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On Strong Convergence Theorems for a Viscosity-Type Extragradient Method

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Abstract. In this paper, we introduce a general viscosity-type extragradient method for solving the fixed point problem of an asymptotically nonexpansive mapping and the variational inclusion problem with two accretive operators. We obtain a strong convergence theorem in the setting of Banach spaces. In terms of this theorem, we establish the strong convergence result for solving the fixed point problem (FPP) of an asymptotically nonexpansive mapping and the variational inequality problem (VIP) for an inverse-strongly monotone mapping in the framework of Hilbert spaces. Finally, this result is applied to deal with the VIP and FPP in an illustrating example.

1. Introduction

Let H be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Given a nonempty closed convex subset $C \subset H$. Let P_C be the metric projection of H onto C. Consider the classical variational inequality problem (VIP) of finding a point $z \in C$ such that $\langle Az, x - z\rangle \geq 0$, $\forall x \in C$, where $A : C \to H$ is an operator. The set of solutions of the VIP is denoted by VI(C, A). It is known well that the variational inequality theory has been widely applicable for diverse disciplines in pure and applied sciences, for example, differential equations, time-optimal control, optimization, mathematical programming, mechanics, economics and other applied science fields; see, e.g., [2–7, 21, 24, 25]. In the past few decades, many methods have been suggested and improved for solving the VIP. Among these methods, Korpelevich's extragradient method is one of the most popular ones. In 1976, Korpelevich [15] introduced the extragradient method for solving saddle point problems. Subsequently, this method was successfully extended to the development of solving variational inequalities in both Euclidean and Hilbert spaces. More precisely, Korpelevich's extragradient method is specified as follows: for any given $u_0 \in C$, $\{u_k\}$ is the sequence generated by

$$\begin{cases} v_k = P_C(u_k - \lambda A u_k), \\ u_{k+1} = P_C(u_k - \lambda A v_k) \quad \forall k \geq 0, \end{cases} \tag{1.1}$$

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with constant $\lambda \in (0, \frac{1}{L})$. It is worth pointing out that the convergence of the sequence $\{u_k\}$ only requires that the operator A is monotone and Lipschitz continuous. Meantime, the sequence $\{u_k\}$ has only weak convergence. In recent years, Korpelevich's extragradient method has received great attention given by many authors, who improved and modified it in various ways; see e.g., [8, 11, 12, 17, 20, 26] and references therein.

Recently, many authors investigated the problem of finding

$$z \in \mathrm{Fix}(S) \cap (A+B)^{-1}0, \tag{1.2}$$

where $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and $S : C \to C$ is a nonexpansive mapping; see, [9, 13, 14, 19, 22, 23] and the references therein. In 2011, Manaka and Takahashi [18] introduced the following iterative process: for any given $x_0 \in C$, $\{x_k\}$ is the sequence generated by

$$x_{k+1} = \alpha_k x_k + (1-\alpha_k) S J^B_{\lambda_k}(x_k - \lambda_k A x_k) \quad \forall k \geq 0, \tag{1.3}$$

where $\{\alpha_k\} \subset (0,1)$ and $\{\lambda_k\} \subset (0,\infty)$. They proved the weak convergence of $\{x_k\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some appropriate assumptions.

Recently, Takahashi et al. [28] invented a Mann-type Halpern iterative scheme for finding a common solution of the FPP of a nonexpansive mapping $S : C \to C$ and the VI for an α-inverse-strongly monotone mapping $A : C \to H$ and a maximal monotone operator $B : D(B) \subset C \to H$, i.e., for any given $x_1 = x \in C$, $\{x_k\}$ is the sequence generated by

$$x_{k+1} = \beta_k x_k + (1-\beta_k) S(\alpha_k x + (1-\alpha_k) J^B_{\lambda_k}(x_k - \lambda_k A x_k)) \quad \forall k \geq 1, \tag{1.4}$$

where $\{\lambda_k\} \subset (0, 2\alpha)$ and $\{\alpha_k\}, \{\beta_k\} \subset (0,1)$ are such that (i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$; (ii) $0 < a \leq \lambda_k \leq b < 2\alpha$, $\lim_{k\to\infty}(\lambda_k - \lambda_{k+1}) = 0$; and (iii) $0 < c \leq \beta_k \leq d < 1$. They proved that $\{x_k\}$ converges strongly to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$.

Meantime, let $F : C \to H$ be a monotone and L-Lipschitzian mapping, $A : C \to H$ be an α-inverse strongly monotone mapping, B be a maximal monotone mapping with $D(B) = C$ and $S : C \to C$ be a nonexpansive mapping such that $\Omega := \mathrm{Fix}(S) \cap (A+B)^{-1}0 \cap \mathrm{VI}(C,F) \neq \emptyset$. Ceng et al. [9] introduce the following Mann-type hybrid extragradient algorithm: for any given $x_0 = u \in C$, $\{x_k\}$ is the sequence generated by

$$\begin{cases} y_k = P_C(x_k - \mu_k F x_k), \\ \hat{t}_k = J^B_{\lambda_k}(I - \lambda_k A) P_C(x_k - \mu_k F y_k), \\ z_k = (1 - \alpha_k - \hat{\alpha}_k) x_k + \alpha_k \hat{t}_k + \hat{\alpha}_k S \hat{t}_k, \\ C_k = \{z \in C : \|z_k - z\| \leq \|x_k - z\|\}, \\ Q_k = \{z \in C : \langle x_k - z, x - x_k \rangle \geq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k} u \quad \forall k \geq 0, \end{cases} \tag{1.5}$$

where $J^B_{\lambda_k} = (I + \lambda_k B)^{-1}$, $\{\mu_k\} \subset (0, \frac{1}{L})$, $\{\lambda_k\} \subset (0, 2\alpha]$ and $\{\alpha_k\}, \{\hat{\alpha}_k\} \subset (0,1]$ such that $\alpha_k + \hat{\alpha}_k \leq 1$. They proved strong convergence of $\{x_k\}$ to the point $P_\Omega u$ under some appropriate conditions.

