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Some Results on q-Multiple Harmonic Sums

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Abstract. In this paper the author present some new identities for q-analogues of multiple harmonic (star) sums whose indices are the sequences $(\{p\}_a)$ and $(\{p\}_a, p+1, \{p\}_b)$. Then we use these formulas to establish some relations between multiple harmonic (star) sums and classical harmonic numbers and binomial coefficients. As an application we give some explicit formulas involving multiple zeta star values. Some interesting consequences and illustrative examples are also given.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers and let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the set of positive integers and $\mathbb{N} \setminus \{1\} := \{2, 3, 4, \cdots\}$. For any multi-index $\mathbf{S} := (s_1, s_2, \cdots, s_k)$ $(s_i \in \mathbb{C},\ k \in \mathbb{N})$, the general multiple harmonic sum $\zeta_n(\mathbf{S})$ and the multiple harmonic star sum $\zeta_n^\star(\mathbf{S})$ are defined, respectively, by convergent series ([3, 10])

$$\zeta_n(\mathbf{S}) \equiv \zeta_n(s_1, s_2, \cdots, s_k) := \sum_{n \geq n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}, \tag{1.1}$$

$$\zeta_n^\star(\mathbf{S}) \equiv \zeta_n^\star(s_1, s_2, \cdots, s_k) := \sum_{n \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}. \tag{1.2}$$

where $s_1 + \cdots + s_k$ is called the weight and k is the depth. When $n < k$, then $\zeta_n(s_1, s_2, \cdots, s_k) = 0$, and $\zeta_n(\emptyset) = \zeta_n^\star(\emptyset) = 1$. For convenience, we let $\{a\}_k$ be the k repetitions of a such that

$$\zeta_n(4, 3, \{1\}_2) = \zeta_n(4, 3, 1, 1),\ \zeta_n^\star(5, 2, \{1\}_3) = \zeta_n^\star(5, 2, 1, 1, 1).$$

If $k = 1$ in (1.1) and (1.2), then the sums reduce to classical harmonic numbers, which are defined by (see [1, 2, 11, 18, 19])

$$H_n^{(p)} \equiv \zeta_n(p) \equiv \zeta_n^\star(p) := \sum_{j=1}^{n} \frac{1}{j^p} \quad (p, n \in \mathbb{N}).$$

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$$H_n = H_n^{(1)} = \sum_{j=1}^{n} \frac{1}{j}.$$

The limit cases of multiple harmonic sums and multiple harmonic star sums give rise to multiple zeta values (MZVs) and multiple zeta star values (MZSVs) (see [3, 12, 14]):

$$\zeta(s_1, s_2, \cdots, s_k) = \lim_{n\to\infty} \zeta(s_1, s_2, \cdots, s_k), \tag{1.3}$$

$$\zeta^\star(s_1, s_2, \cdots, s_k) = \lim_{n\to\infty} \zeta^\star(s_1, s_2, \cdots, s_k) \tag{1.4}$$

defined for positive integers $s_1, s_2, \ldots, s_k \geq 1$ and $s_1 \geq 2$ to ensure convergence of the series. The origin of these numbers goes back to the correspondence of Euler with Goldbach in 1742-1743 (see [12]) and Euler's paper [9] that appeared in 1776. Euler studied double zeta values and established some important relation formulas for them. For example, he proved that (see [11])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^k} = \frac{1}{2}\left\{(k+2)\zeta(k+1) - \sum_{i=1}^{k-2} \zeta(k-i)\zeta(i+1)\right\}. \tag{1.5}$$

It has been discovered in the course of the years that many multiple zeta (star) values admit expressions involving finitely "zeta values", that is say values of the Riemann zeta function,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1).$$

with positive integer arguments. The relationship between the values of the Riemann zeta values and multiple zeta (star) values has been studied by many authors. For details and historical introductions, please see [3–5, 11, 18, 19] and references therein.

The purpose of the present paper is to establish some identities of q-analogues of multiple harmonic (star) sums and q-harmonic numbers. We then apply it to obtain a family of identities relating multiple zeta (star) values to classical zeta values.

