



Radon's and Helly's Theorems for \mathbb{B}^{-1} -Convex Sets

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Abstract. Helly's, Radon's, and Caratheodory's theorems are the basic theorems of convex analysis and have an important place. These theorems have been studied by different authors for different classes of convexity.

Caratheodory's theorem for \mathbb{B}^{-1} -convex sets has been proved before by Adilov and Yeşilce. In this article, Helly's and Radon's theorems are discussed and examined for these sets.

1. Introduction

Convex analysis, which is one of the important fields of mathematics, has gained momentum and become open to innovations especially as a result of the studies conducted in recent years. Examples of application areas of convex analysis are optimization theory, inequality theory, mathematical economics and operations research. Convexity draws upon geometry, analysis, linear algebra, and topology and also plays important role in various areas of mathematics such as number theory, classical extremum problems, combinatorial geometry, game theory, polytopes, and linear programming. In this sense, convex analysis is of interest as a field of study. Convex geometry also examines the geometric properties of convex sets and convex functions. Many authors have contributed to the foundation and development of the field by carrying out studies. The leading ones of these researchers are Hermann Brunn, Hermann Minkowski, Werner Fenchel, Constantin Caratheodory, and Eduard Helly. There are many theorems which have important result in convex geometry, such as, Caratheodory's, Helly's and Radon's theorems.

Caratheodory's theorem is the fundamental dimensionality result in convexity theory, and it is the source of many other results in which dimensionality is prominent [4]. Helly's theorem was discovered by the Austrian mathematician Eduard Helly. After its discovery, it became the subject of research articles by hundreds of authors [4, 8, 9, 13]. It gives sufficient conditions for a family of convex sets to have a non-empty intersection. Radon's theorem constitutes an important step for a standard proof of Helly's theorem about the intersections of convex sets; this was the motivation of Radon's original discovery of Radon's theorem. After the discovery of Radon's theorem, studies on this theorem were carried out [7, 12, 17]. Three basic theorems, and particularly that of Helly, have been studied, applied, and generalized by many authors.

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Especially in recent years there has been a steady flow of publications concerning Helly's theorem and its relatives [1, 4, 5, 7–9, 11–13, 17]. The most pleasant aspect of Helly's theorem is the ease of incorporation into a wide range of impressive applications, which allows new discoveries to be made and forms the basis of the first short proof of some classical results.

Recently, the concept of abstract convexity, which is a generalization of the classical convexity, has attracted a considerable attention. Convexity is important phenomena in optimization theory. The basic concepts of convexity such as convex sets, convex functions, subdifferentials, conjugate functions have specially played major role on the solution methods of optimization problems since these problems involve convex functions and convex sets. Overtime, after a certain development stage in the optimization theory, there has existed the need of solution methods for optimization problems involving "nonconvex" sets and "nonconvex" functions such as the minimizing an arbitrary function over an arbitrary set without assuming any one of the structures topologically or algebraically. Therefore, the extension of classical convexity concepts into the "nonconvex" sets and "nonconvex" functions in classical sense yielded the concepts of abstract convexity. Consequently, many authors have studied the suitable extension methods of convex analysis and defined new types of convexities (See [14, 15, 18] and references therein). There are two main approaches in obtaining abstract convexity; topological abstract convexity and functional abstract convexity. Thus, many abstract convexity classes have emerged and many studies have been done in these classes [3, 14, 15, 18]. \mathbb{B} -convexity and \mathbb{B}^{-1} -convexity are two examples for abstract convexity classes.

The concept of \mathbb{B} -convexity first appeared in an article published by W.Briech and C.D.Horvath in 2004. Caratheodory's, Helly's and Radon's theorems for \mathbb{B} -convex sets are examined in this article [5]. \mathbb{B} -convexity is obtained from the usual linear convexity through the formal substitution maximum for addition.

The concept of \mathbb{B}^{-1} -convexity first appeared in an article published by Adilov and Yeşilce in 2012 [2, 3]. \mathbb{B} -convexity and \mathbb{B}^{-1} -convexity are compared in [10]. Then, \mathbb{B}^{-1} -convex sets and their properties are examined in [3, 10]. Then, separation of \mathbb{B}^{-1} -convex sets by \mathbb{B}^{-1} -measurable maps is studied in [16]. Also, the applications of \mathbb{B}^{-1} -convexity on mathematical economy are introduced in [6]. Then, one of the important theorems, Caratheodory's theorem is proved for \mathbb{B}^{-1} -convex sets [1]. In this article, Helly's and Radon's theorems are discussed and examined for \mathbb{B}^{-1} -convex sets.