Let C be a nonempty closed convex set in a real Banach space E with the dual E^*. Given a self-mapping T on C. We use the notation $\mathrm{Fix}(T)$ to stand for the set of fixed points of T. Recall that T is said to be asymptotically nonexpansive if $\exists\{\theta_n\}$ s.t. $\lim_{k\to\infty} \theta_k = 0$ and $\|T^k u - T^k v\| \leq (1+\theta_k)\|u - v\|$ $\forall u, v \in C$, $k \geq 0$. In particular, if $\theta_k = 0$ $\forall k \geq 1$, then T is said to be nonexpansive. A mapping $f : C \to C$ is called a contraction if $\exists \rho \in [0,1)$ s.t. $\|f(u) - f(v)\| \leq \rho\|u - v\|$ $\forall u, v \in C$. Recall that the normalized duality mapping J from E into the family of nonempty (by Hahn-Banach's theorem) weak* compact subsets of E^* satisfies $J(\tau u) = \tau J(u)$ and $J(-u) = -J(u)$ for all $\tau > 0$ and $u \in E$.

The modulus of convexity of E is the function $\delta : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|u+v\|}{2} : u, v \in E,\ \|u\| = \|v\| = 1,\ \|u - v\| \geq \epsilon\}.$$

The modulus of smoothness of E is the function $\rho_E : \mathbf{R}_+ := [0,\infty) \to \mathbf{R}_+$ defined by

$$\rho_E(\tau) = \sup\{\frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 : u, v \in E,\ \|u\| = \|v\| = 1\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0\ \forall \epsilon \in (0, 2]$. It is said to be uniformly smooth if $\lim_{\tau\to 0^+} \rho_E(\tau)/\tau = 0$. Also, it is said to be q-uniformly smooth with $q > 1$ if $\exists c > 0$ s.t. $\rho_E(t) \le ct^q\ \forall t > 0$. If E is q-uniformly smooth, then $q \le 2$ and E is also uniformly smooth and if E is uniformly convex, then E is also reflexive and strictly convex. It is known that Hilbert space H is 2-uniformly smooth. Further, sequence space ℓ_p and Lebesgue space L_p are $\min\{p, 2\}$-uniformly smooth for every $p > 1$. Let $q > 1$. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^q,\ \|\phi\| = \|x\|^{q-1}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. In particular, if $q = 2$, then $J_2 := J$ is called the normalized duality mapping of E. It is known that $J_q(x) = \|x\|^{q-2}J(x)\ \forall x \ne 0$ and that J_q is the subdifferential of the functional $\frac{1}{q}\|\cdot\|^q$. If E is uniformly smooth, the generalized duality mapping J_q is one-to-one and single-valued. Furthermore, J_q satisfies $J_q = J_p^{-1}$, where J_p is the generalized duality mapping of E^* with $\frac{1}{p} + \frac{1}{q} = 1$. From Xu [30], no Banach space is q-uniformly smooth for $q > 2$.

Let $f : E \to E$ be a ρ-contraction and $S : E \to E$ be a nonexpansive operator. Let $A : E \to E$ be an α-inverse-strongly accretive mapping of order q and $B : E \to 2^E$ be an m-accretive operator. Very recently, to solve the FPP of S and the VI of finding $z \in E$ s.t. $0 \in (A + B)z$. Sunthrayuth and Cholamjiak [27] suggested a modified viscosity-type extragradient method in the setting of uniformly convex and q-uniformly smooth Banach space E with q-uniform smoothness coefficient κ_q, i.e., for any given $x_0 \in E$, $\{x_k\}$ is the sequence generated by

$$\begin{cases} y_k = J^B_{\lambda_k}(x_k - \lambda_k A x_k), \\ z_k = J^B_{\lambda_k}(x_k - \lambda_k A y_k + r_k(y_k - x_k)), \\ x_{k+1} = \alpha_k f(x_k) + \beta_k x_k + \gamma_k S z_k \quad \forall k \ge 0, \end{cases}$$

where $J^B_{\lambda_k} = (I + \lambda_k B)^{-1}$, $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}, \{r_k\} \subset (0, 1)$ and $\{\lambda_k\} \subset (0,\infty)$ are such that: (i) $\alpha_k + \beta_k + \gamma_k = 1$; (ii) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$; (iii) $\{\beta_k\} \subset [a, b] \subset (0, 1)$; and (iv) $0 < \lambda \le \lambda_k < \lambda_k/r_k \le \mu < (\alpha q/\kappa_q)^{1/(q-1)}$, $0 < r \le r_k < 1$. They proved strong convergence of $\{x_k\}$ to a point of $\mathrm{Fix}(S) \cap (A + B)^{-1}0$, which solves a certain hierarchical variational inequality (HVI).

Inspired and motivated by the above research works, we introduce a general viscosity-type extragradient algorithm in the setting of uniformly convex and q-uniformly smooth Banach space E, which admits a weakly continuous duality mapping. It is proven that the sequence constructed by the suggested algorithm converges strongly to a point of $\mathrm{Fix}(T) \cap (A + B)^{-1}0$ under some suitable assumptions imposed on the parameters. In terms of this theorem, we establish the strong convergence result for solving the fixed point problem (FPP) of an asymptotically nonexpansive mapping and the variational inequality problem (VIP) for an inverse-strongly monotone mapping in the framework of Hilbert spaces. Finally, this result is applied to deal with the VIP and FPP in an illustrating example. Our results improve and extend the corresponding results in [9, 13, 14, 23, 27].

2. Preliminaries

Lemma 2.1. [23] Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then

$$\|x + y\|^q \le \|x\|^q + q\langle y, j_q(x + y)\rangle \quad \forall x, y \in E,\ j_q(x + y) \in J_q(x + y).$$

The following lemma can be obtained from the result in [30].

