence $B^* \subseteq \Gamma_1(S_U)$ if and only if $B \subseteq U$. Indeed $B \subseteq U$ yields that $B \in S_U$, so $B^* \subseteq S_U$. Thus $\Gamma_1(S_U) = S_U$, and by Corollary 1.3, $\Gamma(S_U) = S_U$. □

Recall that, a topological space X is called compact if any open cover of X has a finite subcover.

Proposition 2.23. *Let (S,τ) be a topological space and let $(S,\mathcal{B})$ be a τ-complete geometric space such that $|\mathcal{F}_{\mathcal{B}}(S)| < \infty$. Then the topological space (S,τ) is compact.*

Proof. Since $(S,\mathcal{B})$ is τ-complete, then $\tau \subseteq \mathcal{F}_{\mathcal{B}}(S)$. Therefore $|\tau| < \infty$, and hence S is compact. □

Proposition 2.24. *Let (S,τ) be a compact topological space and let $(S,\mathcal{B})$ be a τ-open geometric space such that $\mathcal{B}$ is a cover of S. Then $|\mathcal{F}_{\mathcal{B}}(S)| < \infty$.*

Proof. Since $(S,\mathcal{B})$ is τ-open, then $S = \bigcup_{B\in\mathcal{B}} B$ is an open cover for S, so there is a finite subcover $C \subseteq \mathcal{B}$, such that $S = \bigcup_{B\in C} B$. Since $S = \bigcup_{B\in C} B$, then $\mathcal{F}_C(S) = \{\bigcup_{B\in\mathcal{M}} B \mid \mathcal{M} \in \mathcal{P}(C)\}$. Indeed, $\mathcal{P}(C)$ is finite since C is finite. By Proposition 2.5, $\mathcal{F}_{\mathcal{B}}(S) \subseteq \mathcal{F}_C(S)$ and it completes the proof. □

Definition 2.25. A map $f : (S_1,\mathcal{B}_1) \to (S_2,\mathcal{B}_2)$ between the geometric spaces, is called a *good morphism* if

$$x \sim y \Rightarrow f(x) \sim f(y) \quad \text{for all } x, y \in S_1.$$

Theorem 2.26. *Let $f : (S_1,\mathcal{B}_1) \to (S_2,\mathcal{B}_2)$ be a map between the geometric spaces. If f is a good morphism, then*

1. *$x \approx y \Rightarrow f(x) \approx f(y)$ for all $x, y \in S_1$,*
2. *$f : (S_1, \mathcal{T}_{\mathcal{B}_1}(S_1)) \to (S_2, \mathcal{T}_{\mathcal{B}_2}(S_2))$ is continuous.*

Proof. 1. Let $x, y \in S_1$ with $x \approx y$. Then there exist $n \in \mathbb{N}$ and $(z_1, z_2, \ldots, z_n) \in S^n$ such that $x \sim z_1 \sim z_2 \sim \cdots \sim z_n \sim y$. By the hypothesis $f(x) \sim f(z_1) \sim f(z_2) \sim \cdots \sim f(z_n) \sim f(y)$, we have $f(x) \approx f(y)$.

2. By the first part, we have $f(\Gamma(X)) \subseteq \Gamma(f(X))$. It means that $f(\overline{X}) \subseteq \overline{f(X)}$ in the corresponding closed topology. Therefore f is continuous.

Let $X \in \mathcal{T}^o_{\mathcal{B}_2}(S_2)$; then $X = \Gamma(X)$. By part (1) for each $x \in X$ and every $y \in f^{-1}(x)$, we have

$$f(\Gamma(y)) \subseteq \Gamma(f(y)) = \Gamma(x) \subseteq \Gamma(X) = X,$$

so $f^{-1}(X) \in \mathcal{T}^o_{\mathcal{B}_1}(S_1)$, and hence f is continuous. □

Theorem 2.27. *Let $(S_i, \mathcal{B}_i)$ be a geometric space with topology τ_i on S_i for $i = 1, 2$, and let $f : (S_1, \mathcal{B}_1) \to (S_2, \mathcal{B}_2)$ be a good morphism. If $(S_1, \mathcal{B}_1)$ is τ_1-open such that $S_1 = \bigcup_{B \in \mathcal{B}_1} B$ and $(S_2, \mathcal{B}_2)$ is τ_2-complete, then $f : (S_1, \tau_1) \to (S_2, \tau_2)$ is continuous.*

Proof. Let $U \subseteq \tau_2$ and let $x \in U$. For each $y \in f^{-1}(x)$, there exists $B \in \mathcal{B}_1$ such that $y \in B$ (since $\mathcal{B}_1$ is a cover for S_1). By Corollaries 1.2 and 2.12, $\Gamma(y) = \Gamma(B)$ is open and we have

$$f(\Gamma(y)) \subseteq \Gamma(f(y)) = \Gamma(x) \subseteq \Gamma(U) = U,$$

in which the latest equality holds because of τ_2-completeness of S_2. Thus $f^{-1}(U)$ is open, and hence f is continuous. □

3. Hypergroups and H_v-Groups

In this section, we define some topologies on H_v-groups by employing the concept of geometric space and then generalize the concept of topological hypergroup to topological H_v-group. Firstly, we remind some relative concepts of hypergroups and H_v-groups in algebra; see, for more information, [7, 8].

