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Preservers of Partial Orders on the Set of all Variance-Covariance Matrices

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Abstract. Let $H_n^+(\mathbb{R})$ be the cone of all positive semidefinite $n \times n$ real matrices. Two of the best known partial orders that were mostly studied on subsets of square complex matrices are the Löwner and the minus partial orders. Motivated by applications in statistics we study these partial orders on $H_n^+(\mathbb{R})$. We describe the form of all surjective maps on $H_n^+(\mathbb{R})$, $n > 1$, that preserve the Löwner partial order in both directions. We present an equivalent definition of the minus partial order on $H_n^+(\mathbb{R})$ and also characterize all surjective, additive maps on $H_n^+(\mathbb{R})$, $n \geq 3$, that preserve the minus partial order in both directions.

1. Introduction

Let $M_{m,n}(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ be the set of all $m \times n$ real or complex matrices, let $A^t \in M_{n,m}(\mathbb{F})$ denote the transpose, $A^* \in M_{n,m}(\mathbb{F})$ the conjugate transpose, $\operatorname{Im} A$ the image (i.e. the column space), and $\operatorname{Ker} A$ the kernel (the nullspace) of $A \in M_{m,n}(\mathbb{F})$. Any matrix which is a solution $X = A^- \in M_{n,m}(\mathbb{F})$ to the equation $AXA = A$ is called an inner generalized inverse of $A \in M_{m,n}(\mathbb{F})$. Note that every matrix $A \in M_{m,n}(\mathbb{F})$ has an inner generalized inverse (see e.g. [24]). If $m = n$, then we will write $M_n(\mathbb{F})$ instead of $M_{n,n}(\mathbb{F})$. We say that $A \in M_n(\mathbb{F})$ is symmetric if $A = A^t$ and Hermitian (or self-adjoined) if $A = A^*$. A symmetric matrix $A \in M_n(\mathbb{R})$ is said to be positive semidefinite if $x^t A x \geq 0$ for every $x \in \mathbb{R}^n$. More generally, a Hermitian matrix $A \in M_n(\mathbb{C})$ is said to be positive semidefinite if $z^* A z \geq 0$ for every $z \in \mathbb{C}^n$. The study of positive semidefinite matrices is a flourishing area of mathematical investigation (see e.g. the monograph [1] and the references therein). Moreover, positive semidefinite matrices have become fundamental computational objects in many areas of statistics, engineering, quantum information, and applied mathematics. They appear as variance-covariance matrices (also known as dispersion or covariance matrices) in statistics, as elements of the search space in convex and semidefinite programming, as kernels in machine learning, as density matrices in quantum information, and as diffusion tensors in medical imaging. It is known (see e.g. [6]) that every variance-covariance matrix is positive semidefinite, and that every (real) positive semidefinite matrix is a variance-covariance matrix of some multivariate distribution.

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There are many partial orders which may be defined on various sets of matrices. We will next present two of the best known. Let $A, B \in M_n(\mathbb{R})$ be symmetric matrices. Then we say that A is below B with respect to the Löwner partial order and write

$$A \leq^L B \quad \text{if} \quad B - A \quad \text{is positive semidefinite.} \tag{1}$$

Löwner partial order has many applications in statistics especially in the theory of linear statistical models. Let

$$y = X\beta + \epsilon$$

be the matrix form of a linear model. Here y is a real $n \times 1$ random vector of observed quantities which we try to explain with other quantities that determine the matrix $X \in M_{n,p}(\mathbb{R})$. It is assumed that $E(\epsilon) = 0$ and $V(\epsilon) = \sigma^2 D$, i.e. the errors have the zero mean and covariances are known up to a scalar (real number). Here V denotes the variance-covariance matrix. The nonnegative parameter σ^2 and the vector of parameters (real numbers) β are unspecified, and $D \in M_n(\mathbb{R})$ is a known positive semidefinite matrix. We denote this linear model with the triplet $(y, X\beta, \sigma^2 D)$.

Classical inference problems related to the linear model $(y, X\beta, \sigma^2 D)$ usually concern a vector linear parametric function (LPF), $A\beta$ (here A is a real matrix with p columns). We try to estimate it by a linear function of the response Cy (here C is a real matrix with n columns). We say that the statistic Cy is a linear unbiased estimator (LUE) of $A\beta$ if $E(Cy) = A\beta$ for all possible values of $\beta \in \mathbb{R}^p$. A vector LPF is said to be estimable if it has an LUE. The best linear unbiased estimator (BLUE) of an estimable vector LPF is defined as the LUE having the smallest variance-covariance matrix. Here, the "variance-covariance" condition is expressed in terms of the Löwner order $\leq^L$. Let $A\beta$ be estimable. Then Ly is said to be BLUE of $A\beta$ if (i) $E(Ly) = A\beta$ for all $\beta \in \mathbb{R}^p$ and (ii) $V(Ly) \leq^L V(My)$ for all $\beta \in \mathbb{R}^p$ and all My satisfying $E(My) = A\beta$.