We begin with some basic notations. The q-analogues of multiple harmonic numbers and q-analogues multiple harmonic star numbers are defined by

$$\zeta_n[s_1, s_2, \cdots, s_k] := \sum_{n \geqslant n_1 > n_2 > \cdots > n_k \geqslant 1} \frac{q^{n_1+n_2+\cdots+n_k}}{[n_1]_q^{s_1}[n_2]_q^{s_2}\cdots[n_k]_q^{s_k}} \quad (s_i \in \mathbb{C},\ k \in \mathbb{N}), \tag{1.6}$$

$$\zeta_n^\star[s_1, s_2, \cdots, s_k] := \sum_{n \geqslant n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1} \frac{q^{n_1+n_2+\cdots+n_k}}{[n_1]_q^{s_1}[n_2]_q^{s_2}\cdots[n_k]_q^{s_k}} \quad (s_i \in \mathbb{C},\ k \in \mathbb{N}), \tag{1.7}$$

with $\zeta_n[s_1, s_2, \cdots, s_k] = 0$ if $n < k$, and $\zeta_n[\emptyset] = \zeta_n^\star[\emptyset] = 1$, where $[n]_q$ denotes the q-analog of a non-negative integer defined by

$$[n]_q := \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q} \quad (0 < q < 1).$$

Let n, m denote integers, the Gaussian q-binomial coefficient is defined by [1]

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[m]_q![n-m]_q!},$$

where $0 \leq m \leq n$ and $[n]_q! = [1]_q[2]_q\cdots[n]_q$ with $\begin{bmatrix} n \\ 0 \end{bmatrix}_q := 1$. Obviously, the q-binomial coefficient tends to the ordinary binomial coefficient when $q \to 1$. Xu et.al [20] define the q-polylogarithm function $\mathrm{Li}_p[x]$ and the partial sum of $\mathrm{Li}_p[x]$ by

$$\mathrm{Li}_p[x] := \sum_{k=1}^{\infty} \frac{x^k}{[k]_q^p} \quad (x \in (-1,1), \Re(p) \geq 1), \tag{1.8}$$

$$\zeta_n[p;x] := \sum_{k=1}^{n} \frac{x^k}{[k]_q^p} \quad (x \in (-1,1), \Re(p) \geq 1). \tag{1.9}$$

When $x = q^s$ in (1.9), then $\zeta_n[p;q^s]$ $(s \in \mathbb{N})$ is called the q-harmonic number. It is obviously that

$$\lim_{q\to 1} \zeta_n[p;q^s] = \zeta_n[p] = H_n^{(p)}.$$

There are many results for sums of the types (1.6) and (1.7). Some related results for q-multiple harmonic (star) sums may be seen in the works of [7, 8, 10, 15, 16, 20] and references therein.

Theorem 1.1. *For positive integer n, m and any sequences $a_i \in \mathbb{C}$ $(i = 1, 2 \ldots, n)$, the following identity holds:*

$$\sum_{k_1=1}^{n} a_{k_1} \sum_{k_2=1}^{k_1} a_{k_2} \cdots \sum_{k_m=1}^{k_{m-1}} a_{k_m} = \sum_{r=1}^{n} \frac{a_r^m}{\prod\limits_{i=1, i\neq r}^{n} \left(1 - \frac{a_i}{a_r}\right)}. \tag{1.10}$$

Proof. We consider the product

$$\frac{1}{\prod\limits_{i=1}^{n}(1-a_i t)} = \sum_{m=0}^{\infty} A_m(n)\, t^m, \; A_0(n) = 1, \tag{1.11}$$

where

$$|t| < \min\left\{|a_1|^{-1}, \cdots, |a_n|^{-1}\right\}.$$

By a direct calculation, the following identity is easily derived

$$A_m(n) := \sum_{k_1=1}^{n} a_{k_1} \sum_{k_2=1}^{k_1} a_{k_2} \cdots \sum_{k_m=1}^{k_{m-1}} a_{k_m}. \tag{1.12}$$

Expanding the product on the left hand side of (1.11), we deduce that

$$\frac{1}{\prod\limits_{i=1}^{n}(1-a_i t)} = \sum_{r=1}^{n} \frac{B_r(n)}{1-a_r t} = \sum_{m=0}^{\infty} \left\{ \sum_{r=1}^{n} B_r(n)\, a_r^m \right\} t^m \quad (n \in \mathbb{N}, a_i \in \mathbb{C}), \tag{1.13}$$

where

$$B_r(n) = \lim_{n \to a_r^{-1}} \frac{1-a_r t}{\prod\limits_{i=1}^{n}(1-a_i t)} = \prod_{i=1, i\neq r}^{n} \left(1 - \frac{a_i}{a_r}\right)^{-1}.$$

By comparing the coefficients of t^m in (1.11) and (1.13), we obtain the desired result. □

