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Two-Step Ulm-Type Method for Solving Nonlinear Operator Equations

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Abstract. In this paper, we present a two-step Ulm-type method to solve systems of nonlinear equations without computing Jacobian matrices and solving Jacobian equations. we prove that the two-step Ulm-type method converges locally to the solution with R-convergence rate 3. Numerical implementations demonstrate the effectiveness of the new method.

1. Introduction

Let X and Y be Banach spaces, $D \in X$ be an open subset and $F : D \in X \to Y$ be a nonlinear operator with the continuous Fréchet derivative denoted by F'. Kantorovich's classical work is ever an important principle of determining the existence for the solution of the equation[5].

$$F(\mathbf{x}) = 0 \tag{1}$$

Without any doubt Newton's method is the most used iterative process to get the approximating a solution $\mathbf{x}^*$ of a nonlinear equation. It is given by the algorithm: $\mathbf{x}_{n+1} = \mathbf{x}_n - F'(\mathbf{x}_n)^{-1}F(\mathbf{x}_n), n \geq 0$ for $\mathbf{x}_0$ given. It converges quadratically under some mild conditions [7, 8, 10]. For recent progress on Newton method, one may refer to [9, 11, 12].

Other methods, such as higher order methods also include in their expression the inverse of the operator F'. To avoid this problem, Newton-type methods: $\mathbf{x}_{n+1} = \mathbf{x}_n - H_nF(\mathbf{x}_n)$, where H_n is an approximation of $F'(\mathbf{x}_n)^{-1}$ are considered. One of these methods was proposed by Moser in [2]. Given $\mathbf{x}_0 \in D$ and $B_0 \in \mathcal{L}(Y, X)$, the following sequences are defined

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n - B_nF(\mathbf{x}_n), \\ B_{n+1} &= 2B_n - B_nF'(\mathbf{x}_n)B_n, \quad n = 0, 1, 2, \dots. \end{aligned} \tag{2}$$

The first equation is similar to Newton's method, but replacing the operator $F'(\mathbf{x}_n)^{-1}$ by a linear operator B_n. The second equation is Newton's method applied to equation $g_n = 0$ where $g_n : \mathcal{L}(Y, X) \to \mathcal{L}(X, Y)$ is defined by $g_n(B) = B^{-1} - F'(\mathbf{x}_n)$. So $\{B_n\}$ gives us an approximation of $F'(\mathbf{x}_n)^{-1}$. it can be shown that the rate of convergence for the above scheme is $(1 + \sqrt{5})/2$, provided the root of (1) is simple [2].

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In [1] Ulm proposed the following iterative method to solve nonlinear equations. Given $\mathbf{x}_0 \in D$ and $B_0 \in \mathcal{L}(Y, X)$, Ulm defines

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n - B_n F(\mathbf{x}_n), \\ B_{n+1} &= 2B_n - B_n F'(\mathbf{x}_{n+1}) B_n, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{3}$$

Notice that, here $F'(\mathbf{x}_{n+1})$ appears instead of $F'(\mathbf{x}_n)$ in (2), This is crucial for obtaining fast convergence. Under the classical assumption that the derivative F' is Lipschitz continuous around the solution, Ulm showed that the method generates successive approximations that converge to a solution of (1), asymptotically as fast as Newton's method.

In [17] Ezquerro and Hernández considered Chebyshev's method and proposed the following iterative method to solve nonlinear equations. Given $\mathbf{x}_0 \in D$ and $B_0 \in \mathcal{L}(Y, X)$, Ulm defines

$$\begin{aligned} \mathbf{y}_n &= \mathbf{x}_n - B_n F(\mathbf{x}_n), \\ \mathbf{x}_{n+1} &= \mathbf{y}_n - B_n F(\mathbf{y}_n), \\ B_{n+1} &= B_n + B_n(2I - F'(\mathbf{x}_{n+1})B_n)(I - F'(\mathbf{x}_{n+1})B_n), \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{4}$$

it does not use any inverse operator in its application and Ezquerro and Hernández showed that the method generates successive approximations that converge to a solution of (1), has cubical convergence. For recent progress on Newton method, one may refer to [3, 4, 6, 13–16, 18, 19].

