Filomat 35:3 (2021), 1001–1013
https://doi.org/10.2298/FIL2103001B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Construction of a New Class of Generating Functions of Binary Products of Some Special Numbers and Polynomials

**Souhila Boughaba[a], Ali Boussayoud[a], Serkan Araci[b], Mohamed Kerada[a], Mehmet Acikgoz[c]**

[a]*LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria*
[b]*Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey*
[c] *Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, 27310 Gaziantep,Turkey*

**Abstract.** In this paper, we derive some new symmetric properties of $k$-Fibonacci numbers by making use of symmetrizing operator. We also give some new generating functions for the products of some special numbers such as $k$-Fibonacci numbers, $k$-Pell numbers, Jacobsthal numbers, Fibonacci polynomials and Chebyshev polynomials.

## 1. Introduction

Srivastava *et al.* [14] introduced and studied a new family of the generalized Hermite polynomials, and considered the polynomials $\{H_n^m(\lambda)\}$ and $\{H_{r,n}^m(\lambda)\}$ to find an explicit formula in terms of the Srivastava–Daoust multivariable hypergeometric functions. He *et al.* [15] presented a further investigation for the classical Frobenius–Euler polynomials. They also obtained some summation formulas for the products of an arbitrary number of the classical Frobenius-Euler polynomials by using the generating function methods and summation transform techniques. In [21], Kumam *et al.* introduced a new family of polynomials, which are called the truncated-exponential based Frobenius-Euler polynomials, based upon an exponential generating function. By making use of this exponential generating function, they obtained their several new properties and explicit summation formulas.

Srivastava *et al.* [17] defined the first and second homogeneous $q$-difference operators and they showed that the generalized Cauchy polynomials can be represented by the first homogeneous $q$-difference operator and derived their generating function.

Srivastava *et al.* [16] introduced a family of the twice-iterated $\Delta_h$-Appell sequences of polynomials based upon the discrete Appell convolution of the $\Delta_h$-Appell sequence of polynomials. They also obtained the corresponding properties for the sequences of the twice-iterated polynomials. In [19], Srivastava *et al.* introduced new families of the $q$-Fibonacci and $q$-Lucas polynomials, and gave several properties and generating functions of each of these families $q$-polynomials.

Let $F_n$, $T_n$ and $U_n$ be the $n$-th Fibonacci number, Chebyshev polynomials of the first and second kinds, respectively. In [8], Boussayoud *et al.* [7] derived new generating functions of square of Fibonacci numbers with products of Chebyshev polynomials of first and second kinds.

2010 *Mathematics Subject Classification*. Primary 05E05; Secondary 11B39
*Keywords*. Symmetric functions; Generating functions; $k$-Fibonacci numbers; $k$-Pell numbers; Jacobsthal numbers.
Received: 19 March 2020; Accepted: 13 January 2021
Communicated by Hari M. Srivastava
*Email addresses:* souhilaboughaba@gmail.com (Souhila Boughaba), aboussayoud@yahoo.fr (Ali Boussayoud), mtsrkn@hotmail.com (Serkan Araci), mkerada@yahoo.fr (Mohamed Kerada), acikgoz@gantep.edu.tr (Mehmet Acikgoz)

In [2], Boussayoud *et al.* considered the following generating series:

$$\sum_{n=0}^{+\infty} F_{k,n}F_{k,n+1}z^n, \sum_{n=0}^{+\infty} F_{k,n}F_{k,n+2}z^n, \sum_{n=0}^{+\infty} P_{k,n}P_{k,n+1}z^n, \sum_{n=0}^{+\infty} P_{k,n}P_{k,n+2}z^n,$$
$$\sum_{n=0}^{+\infty} F_{k,n+1}F_n(x)z^n, \sum_{n=0}^{+\infty} F_{k,n+2}F_n(x)z^n, \sum_{n=0}^{+\infty} P_{k,n+1}F_n(x)z^n, \sum_{n=0}^{+\infty} P_{k,n+2}F_n(x)z^n,$$
$$\sum_{n=0}^{+\infty} F_{k,n+1}U_n(x)z^n, \sum_{n=0}^{+\infty} F_{k,n+2}U_n(x)z^n.$$

A systematic study of orthogonal polynomials, which consists of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product, plays an important role in mathematics. In the literature, the most widely used orthogonal polynomials are the classical orthogonal polynomials, for example, Fibonacci polynomials, Chebyshev polynomials of first and second kinds.

Further in [9], the generating functions of the incomplete Fibonacci and Lucas numbers are determined. In [12], Djordjević gave the incomplete generalized Fibonacci and Lucas numbers. In [13], Djordjević and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers. In [10], the authors gave the incomplete Fibonacci and Lucas numbers. For more information about the applications of generating functions, see [20].