Noting that, let $X_n(m) := \sum\limits_{k=1}^{n} a_k^m$, we can rewrite formula (1.11) as

$$\begin{aligned}
\sum_{m=0}^{\infty} A_m(n)\, t^m &= \frac{1}{\prod\limits_{i=1}^{n}(1-a_i t)} = \exp\left\{-\sum_{k=1}^{n}\ln(1-a_k t)\right\} \\
&= \exp\left\{\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{a_k^m}{m}t^m\right\} = \exp\left\{\sum_{m=1}^{\infty}\frac{X_n(m)}{m}t^m\right\} \\
&= \exp\left\{X_n(1) + \frac{1}{2!}X_n(2) + \frac{1}{3!}X_n(3) + \frac{1}{4!}X_n(4)\cdots\right\} \\
&= 1 + X_n(1)\, t + \frac{1}{2}\left\{X_n^2(1) + X_n(2)\right\} t^2 \\
&\quad + \left\{\frac{1}{3}X_n^3(1) + \frac{1}{2}X_n(1)X_n(2) + \frac{1}{6}X_n(3)\right\} t^3 + \cdots . \qquad (1.14)
\end{aligned}$$

Hence, by comparing the coefficients of t^m in above equation, we deduce the following identities

$$A_0(n) = 1,$$

$$A_1(n) = X_n(1),$$

$$A_2(n) = \frac{X_n^2(1) + X_n(2)}{2}, \qquad (1.15)$$

$$A_3(n) = \frac{X_n^3(1) + 3X_n(1)X_n(2) + 2X_n(3)}{3!}, \qquad (1.16)$$

$$A_4(n) = \frac{X_n^4(1) + 8X_n(1)X_n(3) + 3X_n^2(2) + 6X_n^2(1)X_n(2) + 6X_n(4)}{4!}, \qquad (1.17)$$

$$A_5(n) = \frac{1}{5!}\left\{\begin{array}{l} X_n^5(1) + 10X_n^3(1)X_n(2) + 20X_n^2(1)X_n(3) + 15X_n(1)X_n^2(2) \\ +30X_n(1)X_n(4) + 20X_n(2)X_n(3) + 24X_n(5) \end{array}\right\}, \qquad (1.18)$$

$$A_6(n) = \frac{1}{6!}\left\{\begin{array}{l} X_n^6(1) + 15X_n^4(1)X_n(2) + 40X_n^3(1)X_n(3) + 90X_n^2(1)X_n(4) \\ +144X_n(1)X_n(5) + 45X_n^2(1)X_n^2(2) + 120X_n(1)X_n(2)X_n(3) \\ +40X_n^2(3) + 15X_n^3(2) + 90X_n(2)X_n(4) + 120X_n(6) \end{array}\right\}. \qquad (1.19)$$

From (1.14), we know that the multiple sums $A_m(n)$ can be expressed as a rational linear combination of products of $X_n(j)$. In fact, by using the following recurrence relation of exponential complete Bell polynomials Y_n (see [17])

$$Y_k(x_1, x_2, \cdots, x_k) = \sum_{j=0}^{k-1}\binom{k-1}{j} x_{k-j} Y_j\left(x_1, x_2, \cdots, x_j\right) \quad (k \in \mathbb{N}) \qquad (1.20)$$

and letting $x_m = (k-1)!X_n(m) \quad (m = 1, 2, \ldots, k)$, then with the help of formula (1.14), we deduce that the sums $A_k(n)$ satisfy the recurrence

$$A_k(n) = \frac{1}{k}\sum_{j=0}^{k-1} A_j(n)\, X_n(k-j), \qquad (1.21)$$

$$Y_k(X_n(1), 1!X_n(2), \cdots, (k-1)!X_n(k)) = k!A_k(n). \qquad (1.22)$$

Here the exponential complete Bell polynomial Y_n is defined by [6, 17]

$$\exp\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = 1 + \sum_{k\geq 1} Y_k(x_1, x_2, \cdots, x_k)\frac{t^k}{k!},\ Y_0(\cdot) = 1. \qquad (1.23)$$

Similarly, in the same way as in the proofs of formulas (1.15)-(1.19), by considering the product

$$\prod_{i=1}^{n}(1+a_i t)=\sum_{m=0}^{\infty}\bar{A}_m(n)\,t^m, \tag{1.24}$$

$$\bar{A}_m(n):=\sum_{1\le k_m<\cdots<k_1\le n} a_{k_1}\cdots a_{k_m}\quad (a_k\in\mathbb{C}), \tag{1.25}$$

we can obtain the following formulas

$$\bar{A}_1(n)=X_n(1), \tag{1.26}$$

$$\bar{A}_2(n)=\frac{X_n^2(1)-X_n(2)}{2!}, \tag{1.27}$$

$$\bar{A}_3(n)=\frac{X_n^3(1)-3X_n(1)X_n(2)+2X_n(3)}{3!}, \tag{1.28}$$

$$\bar{A}_4(n)=\frac{X_n^4(1)-6X_n^2(1)X_n(2)+8X_n(1)X_n(3)+3X_n^2(2)-6X_n(4)}{4!}, \tag{1.29}$$

