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Multipliers and Closures of Besov-Type Spaces in the Bloch Space

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Abstract. Let $p > 1$ and let ρ be a non-negative function defined in $\mathbb{R}^+$. A function $f \in H(\mathbb{D})$ belongs to the space $B_p(\rho)$ (see [4]) if

$$\|f\|^p_{B_p(\rho)} = |f(0)|^p + \int_{\mathbb{D}} \left|(1-|z|^2)f'(z)\right|^p \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z) < \infty.$$

In this paper, motivated by the works of Békollé and Bao and Göğüs, under some conditions on the weight function ρ, we investigate the closures $C_{\mathcal{B}}(\mathcal{B} \cap B_p(\rho))$ of the spaces $\mathcal{B} \cap B_p(\rho)$ in the Bloch space. Moreover we prove that interpolating Blaschke products in $C_{\mathcal{B}}(\mathcal{B} \cap B_p(\rho))$ are multipliers of $B_p(\rho) \cap BMOA$.

1. Introduction

We denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{D}$ and its boundary $\{z \in \mathbb{C} : |z| = 1\}$ by $\partial\mathbb{D}$. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$.

Let H^p (see [11]) denote the space of those analytic functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{H^p} = \sup_{0<r<1} M_p^p(r, f) = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let $BMOA$ denote the space of those analytic functions f in the Hardy space H^p whose boundary functions have bounded mean oscillation on $\partial\mathbb{D}$. $BMOA$ ([17, 19]) is a Banach space under the following norm:

$$\|f\|_{BMOA} = |f(0)|^p + \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|^p_{H^p},$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, a, z \in \mathbb{D}$ and $p \geq 1$.

Recall that the Bloch space ([2, 34]) is the class of functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

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Let $p > 1$ and let ρ be a non-negative function defined in $\mathbb{R}^+$. A function $f \in H(\mathbb{D})$ belongs to the space $B_p(\rho)$ if

$$\|f\|^p_{B_p(\rho)} = |f(0)|^p + \int_{\mathbb{D}} \left|(1-|z|^2)f'(z)\right|^p \frac{\rho\left(1-|z|^2\right)}{(1-|z|^2)^2} dA(z) < \infty,$$

where $dA(z)$ is the usual normalized Lebesgue measure on $\mathbb{D}$. This space is introduced by Arcozzi, Rochberg and Sawyer in [4]. They considered Carleson measures for $B_p(\rho)$ spaces under the condition that the weight function ρ is p-admissible, or admissible, that is, ρ satisfies the following conditions:

(i) $\rho(z)$ is regular, i.e., there exist $\epsilon > 0$, $C > 0$ such that $\rho(z) \le C\rho(w)$ when z and w are within hyperbolic distance ϵ. Equivalently, there are $\delta < 1$, $C' > 0$ so that $\rho(z) \le C'\rho(w)$ whenever

$$\left|\frac{z-w}{1-\bar{z}w}\right| \le \delta < 1.$$

(ii) The weight $\rho_p(z) = (1-|z|^2)^{p-2}\rho(z)$ satisfies the Békollé-Bonami $\mathcal{B}_p$ condition([7, 8]): There is a $C(\rho, p)$ so that for all $a \in \mathbb{D}$ we have

$$\left(\int_{S(a)} \rho_p(z) dA(z)\right)\left(\int_{S(a)} \rho_p(z)^{1-q} dA(z)\right)^{1/(q-1)} \le \quad C(\rho, p)\left(\int_{S(a)} dA(z)\right)^p.$$

Where $1/p + 1/q = 1$, and

$$S(a) = \{z \in \mathbb{D} : 1-|z| \le 1-|a|, \left|\frac{\arg(a\bar{z})}{2\pi}\right| \le \frac{1-|a|}{2}\}, \quad a \in \mathbb{D}.$$

In the case $\rho(t) = t^s$, $0 \le s < \infty$, the space $B_p(\rho)$ gives the usual Besov type space $B_p(s)$. In particular, if $s = 0$, this gives the classical Besov space B_p. We refer to [5], [9], [10] and [12] for $B_p(s)$ spaces and [30], [31] and [32] for $B_2(s) = D_s$ spaces. The space $B_p(\rho)$ has been extensively studied. For example, under some conditions on ρ, N. Arcozzi, R. Kerman and E. Sawyer [4] give many results on $B_p(\rho)$ spaces using Carleson measures. When $p = 2$, $B_2(\rho) = D_\rho$, weighted Dirichlet spaces. Using maximal operators, R. Kerman and E. Sawyer [21] characterized the Carleson measures and multipliers of D_ρ spaces. For more informations on D_ρ spaces, we refer to [1] and the paper referinthere.