The second partial order which also has many applications in statistics (see [24, Sections 15.3, 15.4]) may be defined on the full set $M_{m,n}(\mathbb{R})$. For $A, B \in M_{m,n}(\mathbb{R})$ we say that A is below B with respect to the minus partial order (know also as the rank substractivity partial order) and write

$$A \leq^- B \quad \text{when} \quad A^-A = A^-B \text{ and } AA^- = BA^-$$

for some inner generalized inverse A^- of A. It is known (see e.g. [24]) that for $A, B \in M_{m,n}(\mathbb{R})$,

$$A \leq^- B \quad \text{if and only if} \quad \operatorname{rank}(B - A) = \operatorname{rank}(B) - \operatorname{rank}(A). \tag{2}$$

Note that both orders may be defined in the same way on sets of complex matrices [24]. Moreover, the minus partial order was introduced by Hartwig in [11] and independently by Nambooripad in [27] on a general regular semigroup however it was mostly studied on $M_n(\mathbb{F})$ (see [23] and the references therein). More recently, Šemrl generalized in [34] this order to $B(\mathcal{H})$, the algebra of all bounded linear opearators on a Hilbert space $\mathcal{H}$, and studied preservers of this order (see also [18]). Let $\mathcal{A}$ be some subset of $B(\mathcal{H})$ and denote by $\leq$ one of the above orders (i.e. $\leq^L$ or $\leq^-$). We say that that a map $\varphi : \mathcal{A} \to \mathcal{A}$ preserves an order $\leq$ in both directions when

$$A \leq B \quad \text{if and only if} \quad \varphi(A) \leq \varphi(B)$$

for every $A, B \in \mathcal{A}$.

Motivated by applications in quantum mechanics and quantum statistics Molnár studied preservers that are connected to certain structures of bounded linear operators which appear in mathematical foundations of quantum mechanics, i.e. he studied automorphisms of the underlying quantum structures or, in other words, quantum mechanical symmetries. Let A^* be the adjoint operator of $A \in B(\mathcal{H})$, and let

$$B^+(\mathcal{H}) = \{A \in B(\mathcal{H}) : A = A^* \text{ and } \langle Ax, x\rangle \geq 0 \text{ for every } x \in \mathcal{H}\}$$

be the set of all positive operators in $B(\mathcal{H})$. Note that in case when $\dim\mathcal{H} < \infty$, the set $B^+(\mathcal{H})$ may be identified with the set of all positive semidefinite $n \times n$ matrices. Note also that we may generalize definition (1) to the set of all self-adjoined operators in $B(\mathcal{H})$ in the following way: For two self-adjoined operators $A, B \in B(\mathcal{H})$ we write $A \leq^L B$ when $B - A \in B^+(\mathcal{H})$. Under assumption that $\mathcal{H}$ is a complex Hilbert space with $\dim\mathcal{H} > 1$, Molnár described in [25] the form of all bijective maps on $B^+(\mathcal{H})$ that preserve the Löwner partial order in both directions. It turns out that every such a map φ is of the form

$$\varphi(A) = TAT^*, \quad A \in B^+(\mathcal{H}) \tag{3}$$

where $T : \mathcal{H} \to \mathcal{H}$ is an invertible bounded either linear or conjugate-linear operator. Since we expect that maps preserving the Löwner order in both directions on the set of all real positive semidefinite matrices may have interesting applications in statistics (e.g. in the theory of comparison of linear models [32]), we will study such maps in Section 3. We will show that a similar result to Molnár's Theorem 1 from [25] holds also in the real matrix case, i.e. we will characterize surjective maps (omitting the injectivity assumption) on the set of all $n \times n$, $n \geq 2$, positive semidefinite real matrices that preserve the order $\leq^L$ in both directions. In Section 4, we will study the minus partial order, search for applications of this order in statistics, and describe the form of all surjective, additive maps on the set of all $n \times n$, $n \geq 3$, positive semidefinite real matrices that preserve the minus partial order in both directions.