Lemma 2.2. Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exist strictly increasing, continuous and convex functions $g, h : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ and $h(0) = 0$ such that

(a) $\|\mu u + (1-\mu)v\|^q \le \mu\|u\|^q + (1-\mu)\|v\|^q - \mu(1-\mu)g(\|u-v\|)$ with $\mu \in [0,1]$;

(b) $h(\|u-v\|) \le \|u\|^q - q\langle u, j_q(v)\rangle + (q-1)\|v\|^q$

for all $u, v \in B_r$ and $j_q(v) \in J_q(v)$, where $B_r := \{y \in E : \|y\| \le r\}$.

Lemma 2.3. [23] Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exists a strictly increasing, continuous and convex function $g : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ such that $\|\lambda u + \mu v + \nu w\|^q \le \lambda\|u\|^q + \mu\|v\|^q + \nu\|w\|^q - \lambda\mu g(\|u-v\|)$ for all $u, v, w \in B_r$ and $\lambda, \mu, \nu \in [0,1]$ with $\lambda + \mu + \nu = 1$.

Proposition 2.1 [30] Let $q \in (1,2]$ a fixed real number and let E be q-uniformly smooth. Then $\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q \ \forall x, y \in E$, where κ_q is the q-uniform smoothness coefficient of E.

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \ge 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. Let E be smooth, D be a nonempty subset of C and Π be a retraction of C onto D. Then the following are equivalent: (i) Π is sunny and nonexpansive; (ii) $\|\Pi(x) - \Pi(y)\|^2 \le \langle x - y, J(\Pi(x) - \Pi(y))\rangle \ \forall x, y \in C$; (iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \le 0 \ \ \forall x \in C, y \in D$.

Let $B : C \to 2^E$ be a set-valued operator with $Bx \ne \emptyset \ \forall x \in C$. Let $q > 1$. An operator B is said to be accretive if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x-y)\rangle \ge 0 \ \ \forall u \in Bx, v \in By$. An accretive operator B is said to be α-inverse-strongly accretive of order q if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x-y)\rangle \ge \alpha\|u-v\|^q \ \forall u \in Bx, v \in By$ for some $\alpha > 0$. If $E = H$ a Hilbert space, then B is called α-inverse-strongly monotone. An accretive operator B is said to be m-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. For an accretive operator B, we define the mapping $J^B_\lambda : (I + \lambda B)C \to C$ by $J^B_\lambda = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J^B_λ is called the resolvent of B for $\lambda > 0$.

Lemma 2.4 [16]Let $B : C \to 2^E$ be an m-accretive operator. Then the following statements hold:

(i) the resolvent identity: $J^B_\lambda x = J^B_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J^B_\lambda x) \ \ \forall \lambda, \mu > 0, \ x \in E$;

(ii) if J^B_λ is a resolvent of B for $\lambda > 0$, then J^B_λ is a firmly nonexpansive mapping with $\mathrm{Fix}(J^B_\lambda) = B^{-1}0$, where $B^{-1}0 = \{x \in C : 0 \in Bx\}$;

(iii) if $E = H$ a Hilbert space, B is maximal monotone.

Let $A : C \to E$ be an α-inverse-strongly accretive mapping of order q and $B : C \to 2^E$ be an m-accretive operator. In the sequel, we will use the notation $T_\lambda := J^B_\lambda(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A) \ \ \forall \lambda > 0$.

Proposition 2.2 [23] The following statements hold:

(i) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0 \ \forall \lambda > 0$;

(ii) $\|y - T_\lambda y\| \le 2\|y - T_r y\|$ for $0 < \lambda \le r$ and $y \in C$.

Lemma 2.5 [23] Let $q \in (1,2]$ and E be q-uniformly smooth. Suppose that $A : C \to E$ is an α-inverse-strongly accretive mapping of order q. Then, for any given $\lambda \ge 0$,

$$\|(I - \lambda A)u - (I - \lambda A)v\|^q \le \|u - v\|^q - 2\lambda(\alpha q - \kappa_q\lambda^{q-1})\|Au - Av\|^q \quad \forall u, v \in C,$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 \le \lambda \le (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.6 [1] Let E be smooth, $A : C \to E$ be accretive and Π_C be a sunny nonexpansive retraction from

E onto C. Then $\mathrm{VI}(C,A) = \mathrm{Fix}(\Pi_C(I-\lambda A))\ \forall\lambda > 0$, where $\mathrm{VI}(C,A)$ is the solution set of the VIP of finding $z \in C$ s.t. $\langle Az, J(z-y)\rangle \le 0\ \forall z \in C$.

Lemma 2.7 [10] Let E be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of E, and $T : C \to C$ be an asymptotically nonexpansive mapping with a fixed point. Then $I - T$ is demiclosed at zero, i.e., if the sequence $\{x_n\} \subset C$ satisfies $x_n \rightharpoonup x \in C$ and $(I-T)x_n \to 0$, then $(I-T)x = 0$, where I is the identity mapping of E.

Lemma 2.8 [29] Let $\{a_n\}$ be a sequence in $[0,\infty)$ such that $a_{n+1} \le (1-s_n)a_n + s_n\nu_n\ \forall n \ge 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions: (i) $\{s_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} s_n = \infty$; (ii) $\limsup_{n\to\infty} \nu_n \le 0$ or $\sum_{n=0}^{\infty} |s_n\nu_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Throughout this section, we assume that E is a q-uniformly smooth and uniformly convex Banach space with $q \in (1,2]$, which admits a weakly continuous duality mapping. Let C be a nonempty convex closed set in E, $f : C \to C$ be a δ-contraction with constant $\delta \in [0,1)$ and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$. Let $A : C \to E$ and $B : C \to 2^E$ be an α-inverse-strongly accretive mapping of order q and an m-accretive operator, respectively. Assume that $\Omega := \mathrm{Fix}(T) \cap (A+B)^{-1}0 \ne \emptyset$.

Algorithm 3.1. General viscosity-type extragradient method for the VI and FPP.