$$\bar{A}_5(n)=\frac{1}{5!}\left\{\begin{array}{l} X_n^5(1)-10X_n^3(1)X_n(2)+20X_n^2(1)X_n(3)+15X_n(1)X_n^2(2)\\ -30X_n(1)X_n(4)-20X_n(2)X_n(3)+24X_n(5)\end{array}\right\}, \tag{1.30}$$

$$\bar{A}_6(n)=\frac{1}{6!}\left\{\begin{array}{l} X_n^6(1)-15X_n^4(1)X_n(2)+40X_n^3(1)X_n(3)-90X_n^2(1)X_n(4)\\ +144X_n(1)X_n(5)+45X_n^2(1)X_n^2(2)-120X_n(1)X_n(2)X_n(3)\\ +40X_n^2(3)-15X_n^3(2)+90X_n(2)X_n(4)-120X_n(6)\end{array}\right\}, \tag{1.31}$$

where $\bar{A}_0(n)=A_0(n)=1$. If $n<m$, then $\bar{A}_m(n)=0$. Moreover, the multiple sums $\bar{A}_m(n)$ can also be expressed as a rational linear combination of products of $X_n(j)$. In fact, by using the definition of Bell polynomial Y_n again, we can find that

$$Y_k\left(X_n(1),-1!X_n(2),\cdots,(-1)^{k-1}(k-1)!X_n(k)\right)=k!\bar{A}_k(n). \tag{1.32}$$

Hence, letting $x_k=(-1)^{k-1}(k-1)!X_n(k)$ in (1.20) and combining above identity (1.32), we deduce the following recurrence relation

$$\bar{A}_k(n)=\frac{(-1)^{k-1}}{k}\sum_{j=0}^{k-1}(-1)^j\bar{A}_j(n)\,X_n(k-j). \tag{1.33}$$

2. Main Theorems

The main result can be stated as follows.

Theorem 2.1. *For integers $p\in\mathbb{N}$, $m\in\mathbb{N}$ and $a,b\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, the following identity holds:*

$$\begin{aligned}\sum_{a+b=m-1}\zeta_n^\star\left[\{p\}_a,p+1,\{p\}_b;x\right]=&\,m\sum_{r=1}^{n}\frac{x^{mr}}{[r]_q^{pm+1}\prod\limits_{i=1,i\ne r}^{n}\left(1-\frac{[r]_q^p}{[i]_q^p}x^{i-r}\right)}\\ &+\sum_{r=1}^{n}\frac{x^{(m-1)r}\left(\sum\limits_{i=1,i\ne r}^{n}\frac{[r]_q-[i]_q}{([i]_q^p-[r]_q^p x^{i-r})}x^i\right)}{[r]_q^{p(m-1)+1}\prod\limits_{i=1,i\ne r}^{n}\left(1-\frac{[r]_q^p}{[i]_q^p}x^{i-r}\right)},\end{aligned} \tag{2.1}$$

where

$$\zeta_n^\star[s_1,\ldots,s_m;x]:=\sum_{1\le k_m\le\cdots\le k_1\le n}\frac{x^{k_1+\cdots+k_m}}{[k_1]_q^{s_1}\cdots[k_m]_q^{s_m}}. \tag{2.2}$$

Proof. Letting $a_k = \frac{x^k}{([k]_q+a)^p}$ in Theorem 1.1 we obtain

$$\sum_{1\le k_m\le\cdots\le k_1\le n} \frac{x^{k_1+\cdots+k_m}}{\left([k_1]_q+a\right)^p\cdots\left([k_m]_q+a\right)^p} = \sum_{r=1}^{n} \frac{x^{mr}}{\left([r]_q+a\right)^{pm}\prod\limits_{i=1,i\ne r}^{n}\left(1-\left(\frac{[r]_q+a}{[i]_q+a}\right)^p x^{i-r}\right)}. \tag{2.3}$$

Upon differentiating both members of (2.3) with respect to a and then setting $a = 0$, we may easily deduce the desired result. □

Theorem 2.2. *For real $p > 1$, integers $m \in \mathbb{N}$ and $a, b \in \mathbb{N}_0$, the sums*

$$\sum_{a+b=m-1} \zeta_n\left[\{p\}_a, p+1, \{p\}_b\right]$$

and

$$\sum_{a+b=m-1} \zeta_n^\star\left[\{p\}_a, p+1, \{p\}_b\right]$$

can be expressed as a rational linear combination of products of q-harmonic numbers.