In [12], Darvishi and Barati gave the two-step Newton-type method, Given $\mathbf{x}_0 \in D$, the following sequences are defined

$$\begin{aligned} \mathbf{y}_n &= \mathbf{x}_n - F'(\mathbf{x}_n)^{-1} F(\mathbf{x}_n), \\ \mathbf{x}_{n+1} &= \mathbf{x}_n - F'(\mathbf{x}_n)^{-1} F(\mathbf{y}_n), \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{5}$$

The purpose of the present paper is, motivated by the two-step Newton-type method to propose the two-step Ulm-type method for solving the nonlinear operator equation $F(\mathbf{x}) = 0$. Given $\mathbf{x}_0 \in D$ and $B_0 \in \mathcal{L}(Y, X)$, the two-step Ulm-type method is defined by

$$\begin{aligned} \mathbf{y}_n &= \mathbf{x}_n - B_n F(\mathbf{x}_n), \\ \mathbf{x}_{n+1} &= \mathbf{y}_n - B_n F(\mathbf{y}_n), \\ A_n &= 2B_n - B_n F'(\mathbf{x}_{n+1}) B_n, \\ B_{n+1} &= 2A_n - A_n F'(\mathbf{x}_{n+1}) A_n, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{6}$$

This method exhibits several attractive features. First, it is inverse free: we do not need to solve a linear equation at each iteration. Second, in addition to solve the nonlinear equation (1), the method produces successive approximations $\{B_n\}$ to the value of $F'(\mathbf{x}^*)^{-1}$, being $\mathbf{x}^*$ a solution of (1). This property is very helpful especially when one investigates the sensitivity of the solution to small perturbations. Further more, under certain assumptions, the radius of the convergence ball for the two-step Ulm-type method is estimated, and the R-convergence rate 3 of the two-step Ulm-type method is proved. Numerical experiment is given in the last section illustrating the convergence performance of the two-step Ulm-type method.

2. Convergence analysis

Let $\mathbf{B}(\mathbf{x}, r)$ stands for the open ball in X with center $\mathbf{x}$ and radius $r > 0$. Let $\mathbf{x}^* \in D$ be a solution of the nonlinear equation (1) such that $F'(\mathbf{x}^*)$ is invertible and that F' satisfies Lipschitz condition on $\mathbf{B}(\mathbf{x}^*, r)$ with the Lipschitz constant L:

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\| \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbf{B}(\mathbf{x}^*, r). \tag{7}$$

Let

$$0 < r_L < \min\Big\{1,\ r,\ \frac{1}{L\|F'(\mathbf{x}^*)^{-1}\|}\Big\};\quad \eta = \frac{\|F'(\mathbf{x}^*)^{-1}\|}{1 - L\|F'(\mathbf{x}^*)^{-1}\|r_L}; \tag{8}$$

$$0 < \alpha \le \min\{\frac{\sqrt{3\sqrt{3}-4}}{12\sqrt{3}\eta L},\ \frac{2\sqrt{2}}{15\eta L}\};\quad 0 < \beta \le \min\{r_L,\ \alpha\};\quad 0 < \xi \le \frac{4\sqrt{3}\eta L\beta}{\sqrt{3\sqrt{3}-4}}. \tag{9}$$

The following lemma is crucial for the proof of the main theorem.