The outline of this article is as follows: In section 2, we recall the definition of \mathbb{B}^{-1} -convex sets and also some its important properties. In section 3, Helly's theorem and Radon's theorems for \mathbb{B}^{-1} -convex sets are proved. Also, Helly's theorem is proved for an infinite family of compact \mathbb{B}^{-1} -convex sets in \mathbb{R}_{++}^n .

2. \mathbb{B}^{-1} -convex sets

Before we define the \mathbb{B}^{-1} -convex sets, let us give some preliminary information that we need to know.

For $r \in \mathbb{Z}^-$, the map $x \mapsto \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $K = \mathbb{R} \setminus \{0\}$ to itself; $x = (x_1, x_2, \dots, x_n) \rightarrow \Phi_r(x) = (\varphi_1(x_1), \varphi_2(x_2), \dots, \varphi_n(x_n))$ is homeomorphism from K^n to itself.

For a finite nonempty set $A = \{x^{(1)}, x^{(2)}, \dots, x^{(m)}\} \subset K^n$ the Φ_r -convex hull (shortly r -convex hull) of A , which we denote $Co^r(A)$ is given by

$$Co^r(A) = \left\{ \Phi_r^{-1} \left(\sum_{i=1}^m t_i \Phi_r(x^{(i)}) \right) : t_i \geq 0, \sum_{i=1}^m t_i = 1 \right\}.$$

We denote by $\bigwedge_{i=1}^m x^{(i)}$ the greatest lower bound with respect to the coordinate-wise order relation of $x^{(1)}, x^{(2)}, \dots, x^{(m)} \in \mathbb{R}^n$, that is,

$$\bigwedge_{i=1}^m x^{(i)} = \left(\min \{x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(m)}\}, \dots, \min \{x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(m)}\} \right)$$

where $x_j^{(i)}$ denotes j th coordinate of the point $x^{(i)}$.

Thus, we can define \mathbb{B}^{-1} -polytopes as follows:

Definition 2.1. [2] The Kuratowski-Painleve upper limit of the sequence of the sets $\{Co^r(A)\}_{r \in \mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$ where A is a finite subset of K^n is called \mathbb{B}^{-1} -polytope of A .

Next, the definition of \mathbb{B}^{-1} -convex sets with the help of \mathbb{B}^{-1} -polytope of A is as follow:

Definition 2.2. [2] A subset S of K^n is called a \mathbb{B}^{-1} -convex if for all finite subsets $A \subset S$ the \mathbb{B}^{-1} -polytope $Co^{-\infty}(A)$ is contained in S .

Definition 2.1 of \mathbb{B}^{-1} -polytope can be expressed in the following form in

$$\mathbb{R}_{++}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, 2, \dots, n\}.$$

Theorem 2.3. [2] For all nonempty finite subsets $A = \{x^{(1)}, x^{(2)}, \dots, x^{(m)}\} \subset \mathbb{R}_{++}^n$ we have

$$Co^{-\infty}(A) = \lim_{r \rightarrow -\infty} Co^r(A) = \left\{ \bigwedge_{i=1}^m t_i x^{(i)} : t_i \geq 1, \min_{1 \leq i \leq m} t_i = 1 \right\}.$$

By Theorem 2.3, we can reformulate the above definition for subsets of \mathbb{R}_{++}^n :

A subset S of \mathbb{R}_{++}^n is \mathbb{B}^{-1} -convex set if and only if for all $x^{(1)}, x^{(2)}, \dots, x^{(m)} \in S$ and all $t_1, t_2, \dots, t_m \in [1, \infty)$ such that $\min\{t_1, t_2, \dots, t_m\} = 1$ one has $\bigwedge_{i=1}^m t_i x^{(i)} \in S$.

The following theorem presents more simple definition of \mathbb{B}^{-1} -convex sets.