Proof. In above section 1 we prove that the multiple sums $A_m(n)$ and $\bar{A}_m(n)$ can also be expressed as a rational linear combination of products of $X_n(j)$. Setting $a_k = \frac{q^k}{([k]_q+a)^p}$ in recurrence formulas (1.21) and (2.32), then

$$A_m(n) = \sum_{1\le k_m\le\cdots\le k_1\le n} \frac{q^{k_1+\cdots+k_m}}{\left([k_1]_q+a\right)^p\cdots\left([k_m]_q+a\right)^p},$$

$$\bar{A}_m(n) = \sum_{1\le k_m<\cdots<k_1\le n} \frac{q^{k_1+\cdots+k_m}}{\left([k_1]_q+a\right)^p\cdots\left([k_m]_q+a\right)^p},$$

$$X_n(j) = \sum_{i=1}^{n} \frac{q^{ij}}{\left([i]_q+a\right)^{pj}}$$

and

$$A_k(n) = \frac{1}{k}\sum_{j=0}^{k-1} A_j(n)\left(\sum_{i=1}^{n}\frac{q^{i(k-j)}}{\left([i]_q+a\right)^{p(k-j)}}\right)$$

$$\bar{A}_k(n) = \frac{(-1)^{k-1}}{k}\sum_{j=0}^{k-1}(-1)^j\bar{A}_j(n)\left(\sum_{i=1}^{n}\frac{q^{i(k-j)}}{\left([i]_q+a\right)^{p(k-j)}}\right).$$

Hence, by differentiating with respect to a and then letting $a = 0$, we obtain the result. □

Therefore, taking $x = 1, q \to 1$ in Theorem 2.1 and $q \to 1, n \to \infty$ in Theorem 2.2, we can give the following corollaries.

Corollary 2.3. *For integers $p \in \mathbb{N}$, $m \in \mathbb{N}$ and $a, b \in \mathbb{N}_0$, the following identity holds:*

$$\sum_{a+b=m-1} \zeta_n^\star\left(\{p\}_a, p+1, \{p\}_b\right) = \left(m + \frac{1}{p}\right)\sum_{r=1}^{n} \frac{1}{r^{pm+1}\prod\limits_{i=1, i\neq r}^{n}\left(1 - \frac{r^p}{i^p}\right)} - \sum_{r=1}^{n} \frac{\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{p} i^{p-j+1} r^{j-p}\right)^{-1}}{r^{pm}\prod\limits_{j=1, j\neq r}^{n}\left(1 - \frac{r^p}{j^p}\right)}. \tag{2.4}$$

Corollary 2.4. *For real $p > 1$, integers $m \in \mathbb{N}$ and $a, b \in \mathbb{N}_0$, the sums*

$$\sum_{a+b=m-1} \zeta\left(\{p\}_a, p+1, \{p\}_b\right)$$

and

$$\sum_{a+b=m-1} \zeta^\star\left(\{p\}_a, p+1, \{p\}_b\right)$$

can be expressed as a rational linear combination of products of Riemann zeta values.

3. Some examples

From Theorems 2.1-2.2 and Corollaries 2.3-2.4 with the help of formulas (1.15)-(1.18), (1.27)-(1.29), we can get the following examples. Here $a, b \in \mathbb{N}_0$.

Example 3.1. *Some illustrative examples follow.*

$$\begin{aligned}
\sum_{a+b=1} \zeta_n\left[\{p\}_a, p+1, \{p\}_b\right] =& \zeta_n[p;q]\,\zeta_n[p+1;q] - \zeta_n\left[2p+1;q^2\right],\\
\sum_{a+b=2} \zeta_n\left[\{p\}_a, p+1, \{p\}_b\right] =& \frac{1}{2}\zeta_n^2[p;q]\,\zeta_n[p+1;q] - \frac{1}{2}\zeta_n[p+1;q]\,\zeta_n\left[2p;q^2\right]\\
&- \zeta_n[p;q]\,\zeta_n\left[2p+1;q^2\right] + \zeta_n\left[3p+1;q^3\right],\\
\sum_{a+b=3} \zeta_n\left[\{p\}_a, p+1, \{p\}_b\right] =& \frac{1}{6}\zeta_n^3[p;q]\,\zeta_n[p+1;q] + \frac{1}{3}\zeta_n[p+1;q]\,\zeta_n\left[3p;q^3\right]\\
&+ \zeta_n[p;q]\,\zeta_n\left[3p+1;q^3\right] + \frac{1}{2}\zeta_n\left[2p;q^2\right]\zeta_n\left[2p+1;q^2\right]\\
&- \frac{1}{2}\zeta_n[p;q]\,\zeta_n[p+1;q]\,\zeta_n\left[2p;q^2\right] - \frac{1}{2}\zeta_n^2[p;q]\,\zeta_n\left[2p+1;q^2\right]\\
&- \zeta_n\left[4p+1;q^4\right],
\end{aligned}$$