Let $(H_1, \circ)$ and $(H_2, \bullet)$ be two hypergroups (H_v-groups resp.). A map $f : H_1 \to H_2$ is called a *homomorphism* or a *good (strong) homomorphism* if

$$f(x \circ y) = f(x) \bullet f(y) \quad \text{for all } x, y \in H_1.$$

The map f is called *weak homomorphism* or *H_v-homomorphism* if

$$f(x \circ y) \cap f(x) \bullet f(y) \neq \emptyset \quad \text{for all } x, y \in H_1.$$

Also f is called an *inclusion homomorphism* if

$$f(x \circ y) \subseteq f(x) \bullet f(y) \quad \text{for all } x, y \in H_1.$$

Moreover, f is an *isomorphism* if it is a one-to-one and onto good homomorphism. If f is an isomorphism, then H_1 and H_2 are said to be *isomorphic*, which is denoted by $H_1 \cong H_2$.

Let $f : (H_1, \circ) \to (H_2, \bullet)$ be a good homomorphism between semihypergroups. The relation $f^{-1} \circ f$ is an equivalence relation ρ on H_1, that is,

$$a \rho b \Leftrightarrow f(a) = f(b).$$

The relation ρ is called the *kernel* of f.

The homomorphism f is one-to-one if and only if its kernel is the diagonal set. Immediately, we have the next proposition.

Proposition 3.1. *Let $(H_1, \circ)$ and $(H_2, \bullet)$ be two hypergroups and let $f : H_1 \to H_2$ be a good homomorphism. Then f is one-to-one if and only if $\mathcal{T}_{\frac{H_1}{ker(f)}}(H_1)$ is the discrete topology.*

For all $n > 1$, the relation β_n is defined on a semihypergroup H, as follows:

$$a \beta_n b \iff \exists (x_1, \ldots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

and $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. Clearly, the relation β is reflexive and symmetric. The transitive closure of β is denoted by β^*.

The relation β^* is the smallest equivalence relation on H such that the quotient $(\frac{H}{\beta^*}, \odot)$ is a semigroup, where

$$\beta^*(x) \odot \beta^*(y) = \beta^*(z), \quad x, y \in H \text{ and } z \in x \circ y.$$

The relation β^* is called the *fundamental equivalence relation* on H and is called the *fundamental semigroup*. If H is a hypergroup, then $\beta = \beta^*$ and $\frac{H}{\beta^*}$ is a group called *fundamental group*.

The relation α_n is defined as follows:

$$\alpha_1 = \{(x, x) \mid x \in H\},$$

and for every integer $n > 1$,

$$x\alpha_n y \iff \exists(z_1, \dots, z_n) \in H^n, \exists\sigma \in S_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

where S_n is the symmetric group of all permutations of the set $\{1, 2, \dots, n\}$.

Obviously, for $n \geq 1$, the relations α_n are symmetric, and the relation $\alpha = \bigcup_{n\geq 1} \alpha_n$ is reflexive and symmetric. Let α^* be the transitive closure of α.

The quotient $\frac{H}{\alpha^*}$ is a commutative semigroup. Furthermore, if H is a hypergroup, then $\frac{H}{\alpha^*}$ is a commutative group.

The *fundamental equivalence relation* on H_v-group $(H, \circ)$, denoted by β^*, is the smallest equivalence relation on H such that the quotient $\frac{H}{\beta^*}$ is a group. If $\mathcal{H}(H)$ denotes the set of all finite products of elements of H (i.e., $\mathcal{H}(H) = \bigcup_{n\in\mathbb{N}} \mathcal{H}_n(H)$, where $\mathcal{H}_n(H)$ is a hyperproduct of n elements of H in which the parentheses are put in all possible ways), then the relation β on H whose transitive closure is the fundamental relation β^*, is defined as follows:

$$x\beta y \iff \text{there exists } B \in \mathcal{H}(H) \text{ such that } \{x, y\} \subseteq B.$$

In other words,

$$a\beta^* b \Leftrightarrow \exists\, z_1, \dots z_{n+1} \in H \text{ with } z_1 = a,\ z_{n+1} = b \text{ and } B_1, \dots,\ B_n \in \mathcal{H}(H) \text{ such that } \{z_i, z_{i+1}\} \subseteq B_i\ (i = 1, \dots, n).$$

Suppose that $\beta^*(a)$ is the equivalence class containing $a \in H$. Then the product on $\frac{H}{\beta^*}$ is defined as follows:

$$\beta^*(a) \odot \beta^*(b) = \beta^*(c) \quad \text{for all } c \in \beta^*(a) \circ \beta^*(b).$$

An element X of H_v-group $(H, \circ)$ is called *single* if its fundamental class is singleton, that is, $\beta^*(x) = \{x\}$. The set of all single elements of H is denoted by S_H.

Clearly if $f : (H_1, \circ) \to (H_2, \bullet)$ is a good homomorphism between two hypergroups or H_v-groups, then f preserves the relation β.

Example 3.2. 1. According to [11], if $(H, \circ)$ is a hypergroup, then $(H, \mathcal{P}^*(H))$ is a geometric space whose points are the elements of H and whose blocks are the hyperproducts of elements of H. Thus, if $B \in \mathcal{P}^*(H)$, then there exists an n-tuple $(z_1, z_2, \dots, z_n) \in H^n$ such that $B = z_1 \circ z_2 \circ \cdots \circ z_n$. By reproducibility of H, for every $x \in H$, a pair (a, b) of elements of H exists such that $x \in a \circ b$, and hence the family $\mathcal{P}^*(H)$ is a covering of H. Moreover, by Proposition 3.1.1 of [12], the geometric space $(H, \mathcal{P}^*(H))$ is strongly transitive. In this case, the relation $\sim$ coincides with the fundamental relation β, therefore β is transitive.