On the other hand, many kinds of generalizations of Fibonacci numbers have been presented in the literature. In particular, one of the well-known generalizations of these numbers is the $k$-Fibonacci numbers given as

$$\begin{cases} F_{k,0} = 1, F_{k,1} = k \\ F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \ (n \geq 1; k \in \mathbb{R}) \end{cases}.$$

The characteristic equation for $k$-Fibonacci numbers is $x^2-kx-1 = 0$ with roots $x_1 = \frac{k+\sqrt{k^2+4}}{2}$, and $x_2 = \frac{k-\sqrt{k^2+4}}{2}$, and $k$-Fibonacci numbers satisfy the following identity:

$$F_{k,n} = \frac{1}{\sqrt{k^2+4}}\left[\left(\frac{k+\sqrt{k^2+4}}{2}\right)^{n+1} - \left(\frac{k-\sqrt{k^2+4}}{2}\right)^{n+1}\right].$$

For any positive real number $k$, the $k$-Pell sequence $(P_{k,n})_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} P_{k,0} = 0, P_{k,1} = 1 \\ P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \ n \geq 1 \end{cases}.$$

The Binet formulas [27] for $k$-Pell sequence and $k$-Pell-Lucas sequence are given by

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ are the roots of characteristic equation of the sequence $(P_{k,n})_{n\in\mathbb{N}}$.

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients. Let $\mathbb{P}'$ be the algebraic linear dual of $\mathbb{P}$. We write $\langle u, p\rangle := u(p)$ $(u \in \mathbb{P}', p \in \mathbb{P})$. A linear functional $u \in \mathbb{P}'$ is said to be regular [31, 34, 35] if it is quasi-definite, i.e., $\det\langle u, x^{i+j}\rangle_{i,j=1,\ldots,n} \neq 0$ for $n \geq 0$. This is equivalent to the existence of a unique sequence of monic polynomials $\{p_n\}_{n\geq 0}$ of degree $n$ such that $\langle u, p_n p_m\rangle = r_n\delta_{n,m},\ n,\ m \geq 0$, with $r_n \neq 0$ $(n \geq 0)$. Then the sequence $\{p_n\}_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials with respect to $u$.

**Proposition 1.1.** *(Favard's Theorem [31]). Let $\{P_n\}_{n\geq 0}$ be a monic polynomial sequence. Then $\{P_n\}_{n\geq 0}$ is orthogonal if and only if there exist two sequences of complex number $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$, such that $\gamma_n \neq 0,\ n \geq 1$ and satisfies the three-term recurrence relation*

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0. \end{cases}$$

The orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ such as Hermite, Laguerre, Bessel or Jacobi polynomials is called classical, if $\{P_n^{[1]}\}_{n\geq 0}$ is also orthogonal, [31, 36, 37]. A second characterization of these polynomials is that they satisfy the solution of the second-order differential equation (Bochner [30])

$$\phi(x)P''_{n+1}(x) - \psi(x)P'_{n+1}(x) = \mu_n P_{n+1}(x),\ n \geq 0,$$

where $\phi, \psi$ are polynomials, $\phi$ is a monic polynomial, $\deg\phi = t \leq 2$, $\deg\psi = 1$ and $\mu_n = (n+1)\Big(\frac{1}{2}\phi''(0)n - \psi'(0)\Big) \neq 0,\ n \geq 0.$

Next, we recall some properties of the classical orthogonal Chebyshev polynomials that we will need in the sequel. The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds are respectively defined by the following formulas:

$$T_n(\cos\theta) = \cos(n\theta),$$
$$U_n(\cos\theta) = \frac{\sin[(n+1)\theta]}{\sin\theta},$$

where $\theta \in [0, \pi]$.

Let $(\alpha)_n$ be a Pochhammer symbol in the ascending factorial of $\alpha$ defined by

$$(\alpha)_n = \prod_{k=0}^{n-1}(\alpha + k).$$

**Definition 1.2.** *[33]The generalized hypergeometric functions ${}_pF_q(.)$ are defined by*

$$\begin{aligned} {}_pF_q[\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;x] &= \sum_{n=0}^{+\infty}\frac{(\alpha_1)_n...(\alpha_p)_n}{(\beta_1)_n...(\beta_q)_n}\frac{x^n}{n!} \\ &= {}_pF_q[\alpha_1,...,\alpha_p;\ \beta_1,...,\beta_q;\ x]. \end{aligned} \tag{1.1}$$

*where $\alpha_1,...,\alpha_p,\beta_1,...,\beta_q, x \in \mathbb{C}$ , $\beta_1,...,\beta_q$ are neither zero nor negative integers.*

In the special case when $p = 2$ and $q = 1$ in (1.1), it yields

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{+\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},$$

which is well-known as Gauss hypergeometric function.

In this part, we give Fibonacci differential equation and the hypergeometric form of the Fibonacci polynomials, Chebyshev polynomials of the first and second kinds.