Lemma 2.1. *If $\mathbf{x}_n \in \mathbf{B}(\mathbf{x}, r_L)$, Then $F'(\mathbf{x}_n)$ is invertible and $\|F'(\mathbf{x}_n)^{-1}\| \le \eta$.*

Proof. By (7), (8) and the assumption that

$$\|F'(\mathbf{x}^*)^{-1}\|\|F'(\mathbf{x}_n) - F'(\mathbf{x}^*)\| \le L\|F'(\mathbf{x}^*)^{-1}\|r_L < 1.$$

Consequently, using Banach's lemma, we can derive that $F'(\mathbf{x}_n)$ is invertible and moreover

$$\|F'(\mathbf{x}_n)^{-1}\| \le \frac{\|F'(\mathbf{x}^*)^{-1}\|}{1 - L\|F'(\mathbf{x}^*)^{-1}\|r_L} = \eta.$$

Note that in the two-step Ulm-type method, sequence $\{B_n\}$ is generated by the algorithm except for B_0. Below, we prove that if B_0 approximates $\|F'(\mathbf{x}_0)^{-1}\|$, then the sequence $\{\mathbf{x}_n\}$ generated by the two-step Ulm-type method converges locally to $\mathbf{x}^*$ with R-convergence rate 3. For this end, let B_0 satisfy that

$$\|I - B_0F'(\mathbf{x}_0)^{-1}\| \le \xi, \tag{10}$$

where ξ is defined in (9).

Theorem 2.2. *Suppose that the Jacobian operator $F'(\mathbf{x}^*)$ is invertible and that F' satisfies the Lipschitz condition* (7) *on $\mathbf{B}(\mathbf{x}^*, r_L)$. Then there exist positive numbers β and ξ such that for any $\mathbf{x}_0 \in \mathbf{B}(\mathbf{x}^*, \beta)$ and B_0 satisfying* (10)*, the sequence $\{\mathbf{x}_n\}$ generated by the two-step Ulm-type method with initial point $\mathbf{x}_0$ converges to $\mathbf{x}^*$. Moreover, the following estimates hold for each $n = 0, 1, ...$*

$$\|\mathbf{x}_n - \mathbf{x}^*\| \le \alpha\Big(\frac{\beta}{\alpha}\Big)^{3^n} \tag{11}$$

and

$$\|I - B_nF'(\mathbf{x}_n)^{-1}\| \le \frac{1}{3}\Big(\frac{\beta}{\alpha}\Big)^{3^n}, \tag{12}$$

where α is defined in (9)*.*

Proof. We proceed by mathematical induction. Clearly, (11) is trivial for $n = 0$ by the assumption. By (9) and (10), we obtain

$$\|I - B_0F'(\mathbf{x}_0)^{-1}\| \le \xi \le \frac{4\sqrt{3}\eta L\beta}{\sqrt{3\sqrt{3}-4}} = \frac{\beta}{3\alpha}.$$

That is, (12) holds for $n = 0$. Now we assume that (11) and (12) hold for $n = m$. Then one has

$$\|\mathbf{x}_m - \mathbf{x}^*\| \le \alpha\Big(\frac{\beta}{\alpha}\Big)^{3^m} \tag{13}$$

and

$$\|I - B_m F'(\mathbf{x}_m)\| \le \frac{1}{3}\Big(\frac{\beta}{\alpha}\Big)^{3^m}. \tag{14}$$

By (9), we get

$$\|\mathbf{x}_m - \mathbf{x}^*\| \le \alpha\Big(\frac{\beta}{\alpha}\Big)^{3^m} < \beta < r_L,$$

It follows from (9), (14) and Lemma 2.1 that

$$\|F'(\mathbf{x}_m)^{-1}\| \le \eta$$

and $\beta \le \alpha$, we have

$$\begin{aligned}\|B_m\| &\le \|B_m F'(\mathbf{x}_m)\|\|F'(\mathbf{x}_m)^{-1}\| \le (1 + \|I - B_m F'(\mathbf{x}_m)\|)\|F'(\mathbf{x}_m)^{-1}\| \\ &\le \eta\Big[1 + \frac{1}{3}\Big(\frac{\beta}{\alpha}\Big)^{3^m}\Big] \le \sqrt{2}\eta. \end{aligned} \tag{15}$$