2. According to [11], if $(H, \circ)$ is a hypergroup, then $(H, P_\sigma(H))$ is a strongly transitive geometric space, where $P_\sigma(H)$ is the family of subsets of H, such that

1. $B(z) = \{z\}$ for all $z \in H$,
2. $B(z_1, z_2, \dots, z_n) = \bigcup\{\prod_{i=1}^{n} z_{\sigma(i)} \mid \sigma \in S_n\}$, if $n \geq 2$, for all n-tuple $(z_1, z_2, \dots, z_n) \in H^n$.

where S_n is the symmetric group of all permutations of the set $\{1, 2, \dots, n\}$. Moreover, the relation $\sim$ coincides with the relation α.

3. If $(G, \circ)$ is a group, then it is a hypergroup, so $(G, \mathcal{H}(G))$ is a geometric space whose blocks are singletons. Hence the topology $\mathcal{T}_{\mathcal{H}(G)}(G)$ is discrete.

4. If $(H, \circ)$ is an H_v-group, then $(H, \mathcal{H}(H))$ is a geometric space whose points are the elements of H and whose blocks are the finite products of elements of H. Thus, if $B \in \mathcal{H}(H)$, then there exist an integer $n \geq 1$

and n-tuple $(z_1, z_2, \ldots, z_n) \in H^n$ such that $B = z_1 \circ z_2 \circ \cdots \circ z_n$. By reproducibility of H, for every $x \in H$, a pair (a, b) of elements of H exists such that $x \in a \circ b$, and hence the family $\mathcal{H}(H)$ is a covering of H. Moreover, the relation $\sim$ coincides with the relation β. By Corollary 3.1 of [8], if $S_H \neq \emptyset$, then $\beta = \beta^*$. In this case, the geometric space $(H, \mathcal{H}(H))$ is strongly transitive.

Since hypergroups are a special class of H_v-groups, so in the following theorem, we try to prove some properties for H_v-groups in general.

According to Example 3.2, any H_v-group induces at least one geometric space, it is natural willing to identify the relation between good homomorphisms of hypergroups and good morphisms of geometric spaces; see the following theorem.

Theorem 3.3. *Let $(H_1, \circ)$ and $(H_2, \bullet)$ be two H_v-groups, and let $f : H_1 \to H_2$ be a good homomorphism. Then f is a good morphism from the geometric space $(H_1, \mathcal{H}(H_1))$ to the geometric space $(H_2, \mathcal{H}(H_2))$, and therefore $f : (H_1, \mathcal{T}_{\mathcal{H}(H_1)}(H_1)) \to (H_2, \mathcal{T}_{\mathcal{H}(H_2)}(H_2))$ is continuous. If f is an isomorphism on H_v-groups, then f is a homeomorphism on the corresponding topological spaces.*

Proof. If $f : (H_1, \circ) \to (H_2, \bullet)$ is a good homomorphism on H_v-groups, then f preserves the relation β, so it preserves the relation $\sim$ on geometric spaces. Hence f is a good morphism and so by Theorem 2.26, f is continuous. If f is isomorphism, then there is a good homomorphism $g : H_2 \to H_1$, such that $g \circ f$ and $f \circ g$ are identity on H_1 and H_2, respectively, which completes the proof. □

Proposition 3.4. *Let H be an H_v-group and let $q : H \to \frac{H}{\beta^*}$ be a natural map; then q is a quotient map.*

Proof. By Theorem 2.26, q is continuous because of preserving $\sim$ and q is natural, so it is surjective. It is remained to show that, it is an open (closed resp.) map. Let X be an open (closed resp.) subset of H in open (closed resp.) corresponding topology. Then by Corollary 2.1, $X = \Gamma(X)$. Therefore $q(\Gamma(X)) = \Gamma(q(X))$, and hence q is open (closed resp.). □

Recall that a *paratopological group* is a pair (G, τ) consisting of a group G and a topology τ on G making the group operation continuous. A paratopological group G with continuous inversion is called a *topological group*. The continuity of the inversion is one of the important research topics in the theory of paratopological groups.

Now we recall the following definition from [13]. Let $(H, \circ)$ be a hypergroup and let (H, τ) be a topological space. Then, the system $(H, \circ, \tau)$ is called a *topological hypergroup* if the following conditions hold:

1. The mapping $(x, y) \mapsto x \circ y$, from $H \times H$ to $\mathcal{P}^*(H)$ is continuous.
2. The mapping $(x, y) \mapsto \frac{x}{y}$, from $H \times H$ to $\mathcal{P}^*(H)$ is continuous, where $\frac{x}{y} = \{z \in H \mid x \in z \circ y\}$.

It is considered the product topology on $H \times H$ and the topology τ^* on $\mathcal{P}^*(H)$. When only the first condition holds, it is called a *paratopological hypergroup*.

Like the above, we can define the concept of topological H_v-group.