**Theorem 1.3.** *[11] The Fibonacci polynomials $F_n(x)$ satisfy the diffrential equation*

$$\left(x^2 + 4\right)y^{''} + 3xy^{'} - (n^2 - 1)y = 0.$$

**Theorem 1.4.** *[11] The Fibonacci polynomials $F_n(x)$ can be written by the hypergeometric function as follows:*

$$F_n(x) = {}_2F_1\left(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4}\right).$$

**Proposition 1.5.** *[32]The hypergeometric form of the Chebyshev polynomials of the first kind, can be written as follows:*

$$T_n(x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right).$$

**Proposition 1.6.** *[32]The hypergeometric form of the Chebyshev polynomials of the second kind, can be written as follows:*

$$U_n(x) = (n+1)\ {}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right).$$

In this paper, we make use of symmetrizing operator, denoted by $\delta_{a_1a_2}^{h+1}$, to formulate, extend and prove new results including the generating functions for generalized of the product of $k$-Fibonacci and $k$-Pell numbers and Chebyshev polynomials of the first and second kinds and Fibonacci polynomials.

In Section 2, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions which are used in the subsequent sections. In Section 3, we prove our main result which relates the symmetric function defined in the previous section with the symmetrizing operator. This main theorem unifies several previously known results about the generating functions. It is then used to find the product of $k$-Fibonacci numbers identities and the generating functions for the product of $k$-Fibonacci numbers and $k$-Pell numbers, in Section 4.

## 2. Definitions, Notations and Preliminaries

In this section, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet $A$ is a function of the letters which is invariant under permutation of the letters of $A$. Taking an extra indeterminate $z$, one has two fundamental series [2]:

$$\lambda_z(A) = \Pi_{a\in A}(1+az),\ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1-az)},$$

the expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$ as follows:

$$\lambda_z(A) = \sum_{n=0}^{+\infty} \Lambda_n(A)z^n,\ \sigma_z(A) = \sum_{n=0}^{+\infty} S_n(A)z^n.$$

Let us now start at the following definition.

**Definition 2.1.** *[1]Let $A$ and $B$ be any two alphabets, then we give $S_n(A-B)$ by the following form:*

$$\frac{\Pi_{b\in B}(1-bz)}{\Pi_{a\in A}(1-az)} = \sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A-B), \tag{2.1}$$

*with the condition $S_n(A-B) = 0$ for $n < 0$.*

**Remark 2.2.** *Taking $A = 0$ in* (2.1) *gives*

$$\Pi_{b\in B}(1-bz) = \sum_{n=0}^{+\infty} S_n(-B)z^n = \lambda_z(-B). \tag{2.2}$$

*Further, in the case $A = 0$ or $B = 0$, we have*

$$\sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A) \times \lambda_z(-B). \tag{2.3}$$

*Thus,*

$$S_n(A-B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B) \text{ (see [1])}. \tag{2.4}$$

**Definition 2.3.** *Let $g$ be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - g(x_1, \cdots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n)}{x_i - x_{i+1}}, \text{ (see [22])}.$$

**Definition 2.4.** *[8] Given an alphabet $A = \{a_1, a_2\}$, the symmetrizing operator $\delta^{h+1}_{a_1 a_2}$ is defined by*

$$\delta^{h+1}_{a_1 a_2} f(a_1) = \frac{a_1^{h+1} f(a_1) - a_2^{h+1} f(a_2)}{a_1 - a_2}, \text{ for all } h \in \mathbb{N}.$$

## 3. Main Results

In this section, we prove the main theorem of the paper which combines all the previously known results in a unified way such that they can be treated as special cases.

**Theorem 3.1.** *Let $A$ and $E$ be two alphabets, respectively, $\{a_1, a_2\}$ and $\{e_1, e_2\}$, then we have for*

$$\sum_{n=0}^{+\infty} S_{n+h}(A) S_n(E) z^n = \frac{S_h(A) - a_1 a_2 (e_1 + e_2) S_{h-1}(A) z + (a_1 a_2)^2 e_1 e_2 S_{h-2}(A) z^2}{\left(\sum_{n=0}^{+\infty} S_n(-E) a_1^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E) a_2^n z^n\right)} \quad (h\ (\geqslant 1) \in \mathbb{N}). \tag{3.1}$$

*Proof.* By applying the operator $\delta^{h+1}_{a_1 a_2}$ to the series $f(a_1 z) = \sum_{n=0}^{+\infty} S_n(E) a_1^n z^n$, we have

$$\begin{aligned}
\delta^{h+1}_{a_1 a_2} f(a_1 z) &= \frac{a_1^{h+1} \sum_{n=0}^{+\infty} S_n(E) a_1^n z^n - a_2^{h+1} \sum_{n=0}^{+\infty} S_n(E) a_2^n z^n}{(a_1 - a_2)} \\
&= \sum_{n=0}^{+\infty} \left( \frac{a_1^{n+h+1} - a_2^{n+h+1}}{a_1 - a_2} \right) S_n(E) z^n \\
&= \sum_{n=0}^{+\infty} S_{n+h}(A) S_n(E) z^n.
\end{aligned}$$

