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Pseudo-Quasi-Conformal Curvature Tensor and Spacetimes of General Relativity

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Abstract. In the present paper, we carried out a systematic investigation of pseudo-quasi-conformal curvature tensor has been made on the four-dimensional spacetime of general relativity. The spacetime fulfilling Einstein's field equations with vanishing of pseudo-quasi-conformal curvature tensor is being considered and existence of Killing and conformal Killing vectors on such spacetime have been established. At last, we extend the similar case for the investigation of cosmological models with dust and perfect fluid spacetime.

1. Introduction

The most modern approaches to mathematical general relativity begin with the concept of a manifold. After Riemannian manifolds, the structure of Lorentzian manifold is the most significant subclass of pseudo-Riemannian manifolds and these are important in applications of cosmology and general relativity. A principle basis of general relativity is that spacetime can be modeled as a 4-dimensional Lorentzian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. Because of the causality that the Lorentzian manifold becomes a convenient choice for the investigation of general relativity.

General relativity takes the structure of field equations, describing the curvature of spacetime and the distribution of matter throughout spacetime. The effects of matter and spacetime on one another are what we see as gravity. The hypothesis of the spacetime continuum already existed, but under general relativity Einstein was able to describe gravity as bending of spacetime geometry. Einstein defined a set of field equations, which represented the way that gravity behaved in response to matter in spacetime. These field equations could be utilized to represent the geometry of spacetime that was at the core of the hypothesis of general relativity. Since spacetime curve, the objects moving through space would follow the "straightest" path along the curve, which explains the motion of the planets. In general relativity the matter content

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of the spacetime is described by the energy-momentum tensor T which is to be determined from physical considerations dealing with the distribution of matter and energy.

The physical motivation for studying different kind of spacetime models in cosmology is to acquire the information of various phases in the development of the universe, which may be classified into three phases, namely, the initial phase, the intermediate phase and the final phase. The initial phase is just after the Big Bang when the effects of both viscosity and heat flux were quite pronounced. The intermediate phase is that when the effect of viscosity was no longer significant but the heat flux was still not negligible. The final phase, which extends to the present state of the universe when both the effects of viscosity and heat flux have become negligible. As it is well known the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models.

As we know that the symmetric spaces play an important role in differential geometry, the geometrical symmetries of the spacetime are expressible through vanishing of the Lie derivative of certain tensors with respect to a vector. These symmetries are also known as collineations were first introduced by Katzin, Levine and Devis ([10]). Further studies of collineations by Z. Ahsan ([24], [25]), M. Ali et al. ([15], [16]), N. A. Pundeer et al. ([19]), among many others. The spacetime symmetries are used in the study of exact solutions of Einstein's field equations in general relativity. A Killing vector field is one of the most important types of symmetries and is defined to be a smooth vector field that preserve the metric tensor.

We have all the tools needed to workout Einstein's field equation, which explains how to metric responds to energy and momentum. The Einstein's equations [2] (p. 337), imply that the energy-momentum tensor is of vanishing divergence. This requirement is satisfied if the energy-momentum tensor is covariant-constant [18]. In [18] M. C. Chaki and Sarbari Ray showed that a general relativistic spacetime with covariant constant energy-momentum tensor is Ricci symmetric, that is,$\nabla S = 0$, where S is the Ricci tensor of the spacetime. Also authors studied spacetimes in different way such as spacetimes with semisymmetric energy momentum tensor in [23], m-projectively semisymmetric at spacetimes by Zengin [7], m-projectively semisymmetric at spacetimes by Prakasha et al. [4], pseudo Z-symmetric spacetimes by Mantica and Suh [3], quasi-conformally, pseudo conhormonically symmetric spacetimes by F. O. Zengin and A. Y. Tasci [8] and pseudo-projectively at spacetimes by Mallick et al. (see, [21]-[22]) and many others.

The authors in [14] introduced and studied the notion of pseudo-quasi-conformal curvature tensor $\tilde{V}$ on a Riemannian manifold of dimension ($n \geq 3$) which includes the projective, quasi-conformal, Weyl conformal and concircular curvature tensor as spacial cases. This tensor is defined as:

$$\begin{aligned}\tilde{V}(X,Y)Z &= (p+d)R(X,Y)Z + (q - \frac{d}{n-1})[S(Y,Z)X - S(X,Z)Y] \\ &+ q[g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n(n-1)}\{p + 2(n-1)q\}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1}$$

where $X, Y, Z \in \chi(M)$, and p, q, d are real constants such that $p^2 + q^2 + d^2 > 0$. In particular, if
i) $p = q = 0,\ d = 1 \Rightarrow$ Projectively curvature,
ii) $p \neq 0,\ q \neq 0, d = 0 \Rightarrow$ Quasi-conformal curvature,
iii) $p = 1,\ q = -\frac{1}{n-2}, d = 0 \Rightarrow$ Conformal Curvature,
iv) $p = 1,\ q = d = 0 \Rightarrow$ Concircular curvature.