Let us recall that a weight ρ is of upper (resp.lower) type γ $(0 \le \gamma < \infty)$ ([20]), if

$$\rho(st) \le Cs^\gamma \rho(t), \;\; s \ge 1 \;\; (\text{resp.} s \le 1) \;\text{ and }\; 0 < t < \infty.$$

We say that ρ is of upper type less than γ if it is of upper type δ for some $\delta < \gamma$ and similarly for lower type greater than δ. From [20], we see that an increasing function ρ is of finite upper type if and only if $\rho(2t) \le C\rho(t)$ for some positive constant C and all t. It is not hard to verify that ρ satisfies (i) and (ii), if ρ is of upper type less than 1 and lower type greater than 0.

In [2], Anderson, Clunie and Pommerenke raised the question of determining the closure of H^∞ in the Bloch norm. Until now, the problem is still unsolved. Jones [3, Theorem 9] gave an unpublished characterization of the closure of BMOA in Bloch space. Zhao [33] studied the closures of some Möbius invariant spaces in the Bloch space. Lou and Chen [22] generalize [33] later. For $1 < p < \infty$, Monreal Galán and Nicolau in [24] characterized the closure in the Bloch norm of the space $H^p \cap \mathcal{B}$. Galanopoulos, Monreal Galán and Pau [16] generalize [24] to $0 < p < \infty$ later. Recently, Bao and Göğüs [6] and Galanopoulos and Girela [15] have investigated the closures in $\mathcal{B}$ of $\mathcal{B} \cap D^p_\alpha$ for certain spaces of Dirichlet Type D^p_α. For more results on closures of analytic function spaces in the Bloch space, we refer to [28] and [29]. In this paper, we study the closures of the $B_p(\rho)$ spaces, generalizing the main results in [6] and [15]. Meanwhile, interpolating Blaschke products in $C_{\mathcal{B}}(\mathcal{B} \cap B_p(\rho))$ as multipliers of $B_p(\rho) \cap BMOA$ are also investigated.

Throughout this paper, let $\rho : [0, \infty) \to [0, \infty)$ be a right continuous and nondecreasing function with $\rho(0) = 0$ and $\rho(t) > 0$ if $t > 0$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a constant C such that $A \le CB$.

Remark 1. *Using [20, Lemma 4], the fact that ρ is increasing, and the above mentioned fact that ρ is of finite upper if and only if $\rho(2t) \le C\rho(t)$ $(t \ge 0)$, we deduce the following:*

If ρ is of finite upper type, then ρ is of upper type less than p if and only if

$$\frac{\rho(t)}{t^p} \approx \int_t^\infty \rho(s)\frac{ds}{s^{1+p}}.$$

Remark 2. *Let $0 \le a < 1$ and $p > 1$. If ρ is of finite upper type a, we can deduce that $B_p(\rho) \subseteq H^p$. Indeed, take b with $a < b < 1$, using Remark 1, we deduce that*

$$\begin{aligned}
&\int_{\mathbb{D}} \left|f'(z)\right|^p (1-|z|^2)^{p-2}\rho\left(1-|z|^2\right) dA(z)\\
=&\int_{\mathbb{D}} \left|f'(z)\right|^p (1-|z|^2)^{p-2}\frac{\rho\left(1-|z|^2\right)}{(1-|z|^2)^b}\left(1-|z|^2\right)^b dA(z)\\
\approx&\int_{\mathbb{D}} \left|f'(z)\right|^p (1-|z|^2)^{p-2}\left(\int_{1-|z|^2}^\infty \rho(s)\frac{ds}{s^{1+b}}\right)\left(1-|z|^2\right)^b dA(z)\\
\gtrsim&\int_{\mathbb{D}} \left|f'(z)\right|^p (1-|z|^2)^{p-2}\left(\int_1^\infty \rho(s)\frac{ds}{s^{1+b}}\right)\left(1-|z|^2\right)^b dA(z)\\
\approx&\rho(1)\int_{\mathbb{D}} \left|f'(z)\right|^p (1-|z|^2)^{p-2}\left(1-|z|^2\right)^b dA(z).
\end{aligned}$$

Thus, $B_p(\rho) \subseteq B_p(b)$. Then the inclusion $B_p(\rho) \subseteq H^p$ follows from the well known fact that $B_p(b) \subseteq H^p$ because $0 < b < 1$.