By (6), we have

$$\begin{aligned}\mathbf{y}_m - \mathbf{x}^* &= \mathbf{x}_m - \mathbf{x}^* - B_m(F(\mathbf{x}_m) - F(\mathbf{x}^*)) \\ &= \mathbf{x}_m - \mathbf{x}^* - \int_0^1 B_m F'(\mathbf{x}_m^\theta)(\mathbf{x}_m - \mathbf{x}^*)d\theta \\ &= \int_0^1 [I - B_m F'(\mathbf{x}_m) + B_m(F'(\mathbf{x}_m) - F'(\mathbf{x}_m^\theta))](\mathbf{x}_m - \mathbf{x}^*)d\theta\end{aligned}$$

where $\mathbf{x}_m^\theta = \mathbf{x}^* + \theta(\mathbf{x}_m - \mathbf{x}^*)$ for $0 \le \theta \le 1$. Since $\|\mathbf{x}_m - \mathbf{x}^*\| \le r_L$ and $\|\mathbf{x}_m^\theta - \mathbf{x}^*\| = \theta\|\mathbf{x}_m - \mathbf{x}^*\| \le \|\mathbf{x}_m - \mathbf{x}^*\| \le r_L$, it follows from (9), (13), (14), (15), $\alpha \le \frac{2\sqrt{2}}{15\eta L}$ and the Lipschitz condition that

$$\begin{aligned}\|\mathbf{y}_m - \mathbf{x}^*\| &\le \int_0^1 (\|I - B_m F'(\mathbf{x}_m)\| + L(1-\theta)\|B_m\|\|\mathbf{x}_m - \mathbf{x}^*\|)\|\mathbf{x}_m - \mathbf{x}^*\|d\theta \\ &= \|I - B_m F'(\mathbf{x}_m)\|\|\mathbf{x}_m - \mathbf{x}^*\| + \frac{L}{2}\|B_m\|\|\mathbf{x}_m - \mathbf{x}^*\|^2 \\ &\le \frac{1}{3}\Big(\frac{\beta}{\alpha}\Big)^{3^m}\alpha\Big(\frac{\beta}{\alpha}\Big)^{3^m} + \frac{\sqrt{2}}{2}L\eta\Big(\alpha\Big(\frac{\beta}{\alpha}\Big)^{3^m}\Big)^2 \\ &= \Big(\frac{1}{3} + \frac{\sqrt{2}}{2}L\eta\alpha\Big)\alpha\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} \le \alpha\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m}\end{aligned} \tag{16}$$

Together with (9) and (13), we get

$$\|\mathbf{x}_m - \mathbf{y}_m\| \le \|\mathbf{x}_m - \mathbf{x}^*\| + \|\mathbf{y}_m - \mathbf{x}^*\| \le \alpha\Big(\frac{\beta}{\alpha}\Big)^{3^m} + \alpha\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} \le 2\alpha\Big(\frac{\beta}{\alpha}\Big)^{3^m}. \tag{17}$$

it follows from (9), (14), (15) and the Lipschitz condition that

$$\begin{aligned}\|I - B_m F'(\mathbf{y}_m)\| &\le \|I - B_m F'(\mathbf{x}_m)\| + \|B_m\|\|F'(\mathbf{x}_m) - F'(\mathbf{y}_m)\| \\ &\le \frac{1}{3}\Big(\frac{\beta}{\alpha}\Big)^{3^m} + 2\sqrt{2}L\eta\alpha\Big(\frac{\beta}{\alpha}\Big)^{3^m} \\ &\le \Big(\frac{1}{3} + 2\sqrt{2}L\eta\alpha\Big)\Big(\frac{\beta}{\alpha}\Big)^{3^m}\end{aligned} \tag{18}$$