On the other hand, we see that

$$
\begin{aligned}
\delta_{a_1a_2}^{h+1}\left(\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n}\right) &= \frac{\frac{a_1^{h+1}}{\sum_{n=0}^{+\infty} S_n(-E)a_1^n z^n} - \frac{a_2^{h+1}}{\sum_{n=0}^{+\infty} S_n(-E)a_2^n z^n}}{a_1 - a_2} \\
&= \frac{a_1^{h+1}\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_2^n z^n - a_2^{h+1}\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n}{(a_1-a_2)\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_2^n z^n\right)} \\
&= \frac{\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n a_2^n \frac{(a_1^{h-n+1}-a_2^{h-n+1})}{a_1-a_2} z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_2^n z^n\right)} \\
&= \frac{\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n a_2^n S_{h-n}(A)\, z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_2^n z^n\right)} \\
&= \frac{\sum\limits_{n=0}^{h} S_n(-E)\, a_1^n a_2^n S_{h-n}(A)\, z^n + \sum\limits_{n=h+1}^{+\infty} S_n(-E)\, a_1^n a_2^n S_{h-n}(A)\, z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_2^n z^n\right)} \\
&= \frac{\sum\limits_{n=0}^{h} S_n(-E)\, a_1^n a_2^n S_{h-n}(A)\, z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_2^n z^n\right)} \\
&= \frac{S_h(A) - a_1a_2(e_1+e_2)S_{h-1}(A)z + (a_1a_2)^2 e_1e_2S_{h-2}(A)z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_2^n z^n\right)}.
\end{aligned}
$$

Therefore

$$
\sum_{n=0}^{+\infty} S_{n+h}(A)\, S_n(E)\, z^n = \frac{S_h(A) - a_1a_2(e_1+e_2)S_{h-1}(A)z + (a_1a_2)^2 e_1e_2S_{h-2}(A)z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E)\, a_2^n z^n\right)}.
$$

Thus, this completes the proof. □

## 4. Generating functions of some well-known numbers and polynomials

In this part, we now derive the new generating functions of the products of some well-known numbers and polynomials.

For the case $A = \{a_1, -a_2\}$ and $E = \{e_1, -e_2\}$ with replacing $a_2$ by $(-a_2)$, $e_2$ by $(-e_2)$ in (3.1), we have

$$\sum_{n=0}^{+\infty} S_{n+h}(a_1+[-a_2])S_n(e_1+[-e_2])z^n$$
$$= \frac{S_h(a_1+[-a_2]) + a_1a_2(e_1-e_2)S_{h-1}(a_1+[-a_2])z - a_1^2a_2^2e_1e_2S_{h-2}(a_1+[-a_2])z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}. \tag{4.1}$$

The Eq. (4.1) consists of five related parts. Firstly, we consider the following conditions

$$\begin{cases} a_1 - a_2 = k \\ a_1a_2 = 1 \end{cases} \text{ and } \begin{cases} e_1 - e_2 = k \\ e_1e_2 = 1 \end{cases},$$

in (4.1). Thus it becomes

$$\sum_{n=0}^{+\infty} S_{n+h}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{S_h(a_1+[-a_2]) + kS_{h-1}(a_1+[-a_2])z - S_{h-2}(a_1+[-a_2])z^2}{1-k^2z-2(k^2+1)z^2-k^2z^3+z^4}$$
$$= \sum_{n=0}^{+\infty} F_{k,n+h}F_{k,n}z^n, \tag{4.2}$$

representing a new generating function for generalized of the product of $k$-Fibonacci numbers, with $F_{k,n+h}F_{k,n} = S_{n+h}(a_1+[-a_2])S_n(e_1+[-e_2])$, [29].

- By putting $h = 1$ and $h = 2$ in the relationship (4.2), we get the following results.

**Corollary 4.1.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} F_{k,n+1}F_{k,n}z^n = \frac{k+kz}{1-k^2z-2(k^2+1)z^2-k^2z^3+z^4}. \tag{4.3}$$

**Corollary 4.2.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} F_{k,n+2}F_{k,n}z^n = \frac{k^2+1+k^2z-z^2}{1-k^2z-2(k^2+1)z^2-k^2z^3+z^4}. \tag{4.4}$$

- Based on the relationships (4.3) and (4.4) and with $k = 1$, we obtain the following table [3, 7]:

**Table 1.** Generating functions of the products of Fibonacci numbers

| The products | The generating functions |
|---|---|
| $\sum_{n=0}^{+\infty} F_n^2z^n$ | $\frac{1-z^2}{1-z-4z^2-z^3+z^4}$ |
| $\sum_{n=0}^{+\infty} F_{n+1}F_nz^n$ | $\frac{1+z}{1-z-4z^2-z^3+z^4}$ |
| $\sum_{n=0}^{+\infty} F_{n+2}F_nz^n$ | $\frac{2+z-z^2}{1-z-4z^2-z^3+z^4}$ |