In 2005, Shaikh and Jana [1] introduced and studied a tensor field, called pseudo-quasi-conformal curvature tensor $\tilde{V}$ on a Riemannian manifold of dimension greater than or equal to 3. Recently, pseudo-quasi-conformal curvature tensor has been studied by many authors in various kind such as Kundu [20] and Prakasha et al., [5] and many others.

Our work structure as follows: After introduction, in Section 2, we characterize a spacetime with vanishing pseudo-quasi-conformal curvature tensor. The relativistic signification of this tensor has been explored in this paper and it is seen that a pseudo-quasi-conformally flat spacetime is an Einstein space and consequently a space of constant curvature. In this section, we prove one of the main results for the energy momentum tensor satisfying Einstein's field equations with cosmological constant is

Theorem 2.1. The energy momentum tensor of the pseudo-quasi-conformally flat spacetime (M^4, g) satisfying Einstein's field equation with a cosmological constant is covariantly constant.

Katzin et al., [10] were the pioneers in carrying out a detailed study of curvature collineation (CC), in the context of the related particle and field conservation laws that may be admitted in the standard form of general relativity. If a spacetime admitting the Killing vector field is curvature collineation, then we obtain the following result.

Theorem 2.2. A spacetime M admitting the pseudo-quasi-conformal curvature tensor with ξ as a Killng vector field is curvature collineation (CC), then the Lie derivative of the pseudo-quasi-conformal curvature tensor vanishes along the vector filed ξ.

Next, we prove the necessary and sufficient condition for a Killing and conformal Killing vector field, when a spacetime obeying Einstein's field equations has vanishing pseudo-quasi-conformally flat with the cosmological constant in the following result.

Theorem 2.3. A spacetime obeying Einstein's field equation has vanishing pseudo-quasi-conformally flat satisfying with cosmological constant admits:
(i) a conformal Killing vector field if and only if the energy momentum tensor has the symmetriy inheritance property.
(ii) a Killing vector field ξ if and only if Lie derivative of the energy momentum tensor with respect to ξ vanishes.

Section 3 deals with the study of perfect fluid cosmological models with vanishing pseudo-quasi-conformal curvature tensor. First we prove the main theorems for case of perfect fluid and secondly, for the case of dust cosmological models are written as:

Theorem 3.1. If a pseudo-quasi-conformally flat perfect fluid spacetime obeys Einstein's field equations without cosmological constant, then the norm of Ricci operator is $||QX||^2 = \frac{k^2(\sigma-3p)^2}{4}$.

Theorem 3.2. If a pseudo-quasi-conformally flat with dust cosmological model satisfying Einstein's field equations without cosmological constant, then the norm of Ricci operator is $||QX||^2 = \frac{k^2\sigma^2}{4}$.

2. Spacetime with vanishing pseudo-quasi-conformal curvature tensor

Let V_4 be the spacetime of general relativity, then from (1), we have

$$\begin{aligned}\tilde{V}(X,Y,Z,W) &= (p+d)R(X,Y,Z,W) + \left(q-\frac{d}{3}\right)[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ &+ q[g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] \\ &- \frac{r}{12}\{p+6q\}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned} \tag{2}$$

where $\tilde{V}(X,Y,Z,W) = g(\tilde{V}(X,Y)Z,W)$ and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$. If $\tilde{V}(X,Y,Z,W) = 0$, then equation (2) leads to

$$\begin{aligned}&(p+d)R(X,Y,Z,W) + (q-\frac{d}{3})[S(Y,Z)g(X,W) - S(X,Z)g(Y,W] \\ +\ & q[g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] \\ -\ & \frac{r}{12}\{p+6q\}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0. \end{aligned} \tag{3}$$

Taking a frame field over X and W, we have from equation (3)

$$(p+2q)S(Y,Z) = (p+2q)\frac{r}{4}g(Y,Z), \tag{4}$$

where S and r are denotes the Ricci tensor and the scalar curvature of the manifold, respectively. Thus we can state the following.