Similar to (16), by (8), (9), (15), (16), (18), $\alpha \le \frac{2\sqrt{2}}{15\eta L}$ and the Lipschitz condition, we get

$$\begin{aligned} \|\mathbf{x}_{m+1} - \mathbf{x}^*\| &= \|I - B_m F'(\mathbf{y}_m)\|\|\mathbf{y}_m - \mathbf{x}^*\| + \frac{L}{2}\|B_m\|\|\mathbf{y}_m - \mathbf{x}^*\|^2 \\ &\le \Big(\frac{1}{3} + 2\sqrt{2}L\eta\alpha\Big)\Big(\frac{\beta}{\alpha}\Big)^{3^m}\alpha\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} + \frac{\sqrt{2}}{2}L\eta\Big(\alpha\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m}\Big)^2 \\ &= \Big(\frac{1}{3} + \frac{5\sqrt{2}}{2}L\eta\alpha\Big)\alpha\Big(\frac{\beta}{\alpha}\Big)^{3^{m+1}} \le \alpha\Big(\frac{\beta}{\alpha}\Big)^{3^{m+1}} \end{aligned} \tag{19}$$

By (9), we get

$$\|\mathbf{x}_{m+1} - \mathbf{x}^*\| \le \alpha\Big(\frac{\beta}{\alpha}\Big)^{3^{m+1}} < \beta < r_L,$$

Consequently, (11) holds for $n = m+1$ and by (9), (16) and (19), we get

$$\|\mathbf{x}_{m+1} - \mathbf{y}_m\| \le \|\mathbf{x}_{m+1} - \mathbf{x}^*\| + \|\mathbf{y}_m - \mathbf{x}^*\| \le \alpha\Big(\frac{\beta}{\alpha}\Big)^{3^{m+1}} + \alpha\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} \le 2\alpha\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m}. \tag{20}$$

Moreover, by (6), we obtain

$$I - B_{m+1}F'(\mathbf{x}_{m+1}) = I - (2A_m - A_m F'(\mathbf{x}_{m+1}A_m)F'(\mathbf{x}_{m+1} = (I - A_{m+1}F'(\mathbf{x}_{m+1}))^2, \tag{21}$$

and

$$I - A_{m+1}F'(\mathbf{x}_{m+1}) = I - (2B_m - B_m F'(\mathbf{x}_{m+1}B_m)F'(\mathbf{x}_{m+1} = (I - B_m F'(\mathbf{x}_{m+1}))^2, \tag{22}$$

Consequently

$$I - B_{m+1}F'(\mathbf{x}_{m+1}) = (I - B_m F'(\mathbf{x}_{m+1}))^4. \tag{23}$$

It is easy to know that

$$\|I - B_m F'(\mathbf{x}_{m+1})\| \le \|I - B_m F'(\mathbf{y}_m)\| + \|B_m\|\|F'(\mathbf{x}_{m+1}) - F'(\mathbf{y}_m)\|$$

it follows from (9), (15), (18), (20) and the Lipschitz condition that

$$\begin{aligned} \|I - B_m F'(\mathbf{x}_{m+1})\|^2 &\le 2\|I - B_m F'(\mathbf{y}_m)\|^2 + 2\|B_m\|^2\|F'(\mathbf{x}_{m+1}) - F'(\mathbf{y}_m)\|^2 \\ &\le 2\Big(\frac{1}{3} + 2\sqrt{2}L\eta\alpha\Big)^2\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} + 4\eta^2L^2 \times 4\alpha^2\Big(\frac{\beta}{\alpha}\Big)^{4\times 3^m} \\ &\le 2\Big(\frac{2}{9} + 16L^2\eta^2\alpha^2\Big)\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} + 4\eta^2L^2 \times 4\alpha^2\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} \\ &\le \Big(\frac{4}{9} + 48L^2\eta^2\alpha^2\Big)\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} \\ &\le \frac{\sqrt{3}}{3}\Big(\frac{\beta}{\alpha}\Big)^{2\times 3^m} \end{aligned}$$

Together with (23), we get

$$\|I - B_{m+1}F'(\mathbf{x}_{m+1})\| \le \|I - B_m F'(\mathbf{x}_{m+1})\|^4 \le \frac{1}{3}\Big(\frac{\beta}{\alpha}\Big)^{4\times 3^m} \le \frac{1}{3}\Big(\frac{\beta}{\alpha}\Big)^{3^{m+1}}.$$

This conforms that (12) holds for $n = m+1$ and the proof is complete.