$$\begin{aligned}\sum_{a+b=4}\zeta_n\left[\{p\}_a,p+1,\{p\}_b\right]=&\frac{1}{24}\zeta_n^4[p;q]\,\zeta_n[p+1;q]-\frac{1}{4}\zeta_n^2[p;q]\,\zeta_n[p+1;q]\,\zeta_n\left[2p;q^2\right]\\&-\frac{1}{6}\zeta_n^3[p;q]\,\zeta_n\left[2p+1;q^2\right]+\frac{1}{3}\zeta_n[p;q]\,\zeta_n[p+1;q]\,\zeta_n\left[3p;q^3\right]\\&+\frac{1}{2}\zeta_n^2[p;q]\,\zeta_n\left[3p+1;q^3\right]+\frac{1}{8}\zeta_n[p+1;q]\,\zeta_n^2\left[2p;q^2\right]\\&+\frac{1}{2}\zeta_n[p;q]\,\zeta_n\left[2p;q^2\right]\zeta_n\left[2p+1;q^2\right]-\frac{1}{4}\zeta_n[p+1;q]\,\zeta_n\left[4p;q^4\right]\\&-\zeta_n[p;q]\,\zeta_n\left[4p+1;q^4\right]-\frac{1}{3}\zeta_n\left[2p+1;q^2\right]\zeta_n\left[3p;q^3\right]\\&-\frac{1}{2}\zeta_n\left[2p;q^2\right]\zeta_n\left[3p+1;q^3\right]+\zeta_n\left[5p+1;q^5\right],\\\sum_{a+b=1}\zeta_n^\star\left[\{p\}_a,p+1,\{p\}_b\right]=&\zeta_n[p;q]\,\zeta_n[p+1;q]+\zeta_n\left[2p+1;q^2\right],\\\sum_{a+b=2}\zeta_n^\star\left[\{p\}_a,p+1,\{p\}_b\right]=&\frac{1}{2}\zeta_n^2[p;q]\,\zeta_n[p+1;q]+\frac{1}{2}\zeta_n[p+1;q]\,\zeta_n\left[2p;q^2\right]\\&+\zeta_n[p;q]\,\zeta_n\left[2p+1;q^2\right]+\zeta_n\left[3p+1;q^3\right],\\\sum_{a+b=3}\zeta_n^\star\left[\{p\}_a,p+1,\{p\}_b\right]=&\zeta_n^3[p;q]\,\zeta_n[p+1;q]+\frac{1}{3}\zeta_n[p+1;q]\,\zeta_n\left[3p;q^3\right]\\&+\zeta_n[p;q]\,\zeta_n\left[3p+1;q^3\right]+\frac{1}{2}\zeta_n\left[2p;q^2\right]\zeta_n\left[2p+1;q^2\right]\\&+\frac{1}{2}\zeta_n[p;q]\,\zeta_n[p+1;q]\,\zeta_n\left[2p;q^2\right]+\frac{1}{2}\zeta_n^2[p;q]\,\zeta_n\left[2p+1;q^2\right]\\&+\zeta_n\left[4p+1;q^4\right],\\\sum_{a+b=4}\zeta_n^\star\left[\{p\}_a,p+1,\{p\}_b\right]=&\frac{1}{24}\zeta_n^4[p;q]\,\zeta_n[p+1;q]+\frac{1}{4}\zeta_n^2[p;q]\,\zeta_n[p+1;q]\,\zeta_n\left[2p;q^2\right]\\&+\frac{1}{6}\zeta_n^3[p;q]\,\zeta_n\left[2p+1;q^2\right]+\frac{1}{3}\zeta_n[p;q]\,\zeta_n[p+1;q]\,\zeta_n\left[3p;q^3\right]\\&+\frac{1}{2}\zeta_n^2[p;q]\,\zeta_n\left[3p+1;q^3\right]+\frac{1}{8}\zeta_n[p+1;q]\,\zeta_n^2\left[2p;q^2\right]\\&+\frac{1}{2}\zeta_n[p;q]\,\zeta_n\left[2p;q^2\right]\zeta_n\left[2p+1;q^2\right]+\frac{1}{4}\zeta_n[p+1;q]\,\zeta_n\left[4p;q^4\right]\\&+\zeta_n[p;q]\,\zeta_n\left[4p+1;q^4\right]+\frac{1}{3}\zeta_n\left[2p+1;q^2\right]\zeta_n\left[3p;q^3\right]\\&+\frac{1}{2}\zeta_n\left[2p;q^2\right]\zeta_n\left[3p+1;q^3\right]+\zeta_n\left[5p+1;q^5\right],\end{aligned}$$