Proposition 2.1. *A pseudo-quasi-conformally flat spacetime is an Einstein spacetime, provided $p + 2q \neq 0$*

Again, from equation (3) and (4), we obtain

$$R(X, Y, Z, W) = \frac{r}{12}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \tag{5}$$

Thus, we can state the following.

Proposition 2.2. *A pseudo-quasi-conformally flat spacetime is a spacetime of constant curvature, provided $p+2q \neq 0$.*

In the subsequent, we shall determine some physical properties of pseudo-quasi-conformally flat spacetime satisfying Einstein's field equation with cosmological constant, that is, in this spacetime, pseudo-quasi-conformal curvature tensor vanishes, then we have from (5)

$$R(X, Y)Z = \frac{r}{12}[g(Y, Z)X - g(X, Z)Y]. \tag{6}$$

According to Karcher ([11]), a Lorentzian manifold is called infinitesimally spatially isotropic relative to a unit timelike vector field U if its Riemann curvature tensor R satisfies the relation

$$R(X, Y)Z = \delta\{g(Y, Z)X - g(X, Z)Y\},$$

for X, Y, $Z \in U^{\perp}$ and $R(X, U)U = \gamma X$ for $X \in U^{\perp}$ where δ and γ are real-valued functions on the manifold. Hence, we arrive at the following.

Proposition 2.3. *Every pseudo-quasi-conformally flat spacetime satisfying Einstein's field equation with the velocity vector field U is infinitesimally spatially isotropic relative to a unit timelike vector field U.*

Proof of Theorem 2.1. The Ricci tensor control the geometry of spacetime whereas energy momentum tensor T signifies the physical aspects of spacetime and in general relativity they are related by Einstein's field equations with cosmological constant [2]

$$S(X, Y) - \frac{r}{2}g(X, Y) + \lambda g(X, Y) = kT(X, Y), \tag{7}$$

for all vector fields X, Y, where S is the Ricci tensor of type $(0, 2)$, r is the scalar curvature, λ is cosmological constant, k is the gravitational constant and T is the energy momentum tensor of type $(0, 2)$.
Now, comparing (4) with (7), we find

$$T(X, Y) = \frac{1}{k}\left[\lambda - \frac{r}{4}\right]g(X, Y). \tag{8}$$

Taking covariant derivative of (8), we obtain

$$(\nabla_Z T)(X, Y) = -\frac{1}{k}dr(Z)g(X, Y). \tag{9}$$

Since pseudo-quasi-conformally flat spacetime is Einstein, therefore the scalar curvature r is constant. Hence

$$dr(Z) = 0, \tag{10}$$

for all X. Finally, using (10) in (9), we get

$$(\nabla_Z T)(X, Y) = 0. \tag{11}$$

The above equation shows that the energy momentum tensor is covariantly constant.

Proof of Theorem 2.2. The geometrical symmetries of a spacetime are expressed through the equation

$$\pounds_\xi A - 2\Omega A = 0, \tag{12}$$

where A represents a geometrical/physical quantity, $£_\xi$ denotes the Lie derivative with respect to the vector field and Ω is a scalar.

One of the most simple and widely used example is the metric inheritance symmetry for which $A = g_{ij}$ in (12), that is,

$$(£_\xi g)(X, Y) = 2\Omega g(X, Y). \tag{13}$$

In this case, ξ is the Killing vector field if Ω is zero and is the conformal Killing vector field if Ω is a scalar. A spacetime M^4 is said to admit a symmetry called a curvature collineation (CC) provided there exists a vector field ξ such that (see, [12]-[13])

$$(£_\xi R)(X, Y)Z = 0, \tag{14}$$

where R is the Riemannian curvature tensor.

Now we shall investigate the role of such symmetry inheritance for the spacetime admitting pseudo-quasi-conformal curvature tensor.

Let us consider a spacetime admitting pseudo-quasi-conformal curvature tensor with a Killing vector field is a CC. Then we have

$$(£_\xi g)(X, Y) = 0. \tag{15}$$

Again, since M admitting a CC, we have from (14)

$$(£_\xi S)(X, Y) = 0, \tag{16}$$

where S is the Ricci tensor of the manifold.

By taking Lie derivative of (1) and using (14), (15) and (16), we obtain

$$(£_\xi \tilde{V})(X, Y)Z = 0. \tag{17}$$

Hence, completes the proof.