3. Numerical experiments

In this section, we report the numerical performance of the two-step Ulm-type method for solving the nonlinear operator equation (1). We compare the two-step Ulm-type method with the Ulm's method [1], the two-step Newton-type method[12] and the Ezquerro and Hernández's method [17]. For convenience, we denote the Ulm's method in [1] as UM , the two-step Newton-type method in [12] as TNM, the Ezquerro and Hernández's method in [17] as EHM, and the two-step Ulm-type method as TUM. All the tests were carried out in `MATLAB 7.10` running on a PC Intel Pentium IV of 3.0 GHz CPU.

we consider the two-point boundary value problem

$$\begin{cases} \mathbf{x}'' + \mathbf{x}^2 = 0, \\ \mathbf{x}(0) = \mathbf{x}(1) = 0. \end{cases} \tag{24}$$

We divide the interval $[0,1]$ into $m+1$ subintervals and we get $h = 1/m+1$. Let $d_0, d_1, \ldots, d_{m+1}$ be the points of subdivision with $0 < d_0 < d_1 < \cdots < d_{m+1} = 1$. An approximation for the second derivative may be chosen as

$$\begin{cases} x_i'' = \frac{x_{i-1}-2x_i+x_{i+1}}{h^2}, \\ x_0 = x_1 = 0. \end{cases} x_i = \mathbf{x}(d_i) \text{ for } i = 1, 2, \ldots, m. \tag{25}$$

Let the operator $\phi : \mathbf{R}^m \to \mathbf{R}^m$ be defined by

$$\phi(\mathbf{x}) = \left(x_1^2, x_2^2, \ldots, x_m^2\right)^T \text{ for } \mathbf{x} = \left(x_1, x_2, \ldots, x_m\right)^T \in \mathbf{R}^m.$$

To get an approximation to the solution of (24), we need to solve the following nonlinear equation:

$$F(\mathbf{x}) := M\mathbf{x} + h^2\phi(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbf{R}^m,$$

where

$$M = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}_{m\times m}$$

Obviously, $\mathbf{x}^* = 0$ is a solution of (25) and

$$F'(\mathbf{x}) = M + 2h^2 \text{diag}(x_1, x_2, \ldots, x_m).$$

Hence $F'(\mathbf{x}^*) = M$. Furthermore, it is easy to verify that

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \le 2h^2\|\mathbf{x} - \mathbf{y}\| \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbf{R}^m.$$

where $\|\cdot\|$ denotes the F-norm. For different choices of m, thanks to the results from Section 2, there exists a radius r_L such that for each $\mathbf{x}_0 \in \mathbf{B}(\mathbf{x}^*, r_L)$, the sequence $\{\mathbf{x}_n\}$ generated by the two-step Ulm-type method converges to $\mathbf{x}^* = 0$ with convergence order 3. Set $B_0 = \|F'(\mathbf{x}_0)^{-1}\|$. the convergence performance of the algorithm are illustrated in the following tables. Here we consider the following three problem sizes: (a) $m = 10$ and $\mathbf{x}_0 = \gamma(1, 1, \ldots, 1)^T$ in Table 1, (b) $m = 100$ and $\mathbf{x}_0 = \gamma(1, 1, \ldots, 1)^T$ in Table 2 and (c) $m = 1000$ and $\mathbf{x}_0 = \gamma(1, 1, \ldots, 1)^T$ in Table 3, where $\gamma = 0.2$ or 0.02. In Table 4, we compare the averaged CPU time of Algorithms UM, EHM, TNM and TUM for ten tests with different (m, γ).

Table 1: value of $\|\mathbf{x}_n - \mathbf{x}^*\|_F$ for case (a).