2. Equivalent Characterizations of closures of $B_p(\rho)$ spaces in Bloch space

Theorem 1. *Let ρ be of finite lower type greater than 0 and upper type less than 1. Suppose that $1 < p < \infty$. Then the following conditions are equivalent.*

(1) $f \in C_{\mathcal{B}}(B_p(\rho) \cap \mathcal{B})$.

(2) For any $\epsilon > 0$,

$$\int_{\Omega_\epsilon(f)} \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z) < \infty,$$

where

$$\Omega_\epsilon(f) = \{z \in \mathbb{D} : (1-|z|^2)|f'(z)| \ge \epsilon\}.$$

(3) For $\epsilon > 0$ and $s > 1$,

$$\int_{\Gamma_{p,\epsilon}(f)} \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z) < \infty,$$

where

$$\Gamma_{p,\epsilon}(f) = \left\{z \in \mathbb{D} : \int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^{p-2}(1-|\varphi_z(w)|^2)^s dA(w) \ge \epsilon\right\}.$$

Proof. (2) $\Rightarrow$ (1). Following [33], without loss of generality, we may assume that $f(0) = 0$. By Proposition 4.27 in [34], we have that

$$f(z) = \frac{1}{(\alpha+1)}\int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)^{1+\alpha}}{\overline{w}(1-z\overline{w})^{2+\alpha}} dA(w), \ \ z \in \mathbb{D},$$

where $\alpha > 0$. Set $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{(\alpha+1)} \int_{\Omega_\epsilon(f)} \frac{f'(w)(1-|w|^2)^{1+\alpha}}{\overline{w}(1-z\overline{w})^{2+\alpha}} dA(w)$$

and

$$f_2(z) = \frac{1}{(\alpha+1)} \int_{\mathbb{D}\setminus\Omega_\epsilon(f)} \frac{f'(w)(1-|w|^2)^{1+\alpha}}{\overline{w}(1-z\overline{w})^{2+\alpha}} dA(w).$$

Clearly,

$$|f_1'(z)| \lesssim \int_{\Omega_\epsilon(f)} \frac{|f'(w)|(1-|w|^2)^{1+\alpha}}{|1-z\overline{w}|^{3+\alpha}} dA(w)$$

and

$$|f_2'(z)| \lesssim \int_{\mathbb{D}\setminus\Omega_\epsilon(f)} \frac{|f'(w)|(1-|w|^2)^{1+\alpha}}{|1-z\overline{w}|^{3+\alpha}} dA(w).$$

Let $F = f_1 - f_1(0)$. Then $F(0) = 0$ and

$$\begin{aligned}
&\|f-F\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f_2'(z)| \\
\lesssim\ & \sup_{z\in\mathbb{D}}(1-|z|^2) \int_{\mathbb{D}\setminus\Omega_\epsilon(f)} \frac{|f'(w)|(1-|w|^2)^{1+\alpha}}{|1-z\overline{w}|^{3+\alpha}} dA(w) \\
\lesssim\ & \epsilon \sup_{z\in\mathbb{D}}(1-|z|^2) \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha}}{|1-z\overline{w}|^{3+\alpha}} dA(w).
\end{aligned}$$

Using [34, Lemma 3.10] with $t = \alpha$ and $c = 1$, we see that $\|f - F\|_{\mathcal{B}} \lesssim \epsilon$. It remains to prove that $F \in B_p(\rho)$. Using Fubini's theorem, we have