Proof of Theorem 2.3. The existence of Killing and conformal Killing vector fields has been established. By Einstein's field equations (4) and (7), we have

$$\left(\lambda - \frac{r}{4}\right)g(X, Y) = kT(X, Y). \tag{18}$$

Taking Lie derivative of both sides of (18) and remembering that the scalar curvature r is constant, we get

$$\left(\lambda - \frac{r}{4}\right)(£_\xi g)(X, Y) = k(£_\xi T)(X, Y). \tag{19}$$

Now, if ξ is comformal Killing vector field, from (13) and (19), we get

$$2(\lambda - \frac{r}{4})\Omega g(X, Y) = k(£_\xi T)(X, Y), \tag{20}$$

where Ω is a scalar function. In a pseudo-quasi-conformally flat spacetime satisfying's field equation with cosmological constant, from (18) and (20), as $k \neq 0$, we get

$$(£_\xi T)(X, Y) = 2\Omega T(X, Y), \tag{21}$$

It follows from (12) that the energy momentum tensor has symmetric inheritance property.

Conversely, if (21) holds, then it follows that (13) holds for some scalar function Ω, that is ξ is a conformal Killing vector field. If we take $\Omega = 0$, then from (20) and (21), a necessary and sufficient condition for ξ to be a Killing vector field is that Lie derivative of the energy momentum tensor with respect to ξ be zero.

Thus completes the proof.

3. Cosmological models with vanishing pseudo-quasi-conformal curvature tensor

In this section we consider a perfect fluid spacetime with vanishing pseudo-quasi-conformal curvature tensor satisfying Einstein's field equation without cosmological constant.
Proof of Theorem 3.1 In a perfect-fluid spacetime, the energy momentum tensor is of the form [2]

$$T(X, Y) = pg(X, Y) + (\sigma + p)A(X)A(Y), \tag{22}$$

where σ, p are the energy density and the isotropic pressure, respectively, and V is the unit timelike flow vector field of the fluid such that

$$g(V, V) = -1, \quad A(X) = g(X, V),$$

for all X. In addition $\sigma + p \neq 0$.

Einstein's field equation without cosmological constant is given by

$$S(X, Y) - \frac{r}{2}g(X, Y) = kT(X, Y). \tag{23}$$

In virtue of equation (22) and (23), we get

$$S(X, Y) - \frac{r}{2}g(X, Y) = k[pg(X, Y) + (\sigma + p)A(X)A(Y)]. \tag{24}$$

Taking $X = Y = e_i$ in (24) and taking summation over i, $1 \leq i \leq 4$, we obtain

$$r = k(\sigma - 3p). \tag{25}$$

In virtue of (4) and (25) the Ricci tensor of a pseudo-quasi-conformally flat spacetime can be written as

$$S(X, Y) = \frac{k(\sigma - 3p)}{4}g(X, Y). \tag{26}$$

Let Q be the Ricci operator is given by

$$g(QX, Y) = S(X, Y) \quad \textit{and} \quad S(QX, Y) = S^2(X, Y). \tag{27}$$

Then we obtain

$$A(QX) = g(QX, V) = S(X, V).$$

Hence, we obtain from (25) that

$$S(QX, Y) = \frac{k^2(\sigma - 3p)^2}{16}g(X, Y). \tag{28}$$

Taking a frame field after contraction over X and Y, we obtain from (28) that

$$||QX||^2 = \frac{k^2(\sigma - 3p)^2}{4}. \tag{29}$$

This establishes the proof.

Proof of Theorem 3.2. The energy-momentum tensor in case of dust model is given by

$$T(X, Y) = \sigma A(X)A(Y), \tag{30}$$

where σ is the energy density and V is the time-like unit flow vector field of the fluid such that

$$g(V, V) = -1 \qquad A(X) = g(X, V), \quad \textit{for all } X.$$

Now, making use of equation (23) and equation (30), we get

$$S(X,Y) - \frac{r}{2}g(X,Y) = k\sigma A(X)A(Y). \tag{31}$$

Putting $X = Y = e_i$ in equation (31), where e_i is the orthonormal basis of the tangent space at each point of the manifold and taking summation over $1 \leq i \leq 4$,

$$r = k\sigma. \tag{32}$$

In view of equations (4) and (32), the Ricci tensor of a pseudo-quasi-conformally flat spacetime may be written as

$$S(X,Y) = \frac{k\sigma}{4}g(X,Y). \tag{33}$$

By virtue of equations (27) and (33) leads to

$$S(QX,Y) = \frac{k^2\sigma^2}{16}g(X,Y). \tag{34}$$

Again, putting $X = Y = e_i$ in equation (34), where e_i is the orthonormal basis of the tangent space at each point of the manifold and taking summation over $1 \leq i \leq 4$, we get

$$||QX||^2 = \frac{k^2\sigma^2}{4}. \tag{35}$$

Thus completes the proof.