Initial Step. Give $x_0 \in C$ arbitrarily.

Iteration Steps. Given the current iterate x_n, compute x_{n+1} as follows:

Step 1. Calculate $y_n = \sigma_n x_n + (1-\sigma_n)J^B_{\lambda_n}(I-\lambda_n A)x_n$;

Step 2. Calculate $z_n = J^B_{\lambda_n}(x_n - \lambda_n Ay_n + r_n(y_n - x_n))$;

Step 3. Calculate $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n z_n$, where $\{r_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\sigma_n\} \subset [0,1)$ and $\{\lambda_n\} \subset (0,\infty)$.

Set $n := n+1$ and go to Step 1.

Theorem 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose that the following conditions hold: (C1) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$; (C2) $0 < a \le \beta_n \le b < 1$ and $0 \le \sigma_n \le d < 1$; (C3) $0 < r \le r_n < 1$ and $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \mu < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$; (C4) $T^n x_n - T^{n+1}x_n \to 0$. Then $x_n \to x^* \in \Omega \Leftrightarrow x_n - x_{n+1} \to 0$, where $x^* \in \Omega$ is only a solution to the HVI: $\langle (I-f)x^*, J(x^*-p)\rangle \le 0\ \forall p \in \Omega$.*

Proof. First of all, we put $u_n := J^B_{\lambda_n}(I-\lambda_n A)x_n\ \forall n \ge 0$. Then $y_n = \sigma_n x_n + (1-\sigma_n)u_n\ \forall n \ge 0$. Since $\lim_{n\to\infty}\frac{\alpha_n}{\theta_n} = 0$, we may assume, without loss of generality, that $\theta_n \le \frac{(1-\delta)\alpha_n}{2}\ \forall n \ge 0$. Also, since $1 < q \le 2$, we get $(1+\theta_n)^q \le 1 + 2\theta_n + \theta_n^2\ \forall n \ge 0$. It is now easy to see that the necessity of the theorem is valid. We show only the sufficiency of the theorem. To the aim, we assume $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and divide the proof of the sufficiency into several steps.

Step 1. We claim that $\{x_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}$ and $\{T^n z_n\}$ are bounded. Indeed, take an element $p \in \Omega$ arbitrarily. Then $Tp = p$ and $p = J^B_{\lambda_n}(I-\lambda_n A)p = J^B_{\lambda_n}((1-r_n)p + r_n(p - \frac{\lambda_n}{r_n}Ap))$ (due to Proposition 2.2 (i)). Using Lemmas 2.4 (ii) and 2.5, we have

$$\|u_n - p\|^q \le \|x_n - p\|^q - \lambda_n(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Ap\|^q, \tag{3.1}$$

which hence leads to $\|u_n - p\| \le \|x_n - p\|$. This immediately implies that $\|y_n - p\| \le \sigma_n\|x_n - p\| + (1-\sigma_n)\|u_n - p\| \le \|x_n - p\|$. Using Lemmas 2.4 (ii) and 2.5 again, from (3.1) and the convexity of $\|\cdot\|^q$ with $q \in (1,2]$, we deduce that

$$\begin{aligned}\|z_n - p\|^q &\le \|((1-r_n)x_n + r_n(y_n - \tfrac{\lambda_n}{r_n}Ay_n)) - ((1-r_n)p + r_n(p - \tfrac{\lambda_n}{r_n}Ap))\|^q \\ &\le (1-r_n)\|x_n - p\|^q + r_n[\|y_n - p\|^q - \tfrac{\lambda_n}{r_n}(\alpha q - \tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ap\|^q] \\ &\le \|x_n - p\|^q - \lambda_n(\alpha q - \tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ap\|^q,\end{aligned} \tag{3.2}$$

which immediately yields $\|z_n - p\| \le \|x_n - p\|$. Thus, we get

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|T^n z_n - p\| \\ &\le [\alpha_n\delta + \beta_n + \gamma_n + \theta_n]\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\le [1 - \tfrac{\alpha_n(1-\delta)}{2}]\|x_n - p\| + \tfrac{\alpha_n(1-\delta)}{2} \cdot \tfrac{2\|f(p)-p\|}{1-\delta} \\ &\le \max\{\|x_n - p\|, \tfrac{2\|f(p)-p\|}{1-\delta}\}.\end{aligned}$$

This implies that $\|x_n - p\| \le \{\|x_0 - p\|, \frac{2\|(I-f)p\|}{1-\delta}\}\ \forall n \ge 0$. Therefore, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}$ and $\{T^n z_n\}$.

Step 2. We claim that $\exists M_0 > 0$ s.t. $\Gamma_{n+1} \le [1 - \frac{\alpha_n(1-\delta)}{2}]\Gamma_n + \delta_n + \theta_n(1+\theta_n)M_0 \quad \forall n \ge 0$, where $\Gamma_n = \|x_n - x^*\|^q$, $\delta_n = q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle$ and $x^* = \Pi_\Omega f(x^*)$ with Π_Ω being the sunny nonexpansive retraction of E onto Ω. Indeed, using Lemmas 2.1 and 2.3, from (3.1) and (3.2) we obtain that

$$\begin{aligned}\|z_n - x^*\|^q &\le (1 - r_n)\|x_n - x^*\|^q + r_n[\sigma_n\|x_n - x^*\|^q + (1 - \sigma_n)\|u_n - x^*\|^q \\ &\quad - \tfrac{\lambda_n}{r_n}(\alpha q - \tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ax^*\|^q] \\ &\le \|x_n - x^*\|^q - r_n(1 - \sigma_n)\lambda_n(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Ax^*\|^q \\ &\quad - \lambda_n(\alpha q - \tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ax^*\|^q,\end{aligned}$$