Theorem 2.4. [2] A subset S of \mathbb{R}_{++}^n is \mathbb{B}^{-1} -convex set if and only if for all $x^{(1)}, x^{(2)} \in S$

$$Co^{-\infty}(\{x^{(1)}, x^{(2)}\}) \subset S$$

that is, subset S of \mathbb{R}_{++}^n is \mathbb{B}^{-1} -convex set if and only if for all $x^{(1)}, x^{(2)} \in S$ and all $t \in [1, \infty)$ one has $tx^{(1)} \wedge x^{(2)} \in S$.

Some of procedures in which \mathbb{B}^{-1} -convexity is preserved are given in [3]:

Theorem 2.5. [2] The following properties hold:

- (i) The empty set, K^n , as well as the singletons are \mathbb{B}^{-1} -convex;
- (ii) If $\{S_\lambda : \lambda \in \Lambda\}$ is an arbitrary family of \mathbb{B}^{-1} -convex sets, then $\bigcap_\lambda S_\lambda$ is \mathbb{B}^{-1} -convex;
- (iii) If $\{S_\lambda : \lambda \in \Lambda\}$ is a family of \mathbb{B}^{-1} -convex such that $\forall \lambda_1, \lambda_2 \in \Lambda, \exists \lambda_3 \in \Lambda$ such that $S_{\lambda_1} \cup S_{\lambda_2} \subset S_{\lambda_3}$, then $\bigcup_\lambda S_\lambda$ is \mathbb{B}^{-1} -convex.

Definition 2.6. [2] Given a set $S \subset K^n$, the intersection of all the \mathbb{B}^{-1} -convex subsets of K^n containing S is called \mathbb{B}^{-1} -convex hull of S and we denote it by $\mathbb{B}^{-1}[S]$.

Theorem 2.7. [2] The following properties hold:

- (i) $\mathbb{B}^{-1}[\emptyset] = \emptyset, \mathbb{B}^{-1}[K^n] = K^n$ for all $x \in K^n, \mathbb{B}^{-1}[\{x\}] = \{x\}$;
- (ii) For all $S \subset K^n, S \subset \mathbb{B}^{-1}[S]$ and $\mathbb{B}^{-1}[\mathbb{B}^{-1}[S]] = \mathbb{B}^{-1}[S]$;
- (iii) For all $S_1, S_2 \subset K^n$, if $S_1 \subset S_2$ then $\mathbb{B}^{-1}[S_1] \subset \mathbb{B}^{-1}[S_2]$;
- (iv) For all $S \subset K^n, \mathbb{B}^{-1}[S] = \bigcup \{\mathbb{B}^{-1}[A] : A \text{ is a finite subset of } S\}$;
- (v) A subset $S \subset K^n$, is \mathbb{B}^{-1} -convex if and only if for all finite subsets A of $S, \mathbb{B}^{-1}[A] \subset S$.

3. Radon's Theorem and Helly's Theorem for \mathbb{B}^{-1} -Convex Sets

Theorem 3.1. [Radon's Theorem] If $A \subset \mathbb{R}_{++}^n$ is a finite set of cardinality at least $n + 2$, then there is a partition $A = A_1 \cup A_2$, in nonempty subsets, such that

$$Co^{-\infty}(A_1) \cap Co^{-\infty}(A_2) \neq \emptyset.$$

Proof. Let us apply Radon’s theorem which is provided for convex sets to operator;

$$A \mapsto Co^r(A) \quad (r \in Z^-).$$

A can be partitioned into two subsets $A_{r,1}, A_{r,2}$ such that

$$Co^r(A_{r,1}) \cap Co^r(A_{r,2}) \neq \emptyset$$

for all $r \in Z^-$. Since A is a finite set, there exist a partition (A_1, A_2) and a sequence $\{r_k\}_{k \in N}$ such that, the following equality, for $k \in N$

$$(A_{r_k,1}, A_{r_k,2}) = (A_1, A_2)$$

is correct. From there, for $k \in N$, a sequence of points $\{a_{r_k}\}_{k \in N}$ is obtained such that

$$a_{r_k} \in Co^{r_k}(A_{r_k,1}) \cap Co^{r_k}(A_{r_k,2}) = Co^{r_k}(A_1) \cap Co^{r_k}(A_2).$$