$$\begin{aligned}
& \int_{\mathbb{D}} |F'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) \\
=\ & \int_{\mathbb{D}} |f_1'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) \\
\le\ & \|f_1\|_{\mathcal{B}}^{p-1} \int_{\mathbb{D}} |f_1'(z)|(1-|z|^2)^{-1} \rho(1-|z|^2) dA(z) \\
\lesssim\ & \int_{\mathbb{D}} (1-|z|^2)^{-1}\rho(1-|z|^2) \left[\int_{\Omega_\epsilon(f)} \frac{|f'(w)|(1-|w|^2)^{1+\alpha}}{|1-z\overline{w}|^{3+\alpha}} dA(w) \right] dA(z) \\
\lesssim\ & \int_{\Omega_\epsilon(f)} |f'(w)|(1-|w|^2)^{1+\alpha} \left[\int_{\mathbb{D}} \frac{\rho(1-|z|^2)}{|1-z\overline{w}|^{\alpha+3}(1-|z|^2)} dA(z) \right] dA(w) \\
\lesssim\ & \|f\|_{\mathcal{B}} \int_{\Omega_\epsilon(f)} (1-|w|^2)^{\alpha} \left[\int_{\mathbb{D}} \frac{\rho(1-|z|^2)}{|1-z\overline{w}|^{\alpha+3}(1-|z|^2)} dA(z) \right] dA(w).
\end{aligned}$$

Since ρ is of finite lower type greater than 0 and upper type less than 1, there exist γ and δ with $0 < \gamma < \delta < 1$, such that

$$\rho(st) \lesssim s^{\gamma} \rho(t), \quad s \le 1 \qquad (A)$$

and

$$\rho(st) \lesssim s^{\delta} \rho(t), \quad s \ge 1, \qquad (B)$$

where $0 < t < \infty$. Using this and [34, Lemma 3.10], we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{\rho(1-|z|^2)}{|1-z\overline{w}|^{\alpha+3}(1-|z|^2)} dA(z) \\
= \;& \rho(1-|w|^2) \int_{\mathbb{D}} \frac{\frac{\rho(1-|z|^2)}{\rho(1-|w|^2)}}{|1-z\overline{w}|^{\alpha+3}(1-|z|^2)} dA(z) \\
\lesssim \;& \rho(1-|w|^2) \int_{\mathbb{D}} \frac{\left(\frac{(1-|z|^2)}{(1-|w|^2)}\right)^{\gamma} + \left(\frac{(1-|z|^2)}{(1-|w|^2)}\right)^{\delta}}{|1-z\overline{w}|^{\alpha+3}(1-|z|^2)} dA(z) \\
\lesssim \;& \frac{\rho(1-|w|^2)}{(1-|w|^2)^{\alpha+2}}.
\end{aligned}$$

Combining this with (2), we have

$$\int_{\mathbb{D}} |F'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) \lesssim \int_{\Omega_\epsilon(f)} \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z) < \infty.$$

Hence, $F \in B_p(\rho)$. This finishes the proof.

(1) ⇒ (3). It is well known that $\|f\|_{\mathcal{B}}$ is equivalent to

$$|||f|||_{\mathcal{B}} = |f(0)| + \left(\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2} (1-|\varphi_z(w)|^2)^s dA(w) \right)^{1/p},$$

where $p > 1$ and $s > 1$. Let $f \in C_{\mathcal{B}}(B_p(\rho) \cap \mathcal{B})$. Then for any $\epsilon > 0$, there exists $g \in B_p(\rho) \cap \mathcal{B}$ such that $|||f-g|||_{\mathcal{B}} \le \frac{\epsilon}{2C}$. For any $z \in \mathbb{D}$, we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2} (1-|\varphi_z(w)|^2)^s dA(w) \\
\le \;& C \int_{\mathbb{D}} |f'(w)-g'(w)|^p (1-|w|^2)^{p-2} (1-|\varphi_z(w)|^2)^s dA(w) + C \int_{\mathbb{D}} |g'(w)|^p (1-|w|^2)^{p-2} (1-|\varphi_z(w)|^2)^s dA(w).
\end{aligned}$$

Thus, $\Gamma_{p,\epsilon}(f) \subseteq \Gamma_{p,\frac{\epsilon}{2C}}(g)$. Note that

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}. \tag{C}$$

Using Fubini's theorem, we have

$$\begin{aligned}
&\int_{\Gamma_{p,\epsilon}(f)} \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z) \\
\le & \frac{2^p C^p}{\epsilon^p} \int_{\Gamma_{p,\frac{\epsilon}{2C}}(g)} (1-|z|^2)^{s-2} \rho(1-|z|^2) \left[\int_{\mathbb{D}} |g'(w)|^p (1-|w|^2)^{p-2} \frac{(1-|w|^2)^s}{|1-\bar{z}w|^{2s}} dA(w) \right] dA(z) \\
\lesssim & \int_{\mathbb{D}} |g'(w)|^p (1-|w|^2)^{p-2+s} \left[\int_{\mathbb{D}} \frac{(1-|z|^2)^{s-2} \rho(1-|z|^2)}{|1-\bar{z}w|^{2s}} dA(z) \right] dA(w).
\end{aligned}$$