Secondly, we consider the following conditions

$$\begin{cases} a_1 - a_2 = 2 \\ a_1a_2 = k \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 2 \\ e_1e_2 = k \end{cases},$$

in (4.1). It yields

$$\sum_{n=0}^{+\infty} S_{n+h-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{S_h(a_1+[-a_2])z+2kS_{h-1}(a_1+[-a_2])z^2-k^3S_{h-2}(a_1+[-a_2])z^3}{1-4z-(2k^2+8k)z^2-4k^2z^3+k^4z^4}$$
$$= \sum_{n=0}^{+\infty} P_{k,n+h}P_{k,n}z^n, \tag{4.5}$$

representing a new generating function for generalized of the product of $k$-Pell numbers with $P_{k,n+h}P_{k,n} = S_{n+h-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])$.

- By putting $h = 1$ and $h = 2$ in the relationship (4.5)we get the following results.

**Corollary 4.3.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Pell numbers is given by*

$$\sum_{n=0}^{+\infty} P_{k,n+1}P_{k,n}z^n = \frac{2z+2kz^2}{1-4z-(2k^2+8k)z^2-4k^2z^3+k^4z^4}. \tag{4.6}$$

**Corollary 4.4.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Pell numbers is given by*

$$\sum_{n=0}^{+\infty} P_{k,n+2}P_{k,n}z^n = \frac{(4+k)z+4kz^2-k^3z^3}{1-4z-(2k^2+8k)z^2-4k^2z^3+k^4z^4}. \tag{4.7}$$

- Based on the relationships (4.6) and (4.7) and with $k = 1$, we obtain the following table [3, 5]:

**Table 2.**Generating functions of the products of Pell numbers

| The products | The generating functions |
|---|---|
| $\sum_{n=0}^{+\infty} P_n^2 z^n$ | $\frac{z-z^3}{1-4z-10z^2-4z^3+z^4}$ |
| $\sum_{n=0}^{+\infty} P_{n+1}P_n z^n$ | $\frac{2z+2z^2}{1-4z-10z^2-4z^3+z^4}$ |
| $\sum_{n=0}^{+\infty} P_{n+2}P_n z^n$ | $\frac{5z+4z^2-z^3}{1-4z-10z^2-4z^3+z^4}$ |

Thirdly, we consider the following conditions

$$\begin{cases} a_1-a_2=1 \\ a_1a_2=2 \end{cases} \text{ and } \begin{cases} e_1-e_2=1 \\ e_1e_2=2 \end{cases},$$

in (4.1). We have

$$\sum_{n=0}^{+\infty} S_{n+h-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{S_h(a_1+[-a_2])z+2S_{h-1}(a_1+[-a_2])z^2-8S_{h-2}(a_1+[-a_2])z^3}{1-z-12z^2-4z^3+16z^4}$$
$$= \sum_{n=0}^{+\infty} J_{n+h}J_nz^n, \tag{4.8}$$

representing a new generating function for generalized of the product of Jacobsthal numbers $J_n$ with $J_{n+h}J_n = S_{n+h-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2]))$.

- By putting $h = 1$ and $h = 2$ in the relationship (4.8), we get the following new results.

**Lemma 4.5.** *For $n, k \in \mathbb{N}$, the new generating function of the product of Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} J_{n+1} J_n z^n = \frac{z + 2z^2}{1 - z - 12z^2 - 4z^3 + 16z^4}.$$

**Lemma 4.6.** *For $n, k \in \mathbb{N}$, the new generating function of the product of Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} J_{n+2} J_n z^n = \frac{3z + 2z^2 - 8z^3}{1 - z - 12z^2 - 4z^3 + 16z^4}.$$

Fourthly, we consider the following conditions

$$\begin{cases} a_1 - a_2 = k \\ a_1 a_2 = 1 \end{cases} \text{ and } \begin{cases} e_1 - e_2 = x \\ e_1 e_2 = 1 \end{cases},$$

in (4.1). It gives

$$\sum_{n=0}^{+\infty} S_{n+h}(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{S_h(a_1 + [-a_2]) + x S_{h-1}(a_1 + [-a_2]) z - S_{h-2}(a_1 + [-a_2]) z^3}{1 - kxz - (k^2 + x^2 + 2) z^2 - kxz^3 + z^4}$$
$$\sum_{n=0}^{+\infty} F_{k,n+h}\, {}_2F_1\left(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4}\right) z^n, \tag{4.9}$$

representing a new generating function for generalized of the product of $k$-Fibonacci numbers with Fibonacci polynomials.

- By putting $h = 1$ and $h = 2$ in the relationship (4.9), we get the following results.