Let us take a prism at \mathbb{R}_{++}^n , such that $A \subset \prod_{i=1}^n [x_i, y_i]$, $x_i, y_i \in \mathbb{R}_{++}^n$, $i = 1, 2, \dots, n$. In this case, for $k \in N$

$$a_{r_k} \in Co^{r_k}(A_1) \cap Co^{r_k}(A_2) \subset Co^{r_k}(A) \subset \prod_{i=1}^n [x_i, y_i]$$

is true. Since $\{a_{r_k}\}_{k \in N}$ is a bounded sequence, it has convergent subsequence (Bolzano-Weierstrass Theorem). Let $\lim a_{r_k} = a^*$ without loss of generality. According to the limit defined by Painleve-Kuratowski,

$$x^* \in Co^{-\infty}(A_1) \cap Co^{-\infty}(A_2)$$

is true \square

Theorem 3.2. [Helly’s Theorem] Let \mathfrak{F} be a finite family of \mathbb{B}^{-1} -convex sets in \mathbb{R}_{++}^n containing at least $n + 1$ members. Consider that every $n + 1$ members of \mathfrak{F} have a non-empty intersection. Then \mathfrak{F} has a non-empty intersection.

Proof. Let us apply the induction method. The theorem is valid if the family of set \mathfrak{F} consists of $n + 1$ sets.

Suppose that the theorem is provided when the number of sets of \mathfrak{F} family is $m \geq n + 1$, and let us show that the theorem is confirmed when the number of sets of the \mathfrak{F} family is $m + 1$.

Let F_i be \mathbb{B}^{-1} -convex sets on \mathbb{R}_{++}^n , for $i = 0, 1, \dots, m$, such that

$$\mathfrak{F} = \{F_0, F_1, \dots, F_m\}$$

and intersection of each $n + 1$ sets is nonempty.

Considering that the theorem is valid for the family consisting of m sets, then there are a_0, a_1, \dots, a_m such that

$$a_i \in F_0 \cap F_1 \cap \dots \cap F_{i-1} \cap F_{i+1} \cap \dots \cap F_m$$

for all $i = 0, 1, \dots, m$.

According to Theorem 3.1, $A = \{a_0, a_1, \dots, a_m\}$ has two partition $A_1 = \{a_j : j \in J\}$ and $A_2 = \{a_k : k \in K\}$ such that

$$Co^{-\infty}(A_1) \cap Co^{-\infty}(A_2) \neq \emptyset.$$

In this case, let

$$a \in Co^{-\infty}(A_1) \cap Co^{-\infty}(A_2). \tag{1}$$

Since $\cap (F_k : k \in K)$ is \mathbb{B}^{-1} -convex set and for all $j \in J$, $a_j \in \cap (F_k : k \in K)$, the following relation holds true:

$$Co^{-\infty}(A_1) \subset \bigcap_{k \in K} F_k. \tag{2}$$

Similarly, it can be shown that

$$Co^{-\infty}(A_2) \subset \bigcap_{j \in J} F_j. \quad (3)$$

is also provided.

Thus, from (1), (2) and (3),

$$a \in Co^{-\infty}(A_1) \cap Co^{-\infty}(A_2) \subset \left[\bigcap_{k \in K} F_k \right] \cap \left[\bigcap_{j \in J} F_j \right] = F_0 \cap F_1 \cap \dots \cap F_m.$$

$$\text{So, } \bigcap_{i=1}^m F_m \neq \emptyset \quad \square$$

If compactness condition is also added on the sets, Helly's theorem is satisfied for infinite family of sets. The following lemma is used to prove this.

Lemma 3.3. [19] Let $\{C_i : i \in I\}$ be family of compact sets in \mathbb{R}^n whose intersection is empty. Then there exists a finite subset $I^* \subset I$ such that the intersection of the family $(C_i : i \in I)$ is empty.

Theorem 3.4. [Helly's Theorem for infinite families] Let \mathfrak{F} be an infinite family of compact B^{-1} -convex sets in \mathbb{R}_{++}^n . Suppose that every $n + 1$ members of \mathfrak{F} have a non-empty intersection. Then \mathfrak{F} has a non-empty intersection.

Proof. Let us assume the opposite. Let the intersection of the family of sets \mathfrak{F} be empty set, even if the conditions of the theorem are verified. In this case, according to Lemma 3.3, \mathfrak{F} has a finite number of subsets $\mathfrak{F}^* \subset \mathfrak{F}$ whose intersections are empty sets.

On the other hand, since \mathfrak{F}^* provides Theorem 3.2, the intersection cannot be an empty set. It is a contradiction \square

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