Definition 3.5. Let $(H, \circ)$ be an H_v-group and let (H, τ) be a topological space. Then, the system $(H, \circ, \tau)$ is called a *topological H_v-group* if the following conditions hold:

1. The mapping $(x, y) \mapsto x \circ y$, from $H \times H$ to $\mathcal{P}^*(H)$ is continuous.
2. The mapping $(x, y) \mapsto \frac{x}{y}$, from $H \times H$ to $\mathcal{P}^*(H)$ is continuous, where $\frac{x}{y} = \{z \in H \mid x \in z \circ y\}$.

We consider the product topology on $H \times H$ and the topology τ^* on $\mathcal{P}^*(H)$. When only the first condition holds, it is called a *paratopological H_v-group*.

Lemma 3.6. *Let $(H, \circ)$ be an H_v-group and let $S_H \neq \emptyset$. Then the hyperoperation $\circ : H \times H \to \mathcal{P}^*(H)$, which maps (x, y) to $x \circ y$, is continuous in the topological space $\left(H, \mathcal{T}_{\mathcal{H}(H)}(H)\right)$. Hence $\left(H, \circ, \mathcal{T}_{\mathcal{H}(H)}(H)\right)$ is a paratopological H_v-group.*

Proof. If we show that $\circ : \big(H, \mathcal{H}(H)\big) \times \big(H, \mathcal{H}(H)\big) \to \big(\mathcal{P}^*(H), (\mathcal{H}(H))^*\big)$ preserves the relation $\sim$, then by Theorem 2.26, it is continuous in the topological space $\big(H, \mathcal{T}_{\mathcal{H}(H)}(H)\big)$. Let $(x_1, y_1) \sim (x_2, y_2)$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$; then $x_1 \sim x_2$ and $y_1 \sim y_2$. Therefore $\Gamma(x_1) = \Gamma(x_2)$ and $\Gamma(y_1) = \Gamma(y_2)$. Since $\frac{H}{\beta^*}$ is a group by the operation $\beta^*(x_1) \odot \beta^*(y_1) = \beta^*(z_1)$, where $z_1 \in \beta^*(x_1) \circ \beta^*(y_1)$, β^* coincides with the relation $\sim$. In the other hand, $\Gamma(x_1) \odot \Gamma(y_1) = \Gamma(z_1)$, thus we have

$$\Gamma(z_1) = \Gamma(x_1) \odot \Gamma(y_1) = \Gamma(x_2) \odot \Gamma(y_2) = \Gamma(z_2),$$

where $z_2 \in x_2 \circ y_2$. Indeed $x_i \circ y_i \in \mathcal{H}(H)$ is a block of geometric space $\big(H, \mathcal{H}(H)\big)$ for $i = 1, 2$, so by Corollary 1.2, $\Gamma(x_i \circ y_i) = \Gamma(z_i)$ for $i = 1, 2$. Hence $\Gamma(x_1 \circ y_1) = \Gamma(x_2 \circ y_2)$. Since $x_i \circ y_i \in \mathcal{H}(H)$ for $i = 1, 2$ are blocks, then by (P1), $x_1 \circ y_1 \cap x_2 \circ y_2 \neq \emptyset$ or there exists a polygonal $(B_1, \dots B_n)$ of $\big(H, \mathcal{H}(H)\big)$ such that $x_1 \circ y_1 \cap B_1 \neq \emptyset$ and $x_2 \circ y_2 \cap B_n \neq \emptyset$. Then the induced blocks $(x_i \circ y_i)^* \in (\mathcal{H}(H))^*$ for $i = 1, 2$ have a nonempty intersection or for the induced polygonal $(B_1^*, \dots B_n^*)$ of $\big(\mathcal{P}^*(H), (\mathcal{H}(H))^*\big)$, we have $(x_1 \circ y_1)^* \cap B_1^* \neq \emptyset$ and $(x_2 \circ y_2)^* \cap B_n^* \neq \emptyset$. Therefore $\Gamma((x_1 \circ y_1)^*) = \Gamma((x_2 \circ y_2)^*)$ and thus by Corollary 1.2, $\Gamma(x_1 \circ y_1) = \Gamma(x_2 \circ y_2)$ (since $x_i \circ y_i \in (x_i \circ y_i)^*$ for $i = 1, 2$) in the geometric space $\big(\mathcal{P}^*(H), (\mathcal{H}(H))^*\big)$.

Since $(\mathcal{H}(H))^*$ is a cover for $\mathcal{H}(H)$, by Theorem 2.16, $\big(\mathcal{P}^*(H), (\mathcal{H}(H))^*\big)$ is a transitive geometric space, so $x_1 \circ y_1 \sim x_2 \circ y_2$. □

In the following lemma, we consider the continuity of inversion of H_v-group.

Lemma 3.7. *Let $(H, \circ)$ be an H_v-group and let $S_H \neq \emptyset$; then the map $f : H \times H \to \mathcal{P}^*(H)$ that maps (x, y) to $\frac{x}{y} = \{z \in H \mid x \in z \circ y\}$ is continuous in the topological space $\big(H, \mathcal{T}_{\mathcal{H}(H)}(H)\big)$.*