Also, with help of (4) and (24), the Einstein's field equations are found as in the following

$$\left[\frac{r}{4} + kp\right]g(X,Y) = -k(\sigma + p)A(X)A(Y). \tag{36}$$

If we set $X = Y = V$ in (36) and using (25), we get

$$\sigma + p = 0, \tag{37}$$

which means that either $\sigma = 0$, $p = 0$ (empty spacetime) or the perfect fluid spacetime satisfies the vacuum-like equation of state [6].

Thus we have the following result.

Theorem 3.3. If a perfect fluid spacetime wih vanishing pseudo-quasi-conformally flat satisfying Einstein's field equation without cosmological constant, then the matter content of M^4 obey the vacuum-like equation of state.

Now, from equations (4), (23) takes the form

$$\frac{r}{4}g(X,Y) = -kT(X,Y). \tag{38}$$

On comparing (37) with (22), we find that

$$T(X,Y) = pg(X,Y). \tag{39}$$

Using (38) and (39), we have

$$\left[\frac{r}{4} + kp\right]g(X,Y) = 0. \tag{40}$$

Taking $X = Y = e_i$ in (40) and summing over, i, $1 \leq i \leq 4$, we get that

$$r = -4kp. \tag{41}$$

In this case, we can say that both the energy density and the isotropic pressure of this space are constants. Since the scalar curvature r of a pseudo-quasi-conformally flat space is constant. Also, M Ali and N. A. Pundeer ([17]) studied the semiconformal curvature tensor in detail and proved that the scalar curvature r of semiconformally flat spacetime is constant. From (41) it follows that isotropic pressure p is a constant and hence from (37), we obtain energy density is a constant which means the fluid behaves as a cosmological constant.

Thus we may state the following.

Theorem 3.4. If a perfect fluid spacetime with vanishing pseudo-quasi-conformally flat satisfying Einstein's field equation without cosmological constant, then the spacetime has constant energy density and isotropic pressure and the spacetime represents inflation and also the fluid behaves as cosmological constant.

Now, making use of equations (4) and (31), we get

$$\frac{r}{4}g(X,Y) = -k\sigma A(X)A(Y). \tag{42}$$

Putting $X = Y = e_i$ in equation (42), where e_i is the orthonormal basis of the tangent space at each point of the manifold and taking summation over $1 \leq i \leq 4$, we get

$$r = k\sigma. \tag{43}$$

Multiplying by $A(Z)$ in equation (42) and then contracting over X and Z, we get

$$r = 4k\sigma. \tag{44}$$

From (43) and (44), we have

$$\sigma = 0. \tag{45}$$

Thus, we may establishes the following result.

Theorem 3.5. A spacetime satisfying Einstein's field equation without cosmological constant and having vanishing pseudo-quasi-conformal curvature tensor represents a dust cosmological model, if the energy density does not vanish.

Furthermore, for a spacetime with radiating perfect fluid the resulting universe be isotropic and homogeneous [9].

Now, making use equations (4) and (7), we get

$$\left(\lambda - \frac{r}{4}\right)g(Y,Z) = kT(Y,Z) \tag{46}$$

In view of equation (22), equation (46) leads to

$$\left(\lambda - \frac{r}{4} - kp\right)g(Y,Z) = k(\sigma + p)A(Y)A(Z). \tag{47}$$

Thus, using the condition of a spacetime with radiative perfect fluid, i.e., $\sigma = 3p$ in equation (47), we have

$$\left(\lambda - \frac{r}{4} - \frac{k\sigma}{3}\right)g(Y,Z) = \frac{4\sigma k}{3}A(Y)A(Z). \tag{48}$$

Putting $X = Y = e_i$ in equation (48), where e_i is the orthonormal basis of the tangent space at each point of the manifold and taking summation over $1 \leq i \leq 4$, we get

$$r = 4\lambda. \tag{49}$$

Mulltiplying by $A(X)$ on both side in equation (48) and then contracting over X & Y, We get

$$r = 4(\lambda + k\sigma) \tag{50}$$

In view of equations (49) and (50), we have

$$\sigma = 0.$$

which is not possible by our assumption.

Thus, we may state the following

Theorem 3.6. A spacetime with vanishing pseudo-quasi-conformal curvature tensor and satisfying Einstein's field equation with cosmological constant is an isotropic and homogeneous spacetime if energy density of the fluid does not vanish.