**Corollary 4.7.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers with Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} F_{k,n+1}\, {}_2F_1\left(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4}\right) z^n = \frac{k + xz}{1 - kxz - (k^2 + x^2 + 2) z^2 - kxz^3 + z^4}. \tag{4.10}$$

**Corollary 4.8.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers with Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} F_{k,n+2}\, {}_2F_1\left(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4}\right) z^n = \frac{k^2 + 1 + kxz - z^2}{1 - kxz - (k^2 + x^2 + 2) z^2 - kxz^3 + z^4}. \tag{4.11}$$

- Based on the relationship (4.10) and (4.11) and with $k = 1$, we obtain the following table :

**Table 3.** Generating functions of the products of Fibonacci numbers with Fibonacci polynomials

| The products | The generating functions |
|---|---|
| $\sum_{n=0}^{+\infty} F_n\, {}_2F_1(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4}) z^n$ | $\frac{1-z^2}{1-xz-(x^2+3)z^2-xz^3+z^4}$ |
| $\sum_{n=0}^{+\infty} F_{n+1}\, {}_2F_1(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4}) z^n$ | $\frac{1+xz}{1-xz-(x^2+3)z^2-xz^3+z^4}$ |
| $\sum_{n=0}^{+\infty} F_{n+2}\, {}_2F_1(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4}) z^n$ | $\frac{2+xz-z^2}{1-xz-(x^2+3)z^2-xz^3+z^4}$ |

Lastly, we consider the following conditions

$$\begin{cases} a_1 - a_2 = 2 \\ a_1 a_2 = k \end{cases} \text{ and } \begin{cases} e_1 - e_2 = x \\ e_1 e_2 = 1 \end{cases},$$

in (4.1). We see that

$$\begin{aligned} \sum_{n=0}^{+\infty} S_{n+h-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n &= \frac{S_h(a_1 + [-a_2])z + kxS_{h-1}(a_1 + [-a_2])z^2 - k^2 S_{h-2}(a_1 + [-a_2])z^3}{1 - 2xz - (kx^2 + 2k + 4)z^2 - 2kxz^3 + k^2 z^4} \\ &= \sum_{n=0}^{+\infty} P_{k,n+h}\ {}_2F_1(\frac{2-n}{2}, \frac{n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4})z^n, \end{aligned} \tag{4.12}$$

representing a new generating function for generalized of the product of $k$-Pell numbers with Fibonacci polynomials, and also we have

$$P_{k,n+h}\ {}_2F_1(\frac{2-n}{2}, \frac{n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4}) = S_{n+h-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2]).$$

- By putting $h = 1$ and $h = 2$ in the relationship (4.12), we get the following results.

**Corollary 4.9.** *For $n, k \in \mathbb{N}$, the new generating function of the product of k-Pell numbers with Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} P_{k,n+1}\ {}_2F_1(\frac{2-n}{2}, \frac{n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4})z^n = \frac{2z + kxz^2}{1 - 2xz - (kx^2 + 2k + 4)z^2 - 2kxz^3 + k^2 z^4}. \tag{4.13}$$

**Corollary 4.10.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Pell numbers with Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} P_{k,n+2}\ {}_2F_1(\frac{2-n}{2}, \frac{n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4})z^n = \frac{(4 + k)z + 2kxz^2 - k^2 z^3}{1 - 2xz - (kx^2 + 2k + 4)z^2 - 2kxz^3 + k^2 z^4}. \tag{4.14}$$

- Based on the relationship (4.13) and (4.14) and with $k = 1$, we obtain the following table [2]:

**Table 4.**Generating functions of the products of Pell numbers with Fibonacci polynomials

| The Products | The Generating Functions |
|---|---|
| $\sum_{n=0}^{+\infty} P_n\ {}_2F_1(\frac{2-n}{2}, \frac{n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4})z^n$ | $\frac{z-z^3}{1-2xz-(x^2+6)z^2-2xz^3+z^4}$ |
| $\sum_{n=0}^{+\infty} P_{n+1}\ {}_2F_1(\frac{2-n}{2}, \frac{n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4})z^n$ | $\frac{2z+xz^2}{1-2xz-(x^2+6)z^2-2xz^3+z^4}$ |
| $\sum_{n=0}^{+\infty} P_{n+2}\ {}_2F_1(\frac{2-n}{2}, \frac{n}{2}; \frac{3}{2}; 1 + \frac{x^2}{4})z^n$ | $\frac{5z+2xz^2-z^3}{1-2xz-(x^2+6)z^2-2xz^3+z^4}$ |

- For the case $A = \{a_1, -a_2\}$ and $E = \{2e_1, -2e_2\}$ with replacing $a_2$ by $(-a_2)$, $e_1$ by $2e_1$ and $e_2$ by $(-2e_2)$ in (4.1), we have

$$\begin{aligned} &\sum_{n=0}^{+\infty} S_{n+h}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\ &= \frac{S_h(a_1 + [-a_2]) + 2a_1a_2(e_1 - e_2)S_{h-1}(a_1 + [-a_2])z - 4a_1^2a_2^2e_1e_2S_{h-2}(a_1 + [-a_2])z^2}{(1 - 2a_1e_1z)(1 + 2a_1e_2z)(1 + 2a_2e_1z)(1 - 2a_2e_2z)}. \end{aligned} \tag{4.15}$$