Combining (A) with (B), we deduce that

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{s-2} \rho(1-|z|^2)}{|1-\bar{z}w|^{2s}} dA(z) \lesssim \frac{\rho(1-|w|^2)}{(1-|w|^2)^s}, \quad s > 1.$$

Thus,

$$\int_{\Gamma_{p,\epsilon}(f)} \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z) \lesssim \int_{\mathbb{D}} |g'(w)|^p (1-|w|^2)^{p-2} \rho(1-|w|^2) dA(w) < \infty.$$

(3) ⇒ (2). Let $E(z,1/2)=\{w\in\mathbb{D}:|\varphi_z(w)|<1/2\}$ be a pseudo-hyperbolic disk of center $z\in\mathbb{D}$ and radius 1/2. Recall that

$$1-|w|\approx|1-\overline{z}w|\approx 1-|z|,\quad w\in E(z,1/2)$$

and $|E(z,1/2)|\approx(1-|z|)^2$ (see [34, Page 69]). Using the subharmonicity of $|f'|^p$, we obtain

$$\begin{aligned}&\int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^{p-2}(1-|\varphi_z(w)|^2)^s dA(w)\\ \gtrsim&\int_{E(z,1/2)}|f'(w)|^2(1-|w|^2)^{p-2}(1-|\varphi_z(w)|^2)^s dA(w)\\ \gtrsim&(1-|z|)^p|f'(z)|^p.\end{aligned}$$

Therefore, $\Omega_\epsilon(f)\subseteq\Gamma_{p,\eta}(f)$. The proof is complete. □

3. Interpolating Blaschke product in $C_{\mathcal{B}}(B_p(\rho)\cap\mathcal{B})$ as multipliers

An analytic function in the unit disc $\mathbb{D}$ is called an inner function if it is bounded and has radial limits of absolute value 1 at almost every point of the boundary $\partial\mathbb{D}$. It is well known that every inner function has a factorization $e^{i\gamma}B(z)S(z)$, where $\gamma\in\mathbb{R}$, $B(z)$ is a Blaschke product and $S(z)$ is a singular inner function. A Blaschke product B with sequence of zeros $\{a_k\}_{k=1}^{\infty}$ is called interpolating if there exists a positive constant δ such that

$$\prod_{j\neq k}|\varphi_{a_j}(a_k)|\geq\delta,\quad k=1,2,\cdots.$$

We also say that $\{a_k\}_{k=1}^{\infty}$ is an interpolating sequence or an uniformly separated sequence. The following notions was introduced by Dyakonov [14]:

Suppose X and Y are two classes of analytic functions on $\mathbb{D}$, and $X\subseteq Y$. Let θ be an inner function, θ is said to be (X,Y)-improving, if every function $f\in X$ satisfying $f\theta\in Y$ must actually satisfy $f\theta\in X$. For more information related to improving multipliers, we refer to [27]. Motivated by the works of Dyakonov and Peláez, we have the following result.

Theorem 2. *Let ρ be of finite lower type greater than 0 and upper type less than 1. Suppose that $1<p<\infty$ and $B(z)$ is an interpolating Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$. Then following are equivalent:*

(1) $B\in C_{\mathcal{B}}(B_p(\rho)\cap\mathcal{B})$.
(2) $\sum_{k=1}^{\infty}\rho(1-|a_k|^2)<\infty$.
(3) B is $(B_p(\rho)\cap BMOA, BMOA)$-improving.
(4) B is $(B_p(\rho)\cap BMOA,\mathcal{B})$-improving.

Before we get into the proof, we need some lemmas.