Proof. By Theorem 2.26, it is enough to show that f preserves the relation $\sim$. Let $(x_1, y_1) \sim (x_2, y_2)$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$; then $x_1 \sim x_2$ and $y_1 \sim y_2$. Indeed, for each $z_1 \in \frac{x_1}{y_1}$ and $z_2 \in \frac{x_2}{y_2}$, we have $x_1 \in z_1 \circ y_1$ and $x_2 \in z_2 \circ y_2$. Then by Corollary 1.2, we have

$$\Gamma(z_1 \circ y_1) = \Gamma(x_1) = \Gamma(x_2) = \Gamma(z_2 \circ y_2).$$

Since $\frac{H}{\beta^*}$ is a group by the operation $\beta^*(z_1) \odot \beta^*(y_1) = \beta^*(x_1)$, where $x_1 \in \beta^*(z_1) \circ \beta^*(y_1)$, and β^* coincides with the relation $\sim$. In the other hand, $\Gamma(z_1) \odot \Gamma(y_1) = \Gamma(x_1)$, and since $\Gamma(y_1) = \Gamma(y_2)$, we have

$$\Gamma(z_1) \odot \Gamma(y_2) = \Gamma(z_1) \odot \Gamma(y_1) = \Gamma(x_1) = \Gamma(x_2) = \Gamma(z_2) \odot \Gamma(y_2).$$

By the properties of group $\frac{H}{\beta^*}$, we have

$$\Gamma(z_1) \odot \Gamma(y_2) = \Gamma(z_2) \odot \Gamma(y_2) \Rightarrow \Gamma(z_1) = \Gamma(z_2).$$

Like the proof of Lemma 3.6, $\Gamma(z_1) = \Gamma(z_2)$ in the geometric space $\big(\mathcal{P}^*(H), (\mathcal{H}(H))^*\big)$, and strongly transitivity of geometric space $\big(\mathcal{P}^*(H), (\mathcal{H}(H))^*\big)$ yields $z_1 \sim z_2$. □

The two above lemmas yield the following theorem.

Theorem 3.8. *Let $(H, \circ)$ be an H_v-group; then $\big(H, \circ, \mathcal{T}_{\mathcal{H}(H)}(H)\big)$ is a topological H_v-group.*

Like the above theorem, to find the condition that the system $(H, \circ, \tau)$ is a topological H_v-group for an arbitrary topology τ on H, we need some concepts and their properties; see, for more information, [7, 8, 13, 17].

Let $(H, \circ)$ be a hypergroupoid and let τ be a topology on H. Then $\circ$ is *τ-closed* when for every $x, y \in H$, $x \circ y$ is a closed subset of (H, τ) (see [17]). Similarly, we can define τ-open.

Remark 3.9. The concept of τ-open (τ-closed resp.) in H_v-groups coincides with the concept of τ-open (τ-closed resp.) in geometric spaces. In fact, the H_v-group $(H, \circ)$ with topology τ on H is τ-open (τ-closed resp.), if and only if the corresponding geometric space is τ-open (τ-closed resp.).

Definition 3.10. ([7, 8]) Let $(H, \circ)$ be an H_v-semigroup. Then a nonempty subset A of H is called a *complete part* of H if for each nonzero natural number n and for all hyperproduct $B \in \mathcal{H}_n(H)$,

$$A \cap B \neq \emptyset \Rightarrow B \subseteq A.$$

The intersection of the complete parts of H containing A is called the *complete closure* of A in H, which is denoted by $C(A)$.

The concept of complete part and complete closure in H_v-groups coincides with the concept of $\mathcal{B}$-part and closure in geometric spaces, respectively. Also $C(A)$ coincides with $\Gamma(A)$.

Definition 3.11. ([7]) A semihypergroup H is *complete*, if it satisfies one of the following conditions:

1. For all $(x, y) \in H^2$ and for all $a \in x \circ y, C(a) = x \circ y$.
2. For all $(x, y) \in H^2, C(x \circ y) = x \circ y$.
3. For all $(m, n) \in \mathbb{N}^2, m, n \geq 2$, for all $(x_1, x_2, \ldots, x_n) \in H^n$, and for all $(y_1, y_2, \ldots, y_m) \in H^m, \prod_{i=1}^n x_i \cap \prod_{j=1}^m y_j \neq \emptyset \Rightarrow \prod_{i=1}^n x_i = \prod_{j=1}^m y_j$.

A hypergroup is *complete* if it is a complete semihypergroup.

Like the concept of complete hypergroup, we can define the concept of complete H_v-groups. In fact, an H_v-group $(H, \circ)$ is complete if every finite product of elements of H is a complete part, that is, $C(B) = B$ for all $B \in \mathcal{H}(H)$.

Recall from [13], that a hypergroup $(H, \circ)$ with a topology τ on H is called a *τ-complete part* if every $U \in \tau$ is a complete part.

Remark 3.12. It is clear that, the concept of complete (τ-complete resp.) H_v-group coincides with the concept of complete (τ-complete resp.) geometric space.

Theorem 3.13. *Let $(H, \circ)$ be an H_v-group and let (H, τ) be a topological space. If H is τ-complete and τ-open, then $(H, \circ, \tau)$ is a topological complete H_v-group.*

Proof. Like the proof of Lemma 3.6, $\circ : H \times H \to \mathcal{P}^*(H)$ is a good morphism. Then by Remarks 3.9 and 3.12, Propositions 2.20 and 2.22, and Theorem 2.27, it is continuous.

Similar to the proof of Lemma 3.7, the map $f : H \times H \to \mathcal{P}^*(H)$ that maps (x, y) to $\frac{x}{y} = \{z \in H \mid x \in z \circ y\}$ is a good morphism. Thus by Remarks 3.9 and 3.12, Propositions 2.20 and 2.22, and Theorem 2.27, f is continuous.