and hence

$$\begin{aligned}&\|x_{n+1} - x^*\|^q \\ &\le \|\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(T^n z_n - x^*)\|^q + q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle \\ &\le \alpha_n\delta\|x_n - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n(1 + \theta_n)^q\|z_n - x^*\|^q - \beta_n\gamma_n g_1(\|x_n - T^n z_n\|) \\ &\quad + q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle \\ &\le [\alpha_n\delta + \beta_n + \gamma_n(1 + \theta_n)^q]\|x_n - x^*\|^q - \gamma_n(1 + \theta_n)^q\{r_n(1 - \sigma_n)\lambda_n(\alpha q - \kappa_q\lambda_n^{q-1}) \\ &\quad \times \|Ax_n - Ax^*\|^q + \lambda_n(\alpha q - \tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ax^*\|^q\} \\ &\quad - \beta_n\gamma_n g_1(\|x_n - T^n z_n\|) + q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle \\ &\le [1 - \tfrac{\alpha_n(1-\delta)}{2}]\|x_n - x^*\|^q + \theta_n(1 + \theta_n)M_0 - \gamma_n(1 + \theta_n)^q\{r_n(1 - \sigma_n)\lambda_n(\alpha q - \kappa_q\lambda_n^{q-1}) \\ &\quad \times \|Ax_n - Ax^*\|^q + \lambda_n(\alpha q - \tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ax^*\|^q\} \\ &\quad - \beta_n\gamma_n g_1(\|x_n - T^n z_n\|) + q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle,\end{aligned} \tag{3.3}$$

where $\sup_{n\ge0}\|x_n - x^*\|^q \le M_0$ for some $M_0 > 0$. For each $n \ge 0$, we put

$$\begin{aligned}&\Gamma_n = \|x_n - x^*\|^2, \epsilon_n = \tfrac{\alpha_n(1-\delta)}{2}, \\ &\delta_n = q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle, \\ &\eta_n = \gamma_n(1 + \theta_n)^q\{r_n(1 - \sigma_n)\lambda_n(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Ax^*\|^q \\ &\qquad + \lambda_n(\alpha q - \tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ax^*\|^q\} + \beta_n\gamma_n g_1(\|x_n - T^n z_n\|).\end{aligned}$$

So it follows from (3.3) that

$$\Gamma_{n+1} \le (1 - \epsilon_n)\Gamma_n - \eta_n + \delta_n + \theta_n(1 + \theta_n)M_0 \quad \forall n \ge 0, \tag{3.4}$$

which hence attains

$$\Gamma_{n+1} \le (1 - \epsilon_n)\Gamma_n + \delta_n + \theta_n(1 + \theta_n)M_0 \quad \forall n \ge 0. \tag{3.5}$$

Step 3. We claim that $x_n - u_n \to 0,\ y_n - x_n \to 0$ and $z_n - x_n \to 0$, where $u_n := J^B_{\lambda_n}(I - \lambda_n A)x_n$. Indeed, using Proposition 2.1 and the assumption $x_n - x_{n+1} \to 0$, we obtain $\Gamma_n - \Gamma_{n+1} \le q\|x_n - x_{n+1}\|\|x_{n+1} - x^*\|^{q-1} + \kappa_q\|x_n - x_{n+1}\|^q \to 0, n \to \infty$. Thus, $\limsup_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) \le 0$. From (3.4) we get $0 \le \eta_n \le \Gamma_n - \Gamma_{n+1} + \delta_n + \theta_n(1 + \theta_n)M_0$.

Since $\theta_n \to 0,\ \delta_n \to 0$ and $\limsup_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) \le 0$, we have $\eta_n \to 0$ $(n \to \infty)$. This immediately implies that

$$\begin{aligned}
&(1-\alpha_n-b)(1+\theta_n)^q\{r(1-d)\lambda(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Ax^*\|^q\\
&\quad+\lambda(\alpha q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q\}+a(1-\alpha_n-b)g_1(\|x_n-T^nz_n\|)\\
&\le\gamma_n(1+\theta_n)^q\{r_n(1-\sigma_n)\lambda_n(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Ax^*\|^q\\
&\quad+\lambda_n(\alpha q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q\}+\beta_n\gamma_n g_1(\|x_n-T^nz_n\|)=:\eta_n\to0\quad(n\to\infty).
\end{aligned}$$

Note that g_1 is a strictly increasing, continuous and convex function with $g_1(0) = 0$. So it follows that

$$\lim_{n\to\infty}\|Ax_n-Ax^*\|=0,\quad \lim_{n\to\infty}\|Ay_n-Ax^*\|=0\quad\text{and}\quad \lim_{n\to\infty}\|x_n-T^nz_n\|=0. \tag{3.6}$$

Since $J^B_{\lambda_n}$ is firmly nonexpansive, by Lemmas 2.2 (b) and 2.5 we get

$$\begin{aligned}
\|y_n-x^*\|^q &\le \sigma_n\|x_n-x^*\|^q+(1-\sigma_n)\|J^B_{\lambda_n}(x_n-\lambda_nAx_n)-J^B_{\lambda_n}(x^*-\lambda_nAx^*)\|^q\\
&\le \sigma_n\|x_n-x^*\|^q+(1-\sigma_n)\tfrac{1}{q}[\|(x_n-\lambda_nAx_n)-(x^*-\lambda_nAx^*)\|^q\\
&\quad+(q-1)\|u_n-x^*\|^q-h_1(\|x_n-\lambda_n(Ax_n-Ax^*)-u_n\|)]\\
&\le \|x_n-x^*\|^q-\tfrac{1-\sigma_n}{q}h_1(\|x_n-\lambda_n(Ax_n-Ax^*)-u_n\|),
\end{aligned}$$

which together with (3.2), implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^q &\le \alpha_n\|f(x_n)-x^*\|^q+\beta_n\|x_n-x^*\|^q+\gamma_n(1+\theta_n)^q\|z_n-x^*\|^q\\
&\le \alpha_n\|f(x_n)-x^*\|^q+\beta_n\|x_n-x^*\|^q+\gamma_n[(1-r_n)\|x_n-x^*\|^q+r_n\|y_n-x^*\|^q]\\
&\quad+\theta_n(2+\theta_n)\|z_n-x^*\|^q\\
&\le \alpha_n\|f(x_n)-x^*\|^q+(\beta_n+\gamma_n)\|x_n-x^*\|^q\\
&\quad-\tfrac{\gamma_nr_n(1-\sigma_n)}{q}h_1(\|x_n-\lambda_n(Ax_n-Ax^*)-u_n\|)+\theta_n(2+\theta_n)\|x_n-x^*\|^q\\
&\le \alpha_n\|f(x_n)-x^*\|^q+\|x_n-x^*\|^q-\tfrac{\gamma_nr_n(1-\sigma_n)}{q}h_1(\|x_n-\lambda_n(Ax_n-Ax^*)-u_n\|)\\
&\quad+\theta_n(2+\theta_n)M_0.
\end{aligned}$$