Example 3.2. *For positive integers n and m, we have*

$$\zeta_n^\star[\{1\}_m]=\sum_{r=1}^{n}(-1)^{r-1}q^{r(r+2m-1)/2}\frac{\begin{bmatrix}n\\r\end{bmatrix}_q}{[r]_q^m},$$

$$\zeta_n^\star[\{2\}_m]=\sum_{r=1}^{n}(-1)^{r-1}q^{r(r+2m-1)/2}(1+q^r)\frac{\begin{bmatrix}n\\r\end{bmatrix}_q}{[r]_q^{2m}\begin{bmatrix}n+r\\r\end{bmatrix}_q},$$

$$\begin{aligned}\sum_{a+b=m-1}\zeta_n^\star\left[\{1\}_a,2,\{1\}_b\right]=&m\sum_{r=1}^{n}(-1)^{r-1}q^{r(r+2m-1)/2}\frac{\begin{bmatrix}n\\r\end{bmatrix}_q}{[r]_q^{m+1}}\\&+\sum_{r=1}^{n}(-1)^{r-1}q^{r(r+2m+1)/2}\frac{\begin{bmatrix}n\\r\end{bmatrix}_q}{[r]_q^{m+1}}\\&-\sum_{r=1}^{n}(-1)^{r-1}q^{r(r+2m-1)/2}\frac{\zeta_n\left[1;q\right]\begin{bmatrix}n\\r\end{bmatrix}_q}{[r]_q^{m}},\\\sum_{a+b=m-1}\zeta_n^\star\left[\{2\}_a,3,\{2\}_b\right]=&m\sum_{r=1}^{n}(-1)^{r-1}q^{r(r+2m-1)/2}\left(1+q^r\right)\frac{\begin{bmatrix}n\\r\end{bmatrix}_q}{[r]_q^{2m+1}\begin{bmatrix}n+r\\r\end{bmatrix}_q}\\&+\sum_{r=1}^{n}(-1)^{r-1}q^{r(r+2m+1)/2}\frac{\begin{bmatrix}n\\r\end{bmatrix}_q}{[r]_q^{2m+1}\begin{bmatrix}n+r\\r\end{bmatrix}_q}\\&-\sum_{r=1}^{n}(-1)^{r-1}q^{r(r+2m-1)/2}\left(1+q^r\right)\frac{\left(\zeta_n\left[1;1\right]-\zeta_{n+r}\left[1;1\right]+\zeta_r\left[1;1\right]\right)\begin{bmatrix}n\\r\end{bmatrix}_q}{[r]_q^{2m}\begin{bmatrix}n+r\\r\end{bmatrix}_q}.\end{aligned}$$

Example 3.3. *For any real $p>1$, the following identities hold:*

$$\begin{aligned}\sum_{a+b=1}\zeta\left(\{p\}_a,p+1,\{p\}_b\right)=&\zeta(p)\zeta(p+1)-\zeta(2p+1),\\\sum_{a+b=2}\zeta\left(\{p\}_a,p+1,\{p\}_b\right)=&\frac{1}{2}\zeta^2(p)\zeta(p+1)-\frac{1}{2}\zeta(p+1)\zeta(2p)-\zeta(p)\zeta(2p+1)\\&+\zeta(3p+1),\\\sum_{a+b=3}\zeta\left(\{p\}_a,p+1,\{p\}_b\right)=&\frac{1}{6}\zeta^3(p)\zeta(p+1)+\frac{1}{3}\zeta(p+1)\zeta(3p)+\zeta(p)\zeta(3p+1)\\&+\frac{1}{2}\zeta(2p)\zeta(2p+1)-\frac{1}{2}\zeta(p)\zeta(p+1)\zeta(2p)\\&-\frac{1}{2}\zeta^2(p)\zeta(2p+1)-\zeta(4p+1),\\\sum_{a+b=4}\zeta\left(\{p\}_a,p+1,\{p\}_b\right)=&\frac{1}{24}\zeta^4(p)\zeta(p+1)-\frac{1}{4}\zeta^2(p)\zeta(p+1)\zeta(2p)-\frac{1}{6}\zeta^3(p)\zeta(2p+1)\\&+\frac{1}{3}\zeta(p)\zeta(p+1)\zeta(3p)+\frac{1}{2}\zeta^2(p)\zeta(3p+1)+\frac{1}{8}\zeta(p+1)\zeta^2(2p)\\&+\frac{1}{2}\zeta(p)\zeta(2p)\zeta(2p+1)-\frac{1}{4}\zeta(p+1)\zeta(4p)-\zeta(p)\zeta(4p+1)\\&-\frac{1}{3}\zeta(2p+1)\zeta(3p)-\frac{1}{2}\zeta(2p)\zeta(3p+1)+\zeta(5p+1),\end{aligned}$$