Lemma 1. ([25, Lemma 2.5]) *Let $s>-1$, $r,t>0$, and $t<s+2<r$. Then*

$$\int_{\mathbb{D}}\frac{(1-|w|^2)^s}{|1-\overline{w}z|^r|1-\overline{w}\zeta|^t}dA(w)\lesssim\frac{(1-|z|^2)^{2+s-r}}{|1-\overline{\zeta}z|^t},\quad z,\zeta\in\mathbb{D}.$$

Lemma 2. *Let ρ be of finite lower type greater than 0 and upper type less than 1. Suppose that $f\in H(\mathbb{D})$ and $a\in\mathbb{D}$, then*

$$\begin{aligned}&\int_{\mathbb{D}}|f(z)-f(0)|^p\frac{\rho\left(1-|\varphi_a(z)|^2\right)}{1-|z|^2}dA(z)\\ \lesssim&\int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)^{p-1}\rho(1-|\varphi_a(z)|^2)dA(z).\end{aligned}$$

Proof. Let $\epsilon > 0$ be sufficiently small. From the proof of Lemma 2.1 of [9], we see that

$$|f(z) - f(0)|^p \lesssim \left(\int_{\mathbb{D}} |f'(w)|^p \frac{(1-|w|^2)^{(1+\epsilon)p+\sigma-\epsilon}}{|1-\overline{w}z|^{2+\sigma}} dA(w) \right) (1-|z|^2)^{-\epsilon(p-1)},$$

where $\sigma - \epsilon > -1$. Using Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z) - f(0)|^p \frac{\rho\left(1-|\varphi_a(z)|^2\right)}{1-|z|^2} dA(z) \\
\lesssim &\int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f'(w)|^p \frac{(1-|w|^2)^{(1+\epsilon)p+\sigma-\epsilon}}{|1-\overline{w}z|^{2+\sigma}} dA(w) \right) \frac{\rho\left(1-|\varphi_a(z)|^2\right)}{(1-|z|^2)^{1+\epsilon(p-1)}} dA(z) \\
= &\int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p+\sigma+\epsilon(p-1)} \left(\int_{\mathbb{D}} \frac{\rho\left(1-|\varphi_a(z)|^2\right)}{(1-|z|^2)^{1+\epsilon(p-1)}|1-\overline{w}z|^{2+\sigma}} dA(z) \right) dA(w).
\end{aligned}$$

Using conditions (A) and (B), combining (C) with Lemma 1, we deduce

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{\rho\left(1-|\varphi_a(z)|^2\right)}{(1-|z|^2)^{1+\epsilon(p-1)}|1-\overline{w}z|^{2+\sigma}} dA(z) \\
= &\rho\left(1-|\varphi_a(w)|^2\right) \int_{\mathbb{D}} \frac{\frac{\rho\left(1-|\varphi_a(z)|^2\right)}{\rho\left(1-|\varphi_a(w)|^2\right)}}{(1-|z|^2)^{1+\epsilon(p-1)}|1-\overline{w}z|^{2+\sigma}} dA(z) \\
\lesssim &\rho\left(1-|\varphi_a(w)|^2\right) \int_{\mathbb{D}} \frac{\left(\frac{1-|\varphi_a(z)|^2}{1-|\varphi_a(w)|^2}\right)^{\gamma} + \left(\frac{1-|\varphi_a(z)|^2}{1-|\varphi_a(w)|^2}\right)^{\delta}}{(1-|z|^2)^{1+\epsilon(p-1)}|1-\overline{w}z|^{2+\sigma}} dA(z) \\
\lesssim &\rho\left(1-|\varphi_a(w)|^2\right) (1-|w|^2)^{-1-\sigma-\epsilon(p-1)},
\end{aligned}$$

where $\gamma + \epsilon(p-1) < \delta + \epsilon(p-1) < 1$. Thus,

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z) - f(0)|^p \frac{\rho\left(1-|\varphi_a(z)|^2\right)}{1-|z|^2} dA(z) \\
\lesssim &\int_{\mathbb{D}} |f'(w)|^2 (1-|w|^2)^{p-1} \rho(1-|\varphi_a(w)|^2) dA(w).
\end{aligned}$$

The proof is complete. □

Lemma 3. ([23]) *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{D}$. Then the measure $d\mu_{a_k} = \sum_{k=1}^{\infty}(1-|a_k|^2)\delta_{a_k}$ is a Carleson measure, i.e.*

$$\sup_{w \in \mathbb{D}} \sum_{k=1}^{\infty} (1 - |\varphi_w(a_k)|^2) < \infty,$$

if and only if $\{a_k\}_{k=1}^{\infty}$ is a finite union of interpolating sequences.