γ	it.	Algorithm UM	Algorithm EHM	Algorithm TNM	Algorithm TUM
0.2	0	$6.3246e-1$	$6.3246e-1$	$6.3246e-1$	$6.3246e-1$
	1	$1.2625e-2$	$5.4276e-4$	$5.4276e-4$	$5.4276e-4$
	2	$2.9655e-5$	$6.1381e-12$	$5.5641e-12$	$3.3657e-13$
	3	$2.6731e-10$	$2.2970e-35$	$9.3920e-34$	
	4	$3.0008e-20$			
0.02	0	$6.3246e-1$	$6.3246e-1$	$6.3246e-1$	$6.3246e-1$
	1	$1.2160e-4$	$4.9840e-7$	$4.9840e-7$	$4.9840e-7$
	2	$2.5862e-9$	$4.5866e-21$	$6.3132e-22$	$3.3066e-22$
	3	$1.9654e-18$			

Table 2: value of $\|\mathbf{x}_n - \mathbf{x}^*\|_F$ for case (b).

γ	it.	Algorithm UM	Algorithm EHM	Algorithm TNM	Algorithm TUM
0.2	0	$2.0000e+0$	$2.0000e+0$	$2.0000e+0$	$2.0000e+0$
	1	$3.8245e-2$	$1.6327e-3$	$1.6327e-3$	$1.6327e-3$
	2	$8.8705e-5$	$1.7773e-11$	$2.8562e-11$	$9.8802e-13$
	3	$7.8135e-10$	$5.7641e-35$	$6.5325e-34$	
	4	$8.3847e-20$			
0.02	0	$2.0000e-1$	$2.0000e-1$	$2.0000e-1$	$2.0000e-1$
	1	$3.6846e-4$	$1.5001e-6$	$1.5001e-6$	$1.5001e-6$
	2	$7.7420e-9$	$1.3307e-20$	$1.8767e-20$	$9.6876e-22$
	3	$5.7553e-18$			

Table 3: value of $\|\mathbf{x}_n - \mathbf{x}^*\|_F$ for case (c).

γ	it.	Algorithm UM	Algorithm EHM	Algorithm TNM	Algorithm TUM
0.2	0	$6.3246e+0$	$6.3246e+0$	$6.3246e+0$	$6.3246e+0$
	1	$1.2040e-1$	$5.1396e-3$	$5.1396e-3$	$5.1396e-3$
	2	$2.7921e-4$	$5.5923e-11$	$4.1096e-11$	$3.1093e-12$
	3	$2.4588e-9$	$1.0346e-34$	$9.6784e-34$	$1.8165e-37$
	4	$2.1700e-19$			
0.02	0	$6.3246e-1$	$6.3246e-1$	$6.3246e-1$	$6.3246e-1$
	1	$1.1600e-3$	$4.7221e-6$	$4.7221e-6$	$4.7221e-6$
	2	$2.4370e-8$	$4.1869e-20$	$7.5248e-21$	$3.0484e-21$
	3	$1.8178e-17$			

Table 4: Averaged CPU time in seconds of Algorithms UM, EHM, TNM and TUM for the ten tests

(m, γ)	(100, 0.2)	(100, 0.02)	(1000, 0.2)	(1000, 0.02)	(2000, 0.2)	(2000, 0.02)
Algorithms UM	$1.23e-2$	$6.21e-3$	1.83	1.09	10.05	6.12
Algorithms EHM	$8.56e-3$	$5.30e-3$	1.65	0.91	9.83	5.73
Algorithms TNM	$9.23e-3$	$5.29e-3$	1.59	0.90	9.50	5.99
Algorithms TUM	$6.43e-3$	$4.96e-3$	1.51	0.83	9.12	5.23

From Tables 1–3, we observe that the Ulm's method converges quadratically and the Ezquerro and Hernández's method the two-step Newton-type method and the two-step Ulm-type method converge cubically in the root sense and the two-step Ulm-type method converges faster than the Ulm's method the two-step Newton-type method and the Ezquerro and Hernández's method. Table 4 shows that the CPU time by Algorithm TUM is cheaper than Algorithm UM, EHM and TNM.

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