$$\sum_{a+b=1}\zeta^{\star}\left(\{p\}_a,p+1,\{p\}_b\right)=\zeta(p)\zeta(p+1)+\zeta(2p+1),$$

$$\sum_{a+b=2}\zeta^{\star}\left(\{p\}_a,p+1,\{p\}_b\right)=\frac{1}{2}\zeta^2(p)\zeta(p+1)+\frac{1}{2}\zeta(p+1)\zeta(2p)+\zeta(p)\zeta(2p+1)+\zeta(3p+1),$$

$$\begin{aligned}\sum_{a+b=3}\zeta^{\star}\left(\{p\}_a,p+1,\{p\}_b\right)=&\frac{1}{6}\zeta^3(p)\zeta(p+1)+\frac{1}{3}\zeta(p+1)\zeta(3p)+\zeta(p)\zeta(3p+1)\\&+\frac{1}{2}\zeta(2p)\zeta(2p+1)+\frac{1}{2}\zeta(p)\zeta(p+1)\zeta(2p)\\&+\frac{1}{2}\zeta^2(p)\zeta(2p+1)+\zeta(4p+1),\end{aligned}$$

$$\begin{aligned}\sum_{a+b=4}\zeta^{\star}\left(\{p\}_a,p+1,\{p\}_b\right)=&\frac{1}{24}\zeta^4(p)\zeta(p+1)+\frac{1}{4}\zeta^2(p)\zeta(p+1)\zeta(2p)+\frac{1}{6}\zeta^3(p)\zeta(2p+1)\\&+\frac{1}{3}\zeta(p)\zeta(p+1)\zeta(3p)+\frac{1}{2}\zeta^2(p)\zeta(3p+1)+\frac{1}{8}\zeta(p+1)\zeta^2(2p)\\&+\frac{1}{2}\zeta(p)\zeta(2p)\zeta(2p+1)+\frac{1}{4}\zeta(p+1)\zeta(4p)+\zeta(p)\zeta(4p+1)\\&+\frac{1}{3}\zeta(2p+1)\zeta(3p)+\frac{1}{2}\zeta(2p)\zeta(3p+1)+\zeta(5p+1).\end{aligned}$$

Example 3.4. *For positive integers m and n, it hold*

$$\zeta_n^{\star}(\{1\}_m)=\sum_{r=1}^{n}(-1)^{r-1}\frac{\binom{n}{r}}{r^m},$$

$$\zeta_n^{\star}(\{2\}_m)=2\sum_{r=1}^{n}(-1)^{r-1}\frac{\binom{n}{r}}{r^{2m}\binom{n+r}{r}},$$

$$\zeta_n^{\star}(\{3\}_m)=3\sum_{r=1}^{n}(-1)^{r-1}\frac{\binom{n}{r}}{r^{3m}\prod_{k=1}^{n}\left(1+\frac{r}{k}+\frac{r^2}{k^2}\right)},$$

$$\zeta_n^{\star}(\{4\}_m)=4\sum_{r=1}^{n}(-1)^{r-1}\frac{\binom{n}{r}}{r^{4m}\binom{n+r}{r}\prod_{k=1}^{n}\left(1+\frac{r^2}{k^2}\right)}.$$

$$\sum_{a+b=m-1}\zeta_n^{\star}(\{1\}_a,2,\{1\}_b)=(m+1)\sum_{r=1}^{n}(-1)^{r-1}\frac{\binom{n}{r}}{r^{m+1}}-\sum_{r=1}^{n}(-1)^{r-1}\frac{H_n\binom{n}{r}}{r^m},$$

$$\sum_{a+b=m-1} \zeta_n^{\star}(\{2\}_a, 3, \{2\}_b) = (2m+1)\sum_{r=1}^{n}(-1)^{r-1}\frac{\binom{n}{r}}{r^{2m+1}\binom{n+r}{r}} - 2\sum_{r=1}^{n}(-1)^{r-1}\frac{(H_n - H_{n+r} + H_r)\binom{n}{r}}{r^{2m}\binom{n+r}{r}}. \tag{3.1}$$

Letting $n \to \infty$ in (3.1), the result is

$$\sum_{a+b=m-1} \zeta^{\star}(\{2\}_a, 3, \{2\}_b) = (2m+1)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{2m+1}} - 2\sum_{n=1}^{\infty}\frac{H_n}{n^{2m}}(-1)^{n-1}. \tag{3.2}$$

Some results of Example 3.2 and 3.4 have been proven in [7, 13, 14].

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