Lemma 4. *Let ρ be of finite lower type greater than 0 and upper type less than 1. Suppose that $1 < p < \infty$, $B(z)$ is an interpolating Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$ and $f \in B_p(\rho)$. If $\sum_{k=1}^{\infty} |f(a_k)|^p \rho(1-|a_k|^2) < \infty$, then $fB \in B_p(\rho)$.*

Proof. Suppose that $f \in B_p(\rho)$ and $B(z)$ is an interpolating Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$. Since

$$\begin{aligned}
&\int_{\mathbb{D}} |(fB)'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) \\
\lesssim &\int_{\mathbb{D}} |f'(z)|^p |B(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) + \int_{\mathbb{D}} |f(z)|^p |B'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) \\
\lesssim &\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) + \int_{\mathbb{D}} |f(z)|^p |B'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z)
\end{aligned}$$

It is enough to prove

$$\int_{\mathbb{D}} |f(z)|^p |B'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) < \infty.$$

Notice the fact that

$$(1-|z|^2)|B'(z)| \lesssim 1$$

and

$$|B'(z)| \le \sum_{k=1}^{\infty} \frac{1-|a_k|^2}{|1-\overline{a_k}z|^2},$$

we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^p |B'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) \\
\lesssim\ &\int_{\mathbb{D}} |f(z)|^p |B'(z)| (1-|z|^2)^{-1} \rho(1-|z|^2) dA(z) \\
\lesssim\ &\sum_{k=1}^{\infty} (1-|a_k|^2) \int_{\mathbb{D}} \frac{|f(a_k)|^p}{|1-\overline{a_k}z|^2 (1-|z|^2)} \rho(1-|z|^2) dA(z) \\
&+ \sum_{k=1}^{\infty} (1-|a_k|^2) \int_{\mathbb{D}} \frac{|f(z)-f(a_k)|^p}{|1-\overline{a_k}z|^2 (1-|z|^2)} \rho(1-|z|^2) dA(z) \\
=\ &M+N.
\end{aligned}$$

Since

$$\int_{\mathbb{D}} \frac{\rho(1-|z|^2)}{|1-\overline{a_k}z|^2 (1-|z|^2)} dA(z) \lesssim \frac{\rho(1-|a_k|^2)}{(1-|a_k|^2)},$$

we deduce that

$$\begin{aligned}
M &=: \sum_{k=1}^{\infty} (1-|a_k|^2) \int_{\mathbb{D}} \frac{|f(a_k)|^p}{|1-\overline{a_k}z|^2 (1-|z|^2)} \rho(1-|z|^2) dA(z) \\
&\lesssim \sum_{k=1}^{\infty} |f(a_k)|^p \rho(1-|a_k|^2) < \infty.
\end{aligned}$$

Making the change of variables $z = \varphi_{a_k}(w)$, we obtain

$$\begin{aligned}
N &=: \sum_{k=1}^{\infty} (1-|a_k|^2) \int_{\mathbb{D}} \frac{|f(z)-f(a_k)|^p}{|1-\overline{a_k}z|^2 (1-|z|^2)} \rho(1-|z|^2) dA(z) \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{D}} |f\circ\varphi_{a_k}(w) - f\circ\varphi_{a_k}(0)|^p \frac{\rho(1-|\varphi_{a_k}(w)|^2)}{(1-|w|^2)} dA(w).
\end{aligned}$$

Using Fubini's theorem and Lemma 2, we have

$$\begin{aligned} N &= \sum_{k=1}^{\infty}\int_{\mathbb{D}}|f\circ\varphi_{a_k}(w)-f\circ\varphi_{a_k}(0)|^p\frac{\rho(1-|\varphi_{a_k}(w)|^2)}{(1-|w|^2)}dA(w)\\ &\lesssim \sum_{k=1}^{\infty}\int_{\mathbb{D}}|(f\circ\varphi_{a_k})'(w)|^p\rho(1-|\varphi_{a_k}(w)|^2)(1-|w|^2)^{p-1}dA(w)\\ &\approx \sum_{k=1}^{\infty}\int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^{p-2}\rho(1-|w|^2)(1-|\varphi_{a_k}(w)|^2)dA(w). \end{aligned}$$

Since $\{a_k\}_{k=1}^{\infty}$ is an interpolating sequences, using Lemma 3, we have $N \lesssim \|f\|_{B_p(\rho)}^p$, that is,

$$\begin{aligned} &\int_{\mathbb{D}}|f(z)|^p|B'(z)|^p(1-|z|^2)^{p-2}\rho(1-|z|^2)dA(z)\\ \lesssim\ &\sum_{k=1}^{\infty}|f(a_k)|^p\rho(1-|a_k|^2)+\|f\|_{B_p(\rho)}^p. \end{aligned}$$

The proof is complete. □

We also need the following lemma.