By Proposition 2.19, $(H, \circ)$ is complete. □

Singha, Das, and Davvaz [25] proved that every topological complete hypergroup is a topological regular hypergroup. Hence by the above theorem, every τ-open and τ-complete hypergroup is a topological complete hypergroup with respect to topology τ.

Example 3.14. 1. Consider the hypergroup $(\mathbb{R}, \circ)$, where $\mathbb{R}$ is the set of real numbers and for each $x, y \in \mathbb{R}$,

$$x \circ y = \begin{cases} (-\infty, x] & \text{if } x = y. \\ \max\{x, y\} & \text{if } x \neq y. \end{cases}$$

Therefore $(\mathbb{R}, \mathcal{B})$ is a geometric space, where $\mathcal{B} = \{\{x\} \mid x \in \mathbb{R}\} \cup \{(-\infty, x] \mid x \in \mathbb{R}\}$. Then $\mathcal{F}_{\mathcal{B}}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$ and $\mathcal{T}_{\mathcal{B}}(\mathbb{R})$ is the discrete topology.

Now consider the upper limit topology τ_{up} on $\mathbb{R}$ (which is the topology generated by $\{(a, b] \mid a, b \in \mathbb{R},\ a \leq b\}$).

Thus $\Gamma((a, b]) = (a, b]$ and $\Gamma(a) = \{a\}$, and hence $(\mathbb{R}, \circ)$ is τ_{up}-complete. Also

$$x \circ y = \begin{cases} (-\infty, x] \in \tau_{up} & \text{if } x = y. \\ \max\{x, y\} \in \tau_{up} \text{ (since } \{x\} = (x, x]) & \text{if } x \neq y. \end{cases}$$

Therefore $(\mathbb{R}, \circ)$ is τ_{up}-open. By Theorem 3.13, $(\mathbb{R}, \circ, \tau_{op})$ is a topological complete hypergroup.

2. Let $(H, *)$ be the total hypergroup (i.e., for all $x, y \in H$, $x * y = H$) with an arbitrary topology τ. Then $(H, \mathcal{B})$ is the corresponding geometric space, where $\mathcal{B} = \{H\}$, $\mathcal{F}_{\mathcal{B}}(H) = \{\emptyset, H\}$, and $\Gamma(H) = H$. Since H is clopen, then $(H, *)$ is τ-open, τ-closed, and τ-complete. Hence $\mathcal{T}_{\mathcal{B}}(H)$ is the trivial topology. So by Theorem 3.13 $(H, *, \tau)$ is a topological complete hypergroup.

3. Consider the hypergroup $(\mathbb{Z}, \star)$, where $\mathbb{Z}$ is the set of all integers and for each $m, n \in \mathbb{Z}$,

$$m \star n = \begin{cases} 2\mathbb{Z} & \text{if } m + n \in \mathbb{Z}. \\ (2\mathbb{Z})^c & \text{otherwise.} \end{cases}$$

The corresponding geometric space is $(\mathbb{Z}, \mathcal{B})$, where $\mathcal{B} = \{2\mathbb{Z}, (a\mathbb{Z})^c\}$. Moreover $\mathcal{B}$ is disjoint, so by Proposition 2.15, $(\mathbb{Z}, \mathcal{B})$ is complete. Also $\mathcal{F}_{\mathcal{B}}(\mathbb{Z}) = \{\emptyset, 2\mathbb{Z}, (a\mathbb{Z})^c, \mathbb{Z}\}$ and by Theorem 3.8, $(\mathbb{Z}, \star, \mathcal{T}_{\mathcal{B}}(\mathbb{Z}))$ is a topological hypergroup.

4. Consider the topological group $(\mathbb{Z}, +, \tau)$, where τ is the inherited topology from the standard topology on $\mathbb{R}$. Let $n \in \mathbb{Z}$ and let $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ be the set of all congruence classes of integers modulo n. Also $(\mathbb{Z}, \bullet)$ is a hypergroup, where $a \bullet b = \overline{a + b}$ for all $a, b \in \mathbb{Z}$. The corresponding geometric space is $(\mathbb{Z}, \mathcal{B})$, where $\mathcal{B} = \mathbb{Z}_n$, which is disjoint, so $(\mathbb{Z}, \mathcal{B})$ is complete. Also

$$\mathcal{F}_{\mathcal{B}}(\mathbb{Z}) = \left\{ U \subseteq \mathbb{Z} \;\middle|\; U \cap \bigcup_{i=0}^{n-1} \overline{i} = \emptyset \text{ or } \bigcup_{\overline{a} \in A} \overline{a} \subseteq U, \text{ for } A \subseteq \mathbb{Z}_n \right\}.$$

Let $\tau_{\mathbb{Z}} = \{\bigcup_{x \in U} \overline{x} \mid U \in \tau\}$. Since $\mathbb{Z}_n \subseteq \mathcal{F}_{\mathcal{B}}(\mathbb{Z})$ and $U \subseteq \mathbb{Z}_n$ for each $U \in \tau_{\mathbb{Z}}$, then $(\mathbb{Z}, \bullet)$ is $\tau_{\mathbb{Z}}$-complete. If $\overline{a} \in \mathbb{Z}_n$, then $\bigcup_{a \in \overline{a}} \left(a - \frac{1}{2}, a + \frac{1}{2}\right) \in \tau$ so $\overline{a} \in \tau_{\mathbb{Z}}$. Thus $\mathcal{B} \subseteq \tau_{\mathbb{Z}}$, and hence $(\mathbb{Z}, \mathcal{B})$ is $\tau_{\mathbb{Z}}$-open. By Theorem 3.13, $(\mathbb{Z}, \bullet, \tau_{\mathbb{Z}})$ is a topological complete hypergroup.