So it follows that

$$\begin{aligned}
&\tfrac{(1-\alpha_n-b)r(1-d)}{q}h_1(\|x_n-\lambda_n(Ax_n-Ax^*)-u_n\|)\\
&\le \Gamma_n-\Gamma_{n+1}+\alpha_n\|f(x_n)-x^*\|^q+\theta_n(2+\theta_n)M_0.
\end{aligned}$$

Since h_1 is a strictly increasing, continuous and convex function with $h_1(0) = 0$, from (3.6) we get $\lim_{n\to\infty}\|x_n-u_n\| = 0$. Noticing $y_n = \sigma_nx_n + (1-\sigma_n)u_n$, we have

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}(1-\sigma_n)\|u_n-x_n\|=0. \tag{3.7}$$

Using Lemmas 2.2 (b) and 2.5 again, we have

$$\begin{aligned}
\|z_n-x^*\|^q &\le \langle(x_n-\lambda_nAy_n+r_n(y_n-x_n))-(x^*-\lambda_nAx^*),J_q(z_n-x^*)\rangle\\
&\le \tfrac{1}{q}[\|(x_n-\lambda_nAy_n+r_n(y_n-x_n))-(x^*-\lambda_nAx^*)\|^q+(q-1)\|z_n-x^*\|^q\\
&\quad-h(\|x_n+r_n(y_n-x_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)],
\end{aligned}$$

which hence yields $\|z_n-x^*\|^q \le \|x_n-x^*\|^q - h(\|x_n+r_n(y_n-x_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)$. So it follows that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^q\\
&\le \alpha_n\|f(x_n)-x^*\|^q+\beta_n\|x_n-x^*\|^q+\gamma_n(1+\theta_n)^q\|z_n-x^*\|^q\\
&\le \alpha_n\|f(x_n)-x^*\|^q+\beta_n\|x_n-x^*\|^q+\gamma_n[\|x_n-x^*\|^q\\
&\quad-h(\|x_n+r_n(y_n-x_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)]+\theta_n(2+\theta_n)\|x_n-x^*\|^q\\
&\le \alpha_n\|f(x_n)-x^*\|^q+\|x_n-x^*\|^q-\gamma_nh(\|x_n+r_n(y_n-x_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)\\
&\quad+\theta_n(2+\theta_n)M_0,
\end{aligned}$$

which immediately attains

$$\begin{aligned}&(1-\alpha_n-b)h(\|x_n+r_n(y_n-x_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)\\ &\le \Gamma_n-\Gamma_{n+1}+\alpha_n\|f(x_n)-x^*\|^q+\theta_n(2+\theta_n)M_0.\end{aligned}$$

Since h is a strictly increasing, continuous and convex function with $h(0)=0$, from (3.6) and (3.7) we have

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.8}$$

Step 4. We claim that $x_n-Tx_n\to 0$ and $x_n-T_\lambda x_n\to 0$ where $T_\lambda := J_\lambda^B(I-\lambda A)$. Indeed, since $T^n x_n - T^{n+1}x_n\to 0$, we obtain from (3.8) and the uniform continuity of T that

$$\begin{aligned}\|T^n z_n-T^{n+1}z_n\| &\le \|T^n z_n-T^n x_n\|+\|T^n x_n-T^{n+1}x_n\|+\|T^{n+1}x_n-T^{n+1}z_n\|\\ &\le (1+\theta_n)(\|z_n-x_n\|+\|Tx_n-Tz_n\|)+\|T^n x_n-T^{n+1}x_n\|\to 0 \quad (n\to\infty).\end{aligned}$$

That is, $\lim_{n\to\infty}\|T^n z_n-T^{n+1}z_n\|=0$. We now observe that

$$\|z_n-T^n z_n\|\le\|z_n-x_n\|+\|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-T^n z_n\|+\beta_n\|x_n-T^n z_n\|.$$

So, from (3.6), (3.8), $\alpha_n\to 0$ and $x_n-x_{n+1}\to 0$, it follows that

$$\lim_{n\to\infty}\|z_n-T^n z_n\|=0. \tag{3.9}$$

Also, note that $\|z_n-Tz_n\|\le\|z_n-T^n z_n\|+\|T^n z_n-T^{n+1}z_n\|+\|T^{n+1}z_n-Tz_n\|$. From (3.9), $T^n z_n-T^{n+1}z_n\to 0$ and the uniform continuity of T, we have

$$\lim_{n\to\infty}\|z_n-Tz_n\|=0. \tag{3.10}$$

Meantime, noticing that $\|x_n-Tx_n\|\le\|x_n-z_n\|+\|z_n-Tz_n\|+\|Tz_n-Tx_n\|$, we deduce from (3.8), (3.10) and the uniform continuity of T that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.11}$$