This case consists of two related parts. Firstly, the substitutions

$$\begin{cases} a_1 - a_2 = k \\ a_1 a_2 = 1 \end{cases} \text{ and } \begin{cases} e_1 - e_2 = x \\ 4e_1 e_2 = -1 \end{cases},$$

in (4.15) gives

$$\sum_{n=0}^{+\infty} S_{n+h}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n = \frac{S_h(a_1 + [-a_2]) + 2xS_{h-1}(a_1 + [-a_2])z + S_{h-2}(a_1 + [-a_2])z^2}{1 - 2kxz - (4x^2 - k^2 - 2)z^2 + 2kxz^3 + z^4}$$
$$= \sum_{n=0}^{+\infty} F_{k,n+h}\ (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n, \tag{4.16}$$

representing a new generating function for generalized of the product of $k$-Fibonacci numbers with Chebyshev polynomials of second kind.

- By putting $h = 1$ and $h = 2$ in the relationship (4.16), we get the following results.

**Corollary 4.11.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers with Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} F_{k,n+1}\ (n+1)\ {}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n = \frac{k + 2xz}{1 - 2kxz - (4x^2 - k^2 - 2)z^2 + 2kxz^3 + z^4}.$$

**Corollary 4.12.** *[2] For $n, k \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers with Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} F_{k,n+2}\,(n+1)\ {}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n = \frac{k^2 + 1 + 2kxz + z^2}{1 - 2kxz - (4x^2 - k^2 - 2)z^2 + 2kxz^3 + z^4}.$$

Before finalizing this paper, we give the following Theorem without proof because its proof can be made similar to that the previous Theorem in this paper.

**Theorem 4.13.** *For $n \in \mathbb{N}$, the new generating function for generalized of the product of k-Fibonacci numbers and Chebyshev polynomials of first kind is given by*

$$\sum_{n=0}^{+\infty} F_{k,n+h}\,(a_1 + [-a_2])\ {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2})z^n = \frac{\begin{array}{c} S_h(a_1 + [-a_2]) + x\,(S_{h-1}(a_1 + [-a_2]) - S_h(a_1 + [-a_2]))\,z \\ +(S_{h-2}(a_1 + [-a_2]) - 2x^2 S_h(a_1 + [-a_2])z^2 - xS_{h-1}(a_1 + [-a_2])z^3 \end{array}}{1 - 2kxz - (4x^2 - k^2 - 2)z^2 + 2kxz^3 + z^4}.$$

**Remark 4.14.** *Some special cases can be investigated as in previous parts of this paper. So it is left to the readers.*

## 5. Conclusion

In this paper, we have derived new theorems in order to determine new generalization of generating functions of $k$-Fibonacci and $k$-Pell numbers and Fibonacci polynomails and Chebyshev polynomials of the first and second kinds. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