References

[1] A. A. Shaikh and S. K. Jana, A pseudo-quasi-conformal curvature tensor on a Riemannian manifolds, South East Asian J. Math. Math. Sci.,**4** (2005), 15-20.

[2] B. O'Neill, Semi-Riemannian Geometry, Academic Press, New York, (1983).

[3] C. A. Mantica and Y. J. Suh, Pseudo-Z symmetric space-times, J. Math. Phys., **55** (2014), no.4, 12.

[4] D. G. Prakasha, F. O. Zengin and V Chavan, On *M*-projectively semisymmetric Lorentzian α-Sasakian Manifolds, Africa Mathematica. DOI 10.1007/s13370-017-0493-9.

[5] D. G. Prakasha, T. R. Shivamurthy and K. K. Mirgi, On pseudo-quasi-conformal curvature tensor of P-sasakian manifolds, Elect. J. Math. Anal. Appl., **5(2)** 2017, 147-155.

[6] D. Kallingas, P. Wesson and C. W. F. Everitte, Flat FRW models with variable G and $\wedge$, Gen. Relativity Gravitation, **24(4)** (1992), 351-357.

[7] F. O. Zengin, *M*-projectively flat spacetimes, Math. Report., **4** (2012), 363-370.

[8] F. O. Zengin and A. Y. Tasci, Pseudo conhormonically symmetric spacetimes, Kragujevac J. Math., **42(4)** (2018), 619-630.

[9] G. F. R. Ellis, Relativistic Cosmology in General Relativity and Cosmology, Edited by R.K. Sachs, Academic Press, London, (1971).

[10] G. H. Katzin, J. Levine and W. R. Davis, Curvature collineations: A fundamental symmetry property of the spacetime of general relativity defined by the vanishing Lie derivative of the Riemannian curvature tensor, J. Math. Phys., **10** (1969) 617-629.

[11] H. Karcher, Infinitesimal characterization of Friedmann universes, Archive Math., **38(1)** (1982), 58-64.

[12] K. L. Duggal, Curvature Collineations and Conservation Laws of General Relativity, presented at the Canadian conference on general relativity and relativistic Astro-physics, Halifax, Canada, (1985).

[13] K. L. Duggal, Curvature inheritance symmetry in Riemannian spaces with applications to fluid space times, J. Math. Phys. **33** (1992), no. 9, 2989-2997.

[14] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, J. Differential Geom., **2**, (1968), 161-184.

[15] M. Ali and Z. Ahsan, Ricci solitons and symmetries of spacetime manifold of general relativity, Global J. Adv. Res. Class. Mod. Geom., **1(2)** (2012), 76-85.

[16] M. Ali, N. A. Pundeer and Y.J. Suh, Proper semiconformal symmetries of spacetimes with divergence-free semiconformal curvature tensor, Filomat, **33(16)** (2019), 5191-5198.

[17] M. Ali and N. A. Pundeer, Semiconformal curvature tensor and Spacetime of General Relativity, Differential Geometry-Dynamical Systems,**21** (2019), 14-22.

[18] M. C. Chaki and S. Roy, Space-times with covariant-constant energy-momentum tensor, Int. J. Theor. Phys., **35** (1996), no. 5, 1027-1032.

[19] N. A. Pundeer, M. Ali, N. Ahmad and Z. Ahsan, Semiconformal symmetry-A new symmetry of the spacetime manifold of the general relativity, J. Math. Comp. Sci., **20(3)** (2020), 241-254.

[20] S. Kundu, On P-Sasakian manifolds, Mathe. Report., **15(65)** (2013), 221-232.

[21] S. Mallick, P. Zhao and U. C. De, Spacetimes admitting quasi-conformal curvature tensor, Bull. Iranian Math. Soc., **42** (2016), 1535-1546.

[22] S. Mallick, Y. J. Suh and U. C. De, A Spacetimes admitting pseudo-projective curvature tensor, J. Math. Phys., **57** (2016), .062501.245-252.

[23] U. C. De and L. Velimirovič, Spacetimes with semisymmetric energy momentum tensor, Int. J. Theor. Phys., **54** (2015), no. 6, 1779-1783.

[24] Z. Ahsan, Symmetries of the electromagnetic fields in general relativity, Acta Phys. Sinica, **4(5)** (1995), 337.
[25] Z. Ahsan, A symmetry property of the spacetime of general relativity in terms of space-matter tensor, Brazilian. J. Phys., **26(3)** (1996), 572-576.