In addition, for each $n\ge 0$, we put $T_{\lambda_n} := J_{\lambda_n}^B(I-\lambda_n A)$. Then from $x_n-u_n\to 0$, we have

$$\lim_{n\to\infty}\|x_n-T_{\lambda_n}x_n\|=0. \tag{3.12}$$

Since $0<\lambda\le\lambda_n\ \forall n\ge 0$, by Proposition 2.2 (ii), we have

$$\|x_n-T_\lambda x_n\|\le 2\|x_n-T_{\lambda_n}x_n\|\to 0 \quad (n\to\infty).$$

That is,

$$\lim_{n\to\infty}\|x_n-T_\lambda x_n\|=0. \tag{3.13}$$

Step 5. We claim that $x_n\to x^*$ where $x^*=\Pi_\Omega f(x^*)$. Indeed, we first show that

$$\limsup_{n\to\infty}\langle (f-I)x^*, J(x_{n+1}-x^*)\rangle\le 0,$$

where $x^*=\Pi_\Omega f(x^*)$. As a matter of fact, it is known that $\exists\{x_{n_i}\}\subset\{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle (f-I)x^*, J(x_n-x^*)\rangle=\lim_{k\to\infty}\langle (f-I)x^*, J(x_{n_k}-x^*)\rangle. \tag{3.14}$$

Taking into account the boundedness of $\{x_n\}\subset C$, we might suppose that $x_{n_k}\rightharpoonup\tilde{x}\in C$. Note that T_λ is nonexpansive and that T is asymptotically nonexpansive. Since $(I-T_\lambda)x_n\to 0$ (due to (3.13)), using Lemma 2.7 we conclude that $\tilde{x}\in\mathrm{Fix}(T_\lambda)=(A+B)^{-1}0$. Also, using (3.11) we obtain $x_{n_k}-Tx_{n_k}\to 0$ for $\{x_{n_k}\}\subset\{x_n\}$. So, by Lemma 2.7 we get $\tilde{x}\in\mathrm{Fix}(T)$. Consequently, $\tilde{x}\in\Omega=\mathrm{Fix}(T)\cap(A+B)^{-1}0$. Note that E admits the weakly sequentially continuous duality mapping $J(\cdot)$. Taking into account the norm-to-norm uniform continuity of $J(\cdot)$ on bounded subsets of E, we deduce from (3.14), $x_n - x_{n+1} \to 0$ and $x_{n_k} \rightharpoonup \tilde{x}$ that

$$\begin{aligned}&\limsup_{n\to\infty}\langle (f-I)x^*, J(x_{n+1}-x^*)\rangle = \limsup_{n\to\infty}\langle (f-I)x^*, J(x_n - x^*)\rangle \\ &= \lim_{k\to\infty}\langle (f-I)x^*, J(x_{n_k}-x^*)\rangle = \langle (f-I)x^*, J(\tilde{x}-x^*)\rangle \le 0.\end{aligned} \tag{3.15}$$

Finally, we show that $x_n \to x^*$. In fact, from Algorithm 3.1 we get

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \\ &\le \alpha_n\|f(x_n)-f(x^*)\|^2 + \beta_n\|x_n-x^*\|^2 + \gamma_n\|T^n z_n - x^*\|^2 + 2\alpha_n\langle (f-I)x^*, J(x_{n+1}-x^*)\rangle \\ &\le (\alpha_n\delta+\beta_n+\gamma_n)\|x_n-x^*\|^2 + \theta_n(2+\theta_n)\|x_n-x^*\|^2 + 2\alpha_n\langle (f-I)x^*, J(x_{n+1}-x^*)\rangle \\ &\le [1-\alpha_n(1-\delta)]\|x_n-x^*\|^2 + \alpha_n(1-\delta)[\tfrac{\theta_n}{\alpha_n}\cdot\tfrac{(2+\theta_n)M_0^{\frac{2}{q}}}{1-\delta} + \tfrac{2\langle (f-I)x^*, J(x_{n+1}-x^*)\rangle}{1-\delta}].\end{aligned} \tag{3.16}$$

Note that $\{\alpha_n(1-\delta)\} \subset [0,1]$, $\sum_{n=0}^{\infty}\alpha_n(1-\delta) = \infty$ and

$$\limsup_{n\to\infty}[\frac{\theta_n}{\alpha_n}\cdot\frac{(2+\theta_n)M_0^{\frac{2}{q}}}{1-\delta} + \frac{2\langle (f-I)x^*, J(x_{n+1}-x^*)\rangle}{1-\delta}] \le 0$$

(due to (3.15)). Therefore, applying Lemma 2.8 to (3.16), we obtain $x_n \to x^*$. This complete the proof. □

Remark 3.2. *Compared with the corresponding results in Sunthrayuth and Cholamjiak [27], and Takahashi et al. [28], our results improve and extend them in the following aspects. The problem of solving the VI for two accretive operators A, B with the FPP constraint of a nonexpansive mapping S in [27, Theorem 3.3] is extended to develop our problem of solving the VI for two accretive operators A, B with the FPP constraint of an asymptotically nonexpansive mapping T. The modified viscosity-type extragradient method in [27, Theorem 3.3] is extended to develop our general viscosity-type extragradient method. The problem of solving the VI for two monotone operators A, B with the FPP constraint of a nonexpansive mapping S in [28, Theorem 3.1] is extended to develop our problem of solving the VI for two accretive operators A, B with the FPP constraint of an asymptotically nonexpansive mapping T. The Mann-type Halpern iterative scheme in [28, Theorem 3.1] are extended to develop our general viscosity-type extragradient method.*

Next we shall utilize the above general viscosity-type extragradient method for solving the fixed point problem of an asymptotically nonexpansive mapping and the variational inequality problem in the framework of Hilbert spaces. Let C be a nonempty convex closed set in a real Hilbert space H, and P_C be the metric projection from H onto C. Let $A: C \to H$ be a nonlinear monotone operator. The variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \ge 0 \quad \forall x \in C.$$