Singha, Das, and Davvaz [25] (Examples 2.7 and 2.12-2.14) proved that these are topological complete hypergroups.

Proposition 3.15. *Let (S, τ) be a topological space and let $\mathcal{B}$ be the family of open (closed resp.) subsets of it. Then $(S, \mathcal{B})$ is a complete geometric space.*

Proof. Clearly $(S, \mathcal{B})$ is a geometric space. Since the element of $\mathcal{B}$ is a (the complement of) element of τ, so τ is equal to $\mathcal{T}^{o}_{\mathcal{B}}(S)$ ($\mathcal{T}^{c}_{\mathcal{B}}(S)$ resp.). Hence $\Gamma(B) = B$ for each $B \in \mathcal{B}$. □

In the relation between the geometric spaces and hypergroups, it remains to show that a geometric space is a hypergroup.

Theorem 3.16. *If $(S, \mathcal{B})$ is a geometric space, then $(S, \circ)$ is a hyper group, where $\circ : S \times S \to \mathcal{P}(S)$ is the hyperoperation defined by $x \circ y = \Gamma(\{x, y\})$. Moreover, $(S, \circ)$ is commutative. Hence the relation α is equal to the relation β.*

Proof. Clearly $\circ$ is a well-defined map. Since by (P4), $\Gamma(\{x, y\}) = \Gamma(x) \cup \Gamma(y)$, it is obvious that $\circ$ is associative. The reproduction axiom holds, since $x \in \Gamma(\{x, a\}) = x \circ a$ for each $x, a \in S$. Hence $(S, \circ)$ is a hypergroup and $x \circ y = \Gamma(\{x, y\}) = \Gamma(\{y, x\}) = y \circ x$ for all $x, y \in S$, yields that $(S, \circ)$ is commutative. □

In this paper, we have proved the following diagram.

$$\begin{array}{ccccc}
\text{Hypergroup } (H,\circ) & \xrightarrow[\mathcal{B}:=\mathcal{P}^*(H)]{S:=H} & \text{Geometric Space } (S,\mathcal{B}) & \xrightarrow{\tau:=\mathcal{T}_{\mathcal{B}}(S)} & \text{Topological Space } (S,\tau) \\
{\scriptstyle \tau:=\mathcal{T}_{\mathcal{P}^*(H)}(H)}\big\downarrow & & {\scriptstyle H:=S}\big\downarrow{\scriptstyle \circ(x,y):=\Gamma(\{x,y\})} & & \big\downarrow{\scriptstyle \mathcal{B}:=\left\{\substack{\text{open}\\ \text{closed}} \text{ subsets of } S\right\}} \\
\text{Topological Hypergroup} & & \text{Commutative Hypergroup} & & \text{Complete Geometric Space} \\
(H,\circ,\tau) & & (H,\circ) & & (S,\mathcal{B})
\end{array}$$

In the following, we present some properties of topological H_v-group $\left(H,\circ,\mathcal{T}_{\mathcal{H}(H)}(H)\right)$ and in general case, for a topology τ on H.

Proposition 3.17. *Let $(H,\circ)$ be an H_v-group and let K be a subset of H. Then $x\circ\Gamma(K)=\Gamma(x\circ K)$ for each $x\in H$.*

Proof. By (P1), we have $x\circ\Gamma(K)\subseteq\Gamma(x\circ K)$. To prove that $\Gamma(x\circ K)\subseteq x\circ\Gamma(K)$, let $t\in\Gamma(x\circ K)$. If $t\in x\circ K$, then it is done. Let $t\notin x\circ K$; then by the reproduction axiom, there exists $y\in H$ such that $t\in x\circ y$. Since $t\in x\circ y\cap\Gamma(x\circ K)$, then $x\circ y\subseteq\Gamma(x\circ K)$, so there exists $z\in K$ such that $x\circ y\approx x\circ z$. Thus

$$\Gamma(x\circ y)=\Gamma(x\circ z)\Rightarrow\Gamma(x)\odot\Gamma(y)=\Gamma(x)\odot\Gamma(z)\Rightarrow\Gamma(y)=\Gamma(z)\Rightarrow y\in\Gamma(z)\subseteq\Gamma(K).$$

□

By the conditions of Proposition 3.17, in the closed corresponding topology, we have $x\circ\overline{K}=\overline{x\circ K}$, while Singha, Das, and Davvaz [25] proved it when $(H,\circ,\tau)$ is a compact Hausdorff topological hypergroup. Since in any topological space H and a subset K of it, $Int(K)=H\setminus(\overline{H\setminus K})$, where $Int(K)$ denotes the interior of the subset K, in the closed corresponding topology, we have

$$x\circ Int(K)=x\circ(H\setminus(\overline{H\setminus K})=(x\circ H)\setminus(\overline{x\circ H\setminus K})=H\setminus(\overline{H\setminus x\circ K})=Int(x\circ K).$$

Since, the open (closed resp.) subsets of open corresponding topology are the closed (open resp.) subsets of the closed corresponding topology, these properties are true for $\mathcal{T}^{o}_{\mathcal{H}(H)}(H)$. Hence we have the following proposition.