2. Preliminaries

Let us present some tools that will be useful throughout the paper. As before, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $H_n(\mathbb{F})$ be the set of all Hermitian (symmetric in the real case) matrices in $M_n(\mathbb{F})$, denote by $H_n^+(\mathbb{F})$ the set of all positive semidefinite matrices in $H_n(\mathbb{F})$ and by $P_n(\mathbb{F})$ the set of all idempotent matrices in $H_n^+(\mathbb{F})$ (i.e. the set of all orthogonal projection matrices in $M_n(\mathbb{F})$). Let V be a subspace of $\mathbb{F}^n$. By $P_V \in P_n(\mathbb{F})$ we will denote the orthogonal projection matrix with $\operatorname{Im} P_V = V$. Recall that a convex cone C is a subset of a vector space $\mathcal{V}$ over an ordered field that is closed under all linear combinations with nonnegative scalars. For every convex cone C, we will from now on assume that $C \cap (-C) = \{0\}$. Observe that then every convex cone C induces a partial ordering $\leq$ on $\mathcal{V}$ so that we write

$$x \leq y \quad \text{when} \quad y - x \in C.$$

Note that $H_n^+(\mathbb{F})$ is a convex cone which is closed in the real normed vector space $H_n(\mathbb{F})$. The following result of Rothaus [31] will be one of the main tools in the proof of our first theorem.

Proposition 2.1. *Let $\mathcal{D}$ be the interior of a closed convex cone C in a real normed vector space $\mathcal{V}$. Suppose $\varphi : \mathcal{D} \to \mathcal{D}$ is a bijective map where*

$$x \leq y \quad \textit{if and only if} \quad \varphi(x) \leq \varphi(y)$$

for every $x, y \in \mathcal{D}$. Then the map φ is linear.

We say that two Hermitian (symmetric) matrices $A, B \in M_n(\mathbb{F})$ are adjacent if $\operatorname{rank}(A - B) = 1$. Huang and Šemrl characterized in [15] maps $\varphi : H_n(\mathbb{C}) \to H_m(\mathbb{C})$, $m, n \in \mathbb{N}$, $n > 1$, such that matrices $\varphi(A)$ and $\varphi(B)$ are adjacent whenever A and B are adjacent, $A, B \in H_n(\mathbb{C})$. In [20] Legiša considered adjacency preserving maps from $H_n(\mathbb{R})$ to $H_m(\mathbb{R})$ and proved the following result.

Proposition 2.2. *Let $n \geq 2$ and let $\varphi : H_n(\mathbb{R}) \to H_m(\mathbb{R})$ be a map preserving adjacency, i.e. if $A, B \in H_n(\mathbb{R})$ and $\operatorname{rank}(A - B) = 1$, then $\operatorname{rank}(\varphi(A) - \varphi(B)) = 1$. Suppose $\varphi(0) = 0$. Then either*

(i) there is a rank-one matrix $B \in H_m(\mathbb{R})$ and a function $f : H_n(\mathbb{R}) \to \mathbb{R}$ such that for every $A \in H_n(\mathbb{R})$

$$\varphi(A) = f(A)B, \textit{or}$$

(ii) there exist $c \in \{-1, 1\}$ and an invertible matrix $R \in M_m(\mathbb{R})$ such that for every $A \in H_n(\mathbb{R})$

$$\varphi(A) = cR \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} R^t.$$

(Obviously, in this case $m \geq n$. If $m = n$, the zeros on the right-hand side of the formula are absent.)

We will conclude this section with an auxiliary result. Note first that for $A, B \in H_n(\mathbb{F})$, $B \leq^L A$ implies $\operatorname{Im} B \subseteq \operatorname{Im} A$ (see e.g. [24, Corollary 8.2.12]).

Lemma 2.3. *Let $A, B \in H_n^+(\mathbb{F})$ and let* $\operatorname{rank}(A) = 1$. *If $B \leq^L A$, then $B = \lambda A$ for some scalar $\lambda \in [0,1]$.*

Proof. Since A is of rank-one and $A \in H_n^+(\mathbb{F})$, it follows by the spectral theorem [7, page 46] that $A = \alpha P$ where $\alpha > 0$ and $P \in P_n(\mathbb{F})$ with $\operatorname{rank}(P) = 1$. Let $B \leq^L A$ for some $B \in H_n^+(\mathbb{F})$. Then $\operatorname{Im} B \subseteq \operatorname{Im} A$ and thus $\operatorname{rank}(B) \leq 1$. Again, by the spectral theorem $B = \beta Q$ for some $\beta \geq 0$ and a rank-one $Q \in P_n(\mathbb{F})$. If $\beta = 0$, then $B = 0$ and thus $B = \lambda A$ for $\lambda = 0$. Suppose $\beta \neq 0$. Since $\operatorname{Im} B \subseteq \operatorname{Im} A$, we have $\operatorname{Im} Q = \operatorname{Im} P$ and thus (since P and Q are orthogonal projection matrices) $P = Q$. Let $\lambda = \frac{\beta}{\alpha}$. We have