The set of solutions of VIP is denoted by $\mathrm{VI}(C, A)$. Let i_C be an indicator function of C given by

$$i_C = \begin{cases} 0, & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

Denote N_C the normal cone of C, that is, $N_C(u) = \{w \in H : \langle w, v-u\rangle \le 0\ \forall v \in C\}$. It is also known that i_C is proper convex lower semicontinuous function and the subdifferential ∂i_C is maximal monotone operator. The resolvent operator $J_\lambda^{\partial i_C}$ of i_C for $\lambda > 0$ is defined as $J_\lambda^{\partial i_C}(x) = (I + \lambda\partial i_C)^{-1}(x)\ \forall x \in H$, where

$$\begin{aligned}\partial i_C(u) &= \{w \in H : i_C(u) + \langle w, v-u\rangle \le i_C(v)\ \forall v \in C\} \\ &= \{w \in H : \langle w, v-u\rangle \le 0\ \forall v \in C\} = N_C(u) \quad \forall u \in C.\end{aligned}$$

Hence, we get

$$u = J_\lambda^{\partial i_C}(x) \Leftrightarrow x - u \in \lambda N_C(u) \Leftrightarrow \langle x-u, v-u\rangle \le 0\ \forall v \in C \Leftrightarrow u = P_C(x).$$

So, it is easy to see that $(A + \partial i_C)^{-1}0 = \mathrm{VI}(C, A)$. Therefore, putting $B = \partial i_C$ in Theorem 3.1, we obtain the following consequence.

Corollary 3.3. *Let $f : C \to C$ be a δ-contraction with constant $\delta \in [0,1)$. Assume that $T : C \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$ and $A : C \to H$ is an α-inverse-strongly monotone mapping such that $\Omega := \mathrm{Fix}(T) \cap \mathrm{VI}(C,A) \neq \emptyset$. For give $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \sigma_n x_n + (1-\sigma_n)P_C(I-\lambda_n A)x_n, \\ z_n = P_C(x_n - \lambda_n A y_n + r_n(y_n - x_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n z_n \quad \forall n \geq 0, \end{cases}$$

where $\{r_n\},\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\sigma_n\} \subset [0,1)$ and $\{\lambda_n\} \subset (0,\infty)$. Suppose that the following conditions hold: (C1) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$; (C2) $0 < a \leq \beta_n \leq b < 1$ and $0 \leq \sigma_n \leq d < 1$; (C3) $0 < r \leq r_n < 1$ and $0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq \mu < 2\alpha$; (C4) $T^n x_n - T^{n+1}x_n \to 0$. Then $x_n \to x^* \in \Omega \Leftrightarrow x_n - x_{n+1} \to 0$, where $x^* \in \Omega$ is only a solution to the HVI: $\langle (I-f)x^*, x^* - p\rangle \leq 0\ \forall p \in \Omega$.

Finally, the above corollary is applied to solve the VIP and FPP in an illustrating example. Let $\lambda_n = \frac{1}{9}$, $\sigma_n = r_n = \beta_n = \frac{1}{2}$, $\alpha_n = \frac{1}{2(n+1)}$ and $\gamma_n = \frac{2n+1}{2(n+1)} - \frac{1}{2}$ for all $n \geq 0$. We first provide an example of α-inverse-strongly monotone mapping $A : C \to H$ and asymptotically nonexpansive mapping $T : C \to C$ with $\Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C,A) \neq \emptyset$. Let $C = [-1,1]$ and $H = \mathbf{R}$ with the inner product $\langle a,b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. Let $f : C \to C$, $A : C \to H$ and $T : C \to C$ be defined as $f(x) = \frac{1}{2}x$, $Ax = x - \frac{1}{2}\sin x$ and $Tx = \frac{2}{3}\sin x$ for all $x \in C$. Then f is a δ-contraction with constant $\delta = \frac{1}{2}$. Moreover, A is α-inverse-strongly monotone with $\alpha = \frac{2}{9}$ since for all $x, y \in C$, we deduce that $\|Ax - Ay\| \leq \frac{3}{2}\|x-y\|$ and

$$\langle Ax - Ay, x - y\rangle = \|x-y\|^2 - \frac{1}{2}\langle \sin x - \sin y, x - y\rangle \geq \frac{1}{2}\|x-y\|^2.$$

Meantime, it is easy to see that T is asymptotically nonexpansive with $\theta_n = (\frac{2}{3})^n\ \forall n \geq 1$ such that $\|T^{n+1}x_n - T^n x_n\| \to 0$ as $n \to \infty$. In fact, observe that

$$\|T^n x - T^n y\| \leq \frac{2}{3}\|T^{n-1}x - T^{n-1}y\| \leq \cdots \leq (\frac{2}{3})^n\|x-y\| \leq (1+\theta_n)\|x-y\|,$$

and hence

$$\|T^{n+1}x_n - T^n x_n\| \leq (\frac{2}{3})^{n-1}\|T^2 x_n - Tx_n\| = (\frac{2}{3})^{n-1}\|\frac{2}{3}\sin(Tx_n) - \frac{2}{3}\sin x_n\| \leq 2(\frac{2}{3})^n \to 0\ (n \to \infty).$$

It is clear that $\mathrm{Fix}(T) = \{0\}$ and $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = \lim_{n\to\infty} \frac{(2/3)^n}{1/2(n+1)} = 0$. Therefore, $\Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C,A) = \{0\} \neq \emptyset$. Take a given $x_0 \in C$ arbitrarily. In this case, the iterative scheme in Corollary 3.3 can be rewritten as:

$$\begin{cases} y_n = \frac{1}{2}x_n + \frac{1}{2}P_C(x_n - \frac{1}{9}Ax_n), \\ z_n = P_C(x_n - \frac{1}{9}Ay_n + \frac{1}{2}(y_n - x_n)), \\ x_{n+1} = \frac{1}{2(n+1)} \cdot \frac{1}{2}x_n + \frac{1}{2}x_n + (\frac{2n+1}{2(n+1)} - \frac{1}{2})T^n z_n \quad \forall n \geq 0. \end{cases}$$

Then, by Corollary 3.3, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C,A)$ if and only if $x_n - x_{n+1} \to 0$ as $n \to \infty$.

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