## References


[1] A. Abderrezzak, Généralisation de la transformation d'Euler d'une série formelle. Adv. Math. **103** (1994) 180–195.
[2] A. Boussayoud, S. Boughaba, M. Kerada, S. Araci, M. Acikgoz, Generating Functions of Binary Products of k-Fibonacci and Orthogonal Polynomials, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.113, (2019), 1579-1505.
[3] A. Boussayoud, M. Kerada, S. Araci, M. Acikgoz, A. Esi, Symmetric Functions of Binary Products of Fibonacci and Orthogonal Polynomials, Filomat. 33(3), (2019), 1495–1504.
[4] A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers, Online J. Anal. Comb. 12, (2017), 1-10.
[5] A. Boussayoud, N.Harrouche, Complete symmetric functions and k-Fibonacci numbers. Commun.Appl.Anal. 20, (2016), 457-465.
[6] A. Boussayoud, M.kerada, M, Boulyer, A simple and accurate method for determination of some generalized sequence of numbers, Int.J.Pure Appl Math. 108, (2016), 503-511.
[7] A. Boussayoud, A. Abderrezzak, M. Kerada, Some applications of symmetric functions, Integers., 15 A#48, (2015), 1-7.
[8] A. Boussayoud, M. Kerada, Symmetric and Generating Functions, Int. Electron. J. Pure Appl. Math. 7, (2014), 195-203.
[9] A. Pintér, H. M. Srivastava, Generating functions of the incomplete Fibonacci and Lucas numbers, Rend. Circ. Mat. Palermo. 48, (1999), 591–596.
[10] D. Tasci and M. Cetin Firengiz, Incomplete Fibonacci and Lucas $p$-numbers, Math. Comput. Modelling. 52, (2010), 1763-1770.
[11] E. E. Duman, H. Ciftci, Fibonacci and Lucas Differential Equations, Appl. Appl. Math.13(2), (2018), 756-763.
[12] G. B. Djordjević, Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers, Fibonacci Q. 42, (2004), 106–113.
[13] G. B. Djordjević and H. M. Srivastava, Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers, Math. Comput. Modelling. 42, (2005), 1049-1056.
[14] H.M. Srivastava and G. B. Djordjević, Some generating functions and other properties associated with the generalized Hermite and related polynomials, Integral Transforms Spec. Funct. 22:12, 895-906, 2011.
[15] Y. He, S. Araci, H. M. Srivastava, Summation formulas for the products of the Frobenius-Euler polynomials, Ramanujan J. 44 (2017), 177-195.
[16] H. M. Srivastava, M. A. Özarslan, B. Y. Yaşar, Difference equations for a class of twice-iterated $\triangle_h$-Appell sequences of polynomials, Rev. R. Acad. Cienc. Exactas Fís, 113 (2019), 1851-1871.
[17] H. M. Srivastava, S. Arjika, A. Sherif Kelil, Some homogeneous $q$-difference operators and the associated generalized Hahn polynomials, Appl. Set-Valued Anal. Optim. 1 (2019), 187–201.
[18] Mohamed R. Ali, Adel R. Hadhoud, H. M. Srivastava, Solution of fractional Volterra–Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method, Adv. Differ. Equ. 2019, 115 (2019).
[19] H. M. Srivastava, N. Tuglu, M. Çetin, Some results on the $q$-analogues of the incomplete Fibonacci and Lucas Polynomials, Miskolc Math. Notes 20 (2019), 511-524.
[20] H. M. Srivastava, H. L. Manocha, A Treatise on Generating Functions, Halsted Press, John Wiley and Sons, New York, 1984.
[21] W. Kumam, H.M. Srivastava, S. A. Wani, S. Araci, P. Kumam, Truncated-exponential-based Frobenius-Euler polynomials. Adv Differ Equ 2019, 530 (2019).
[22] I.G.Macdonald, Symmetric Functions and Hall Polynomials. Oxford University Press, Oxford (1979).
[23] S. Boughaba, A. Boussayoud, On Some Identities And Generating Function of Both K-jacobsthal Numbers and Symmetric Functions in Several Variables, Konuralp J. Math. **7 (2),** 235-242,2019 .
[24] S. Boughaba, A.Boussayoud, Construction of Symmetric Functions of Generalized Fibonacci Numbers, Tamap Journal of Mathematics and Statistics .3**, (**2019), 1-8.
[25] S. Boughaba, A. Boussayoud, Kh. Boubellouta, Generating Functions of Modified Pell Numbers and Bivariate Complex Fibonacci Polynomials, Turkish Journal of Analysis and Number. **7(4), (**2019**),** 113-116 .
[26] S. Falcon , A. Plaza, On the $k$-Fibonacci numbers, Chaos,Sulutions&Fractals. 32, (2007), 1615-1624.
[27] S. Falcon, A. Plaza, The $k$-Fibonacci sequence and the Pascal 2-triangle, Chaos, Sulutions & Fractals. 33, (2008), 38-49.
[28] S. Falcon, A. Plaza, On k-Fibonacci sequences and Polynomials and their derivatives, Chaos, Sulutions & Fractals 39, (2009), 1005-1019.
[29] S. Falcon, On the sequences of products of two k-Fibonacci numbers. Am. Rev. Math. Stat. 2, (2014), 111–120.
[30] S. Bochner, Über sturm-Liouvillesche polynomsysteme, Math Z. 29, (1929), 730–736.
[31] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[32] T. Kim, D. S. Kim, L.C. Jang, D. V. Dolgy, Representation by Chebyshev Polynomials for Sums of Finite Products of Chebyshev Polynomials, Symmetry, 10, 742, 2018.
[33] O.G. Yılmaz, R. Aktaş, F. Taşdelen, On Some Formulas for the k-Analogue of Appell Functions and Generating Relations via $k$-Fractional Derivative, Fractal Fract. 4, 48, 2020,
[34] P. Maroni, Une théorie algébrique des polynômes orthogonaux Applications aux polynômes orthogonaux semi-classiques, In Orthogonal Polynomials and their Applications, C. Brezinski et al. Editors, IMACS Ann. Comput. Appl. Math. 9, (1991), 95–130.

[35] P. Maroni, Fonctions Eulériennes, Polynômes Orthogonaux Classiques. Techniques de l'Ingénieur, Traité Généralités (Sciences Fondamentales) A 154 Paris. (1994), 1–30.
[36] W. Al-Salam and T.S. Chihara, Another characterization of the classical orthogonal polynomials, SIAM J. Math. Anal. 3, (1972), 65–70.
[37] G. Szegö, Orthogonal Polynomials. Fourth edition, Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, Rhode Island, 1975.