Lemma 5. ([13, Theorem 1]) *If $f \in BMOA$ and θ is an inner function, then the following conditions are equivalent:*

(1) $f\theta \in BMOA$;
(2) $\sup_{z\in\mathbb{D}}|f(z)|^2(1-|\theta(z)|^2) < \infty$;
(3) $\sup_{z\in\Omega(\theta,\,\epsilon)}|f(z)| < \infty$, *for every* ϵ, $0 < \epsilon < 1$;
(4) $\sup_{z\in\Omega(\theta,\,\epsilon)}|f(z)| < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

Proof of Theorem 2.

Proof. (1) ⇒ (2). Let B be an interpolating Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$ and $B \in C_{\mathcal{B}}(\mathcal{B}\cap B_p(\rho))$. From [18, Page 681], we know that there exist a $\delta > 0$, such that

$$(1-|z|^2)|B'(z)| \geq \frac{\delta(1-\delta)}{8},\ \ z \in E(a_k,\frac{\delta}{4}).$$

Thus,

$$\bigcup_{k=1}^{\infty}E(a_k,\frac{\delta}{4}) \subseteq \left\{z\in\mathbb{D} : (1-|z|^2)|B'(z)| \geq \frac{\delta(1-\delta)}{8}\right\}.$$

Since $\left\{E(a_k,\frac{\delta}{4})\right\}_{k=1}^{\infty}$ are pairwise disjoint, using the fact that

$$|E(a_k,\frac{\delta}{4})| \approx (1-|z|^2)^2,\quad z \in E(a_k,\frac{\delta}{4}),$$

we obtain

$$\begin{aligned} \sum_{k=1}^{\infty}\rho\left(1-|a_k|^2\right) &\lesssim \sum_{k=1}^{\infty}\int_{E(a_k,\frac{\delta}{4})}\frac{\rho\left(1-|z|^2\right)}{(1-|z|^2)^2}dA(z)\\ &\lesssim \int_{\left\{z\in\mathbb{D}:(1-|z|^2)|B'(z)|\geq\frac{\delta(1-\delta)}{8}\right\}}\frac{\rho\left(1-|z|^2\right)}{(1-|z|^2)^2}dA(z) < \infty. \end{aligned}$$

(2) ⇒ (3). Suppose that $f \in B_p(\rho) \cap BMOA$ and $fB \in BMOA$. We only need to prove that $fB \in B_p(\rho)$. Using Lemma 5, we obtain

$$\sum_{k=1}^{\infty} |f(a_k)|^p \rho(1-|a_k|^2) \leq \sup_{a_k} |f(a_k)|^p \sum_{k=1}^{\infty} \rho(1-|a_k|^2) < \infty.$$

By Lemma 4, we have $fB \in B_p(\rho)$.

(3) ⇒ (4). Let $f \in B_p(\rho) \cap BMOA \subseteq BMOA$ and $fB \in \mathcal{B}$. From [27, Corollary 1], we see that every interpolating Blaschke product B is $(BMOA, \mathcal{B})$-improving. Hence, we have $fB \in BMOA$. Notice that B is $(B_p(\rho) \cap BMOA, BMOA)$-improving, we have $fB \in B_p(\rho) \cap BMOA$. Thus, B is $(B_p(\rho) \cap BMOA, \mathcal{B})$-improving.

(4) ⇒ (1). Suppose that B is $(B_p(\rho) \cap BMOA, \mathcal{B})$-improving. Note that $1 \in B_p(\rho) \cap BMOA$ and $B \in H^\infty \subseteq \mathcal{B}$. Thus, $B \in B_p(\rho) \cap BMOA \subseteq B_p(\rho) \cap \mathcal{B} \subseteq C_{\mathcal{B}}(B_p(\rho) \cap \mathcal{B})$. The proof is complete. □

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