Proposition 3.18. *Let $(H,\circ)$ be an H_v-group and let U be an open (closed resp.) subset of topological space $\left(H,\mathcal{T}_{\mathcal{H}(H)}(H)\right)$. Then $a\circ U$ and $U\circ a$ are open (closed resp.) subsets of H for every $a\in H$.*

Proposition 3.19. *Let $(H,\circ)$ be an H_v-group and let A and B be open (closed resp.) subsets of topological space $\left(H,\mathcal{T}_{\mathcal{H}(H)}(H)\right)$. Then $A\circ B$ is an open (closed resp.) subset of H.*

Proof. By Proposition 3.18, $A\circ b$ is open (closed resp.). Then $A\circ B=\bigcup_{b\in B}A\circ b$ is open. When $A\circ b$ is closed in one of the corresponding topologies, by the argument before the Proposition 3.18, it is open in the other one, then the proof is complete in the closed case. □

Proposition 3.20. *Let $(H,\circ)$ be an H_v-group. If A and B are two nonempty subsets of H, then in the topological space $(H,\mathcal{T}_{\mathcal{H}(H)}(H))$, it follows that*

1. $\overline{A}\circ\overline{B}\subseteq\overline{A\circ B}$.
2. $IntA\circ IntB\subseteq Int(A\circ B)$.

Proof. 1. Since the map $\circ: H\times H\to\mathcal{P}^*(H)$ is continuous, then $\overline{A}\circ\overline{B}=\circ(\overline{A},\overline{B})\subseteq\overline{A\circ B}$.

2. Let $x\in IntA\circ IntB$; then $x\in a\circ b$ for some $a\in IntA$ and $b\in IntB$. Since a and b are interior points of A and B, respectively, there exist $U,V\in\mathcal{T}_{\mathcal{H}(H)}(H)$ such that $a\in U\subseteq A$ and $b\in V\subseteq B$. Then $x\in a\circ b\subseteq U\circ V\subseteq A\circ B$, and therefore $x\in Int(A\circ B)$, since $U\circ V$ is open (by proposition 3.19). Thus, $IntA\circ IntB\subseteq Int(A\circ B)$. □

The latest three propositions were proved in [25], when $(H, \circ, \tau)$ is a topological complete hypergroup and U is a complete part and an open subset.

For an arbitrary topology τ on H, we have the following proposition.

Proposition 3.21. *Let $(H, \circ)$ be an H_v-group with topology τ on H such that $(H, \circ)$ is τ-complete and τ-open, and let U and V be open subsets of H. Then*

1. *$a \circ U$ and $U \circ a$ are open subsets of H for every $a \in H$.*
2. *$U \circ V$ is open.*

Proof. 1. Since H is τ-complete, then $\Gamma(U) = U$, then by Proposition 3.17, $a \circ U = \Gamma(a \circ U)$ is open by Corollary 2.12, since H is τ-open. Similarly $U \circ a$ is open.

2. By the first part, $U \circ a$ is open, and then $U \circ V = \bigcup_{a \in V} U \circ a$ is open. □

Proposition 3.22. *Let $(H, \circ, \tau)$ be a topological H_v-group and let A and B be compact subsets of H. Then, $A \circ B$ is compact.*

Proof. Since A and B are compact subsets of H, it follows that $A \times B$ is a compact subset of $H \times H$ with respect to the product topology induced from the topology τ. Now, the continuity of the map $\circ : H \times H \to \mathcal{P}^*(H)$ completes the proof. □

4. Conclusion

In geometric space $(S, \mathcal{B})$, by the properties of the family of $\mathcal{B}$-parts, two topologies were defined on S, which denoted by $\mathcal{T}^o_{\mathcal{B}}(S)$ and $\mathcal{T}^c_{\mathcal{B}}(S)$. Then it was proved that $\mathcal{B}$ is an open (closed resp.) basis of topology $\mathcal{T}^o_{\mathcal{B}}(S)$ ($\mathcal{T}^c_{\mathcal{B}}(S)$ resp.), where $S = \bigcup_{B \in \mathcal{B}} B$. Afterwards, the good morphism $f : (S_1, \mathcal{B}_1) \to (S_2, \mathcal{B}_2)$ was defined between geometric spaces that preserves the memberships of blocks of B_1. The properties which f is continuous in topological spaces $(S_i, \mathcal{B}_i)$ for $i = 1, 2$, was stated.

It was illustrated that at least one geometric space is induced from each H_v-group; A good homomorphism $f : H_1 \to H_2$ between H_v-groups is a good morphism between corresponding geometric spaces and is continuous, where topology on S_i is the corresponding topology on $\mathcal{B}_i$ for $i = 1, 2$; Moreover, an isomorphism between H_v-groups yields a homeomorphism between corresponding topological spaces.

Afterwards, topological H_v-group was defined as a generalized topological hypergroup. By concept of geometric space, it was shown that each H_v-group is a topological H_v-group with respect to the corresponding topology. In general, an H_v-group is a topological H_v-group with respect to an arbitrary topology τ if it is τ-open and τ-complete.

It is illustrated that each geometric space induces at least one commutative hypergroup. Finally, some propositions presented, some properties of topological H_v-group, and they were compared by the results of previous relevant studies.

Similar to this study, in future, the concept of topological space can be defined for other algebraic structures such as hyperring, H_v-ring and etc. By the concept of induced geometric spaces of them and the results of this paper, it is possible to determine the conditions that these spaces be topological.

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