$$\lambda A = \frac{\beta}{\alpha}\alpha P = \beta P = B.$$

Moreover, from $B \leq^L A$ it clearly follows that $\lambda \in [0,1]$. □

3. Preservers of the Löwner partial order

Let $S \in M_n(\mathbb{R})$ be an invertible matrix and $A, B, C \in H_n(\mathbb{R})$. It is easy to see ([24, Theorem 8.2.7, Remark 8.2.8]) that then

$$A \leq^L B \quad \text{if and only if} \quad SAS^t \leq^L SBS^t. \tag{4}$$

Also, if $A \leq^L B$, then $A + C \leq^L B + C$ and $\lambda A \leq^L \lambda B$ for every $\lambda \geq 0$. Let us now state and prove our main result. The proof will follow some ideas from [25, the proof of Theorem 1] however for the sake of completeness and since we are dealing here with real matrices, we will not skip the details and will present it in its entirety.

Theorem 3.1. *Let $n \geq 2$ be an integer. Then $\varphi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ is a surjective map that preserves the Löwner order $\leq^L$ in both directions if and only if there exists an invertible matrix $S \in M_n(\mathbb{R})$ such that*

$$\varphi(A) = SAS^t$$

for every $A \in H_n^+(\mathbb{R})$.

Proof. If $\varphi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ is of the form $\varphi(A) = SAS^t$, $A \in H_n^+(\mathbb{R})$, where $S \in M_n(\mathbb{R})$ is invertible, than it preserves by (4) the order $\leq^L$ in both directions and is clearly surjective.

Conversely, let $\varphi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ be a surjective map that preserves the Löwner order $\leq^L$ in both directions. We will split the proof into several steps.

1. *φ is bijective.* Let $\varphi(A) = \varphi(B)$ for $A, B \in H_n^+(\mathbb{R})$. The order $\leq^L$ is reflexive so $\varphi(A) \leq^L \varphi(B)$ and $\varphi(B) \leq^L \varphi(A)$. Since φ preserves the order $\leq^L$ in both directions, we have $A \leq^L B$ and $B \leq^L A$. It follows that $A = B$, since $\leq^L$ is antisymmetric. Thus, φ is injective and therefore bijective.

2. $\varphi(0) = 0$. Note that $0 \leq^L A$ for every $A \in H_n^+(\mathbb{R})$. So, on the one hand $0 \leq^L \varphi(0)$ and on the other hand, since φ^{-1} has the same properties as φ, $0 \leq^L \varphi^{-1}(0)$ and thus $\varphi(0) \leq^L 0$.

3. *φ preserves the set of all matrices of rank-one.* Let us first show that $A \in H_n^+(\mathbb{R})$ is of rank-one if and only if for every $B, C \in \left\{D \in H_n^+(\mathbb{R}) : D \leq^L A\right\} \equiv [0, A]$ we have $B \leq^L C$ or $C \leq^L B$, i.e. the order $\leq^L$ is linear on $[0, A]$.

Let $A \in H_n^+(\mathbb{R})$ be of rank-one and suppose first $B, C \in [0, A]$. By Lemma 2.3 we have $B = \lambda A$ and $C = \mu A$ for some $\lambda, \mu \in [0,1]$. If $\lambda = 0$ or $\mu = 0$, then clearly $B \leq^L C$ or $C \leq^L B$. Suppose $\lambda \neq 0$ and $\mu \neq 0$. It follows that $\mu B = \lambda C$ and thus

$$B - C = \left(1 - \frac{\mu}{\lambda}\right)B.$$

Clearly, then $0 \leq^L B - C$ or $0 \leq^L C - B$, i.e. $C \leq^L B$ or $B \leq^L C$.

Conversely, suppose that the order $\leq^L$ is linear on $[0, A]$ and assume that $\mathrm{rank}(A) > 1$. By the spectral theorem there exist $P_1, P_2 \in P_n(\mathbb{R})$ of rank-one with $\mathrm{Im}\, P_1 \cap \mathrm{Im}\, P_2 = \{0\}$, and $\lambda_1, \lambda_2 \in (0, \infty)$, such that $\lambda_1 P_1 \leq^L A$ and $\lambda_2 P_2 \leq^L A$, i.e. $\lambda_1 P_1, \lambda_2 P_2 \in [0, A]$. This yields by assumption $\lambda_1 P_1 \leq^L \lambda_2 P_2$ or $\lambda_2 P_2 \leq^L \lambda_1 P_1$ and therefore in either case $\mathrm{Im}\, P_1 = \mathrm{Im}\, P_2$, a contradiction.

Since φ preserves the order $\leq^L$ in both directions, $[0, A]$ is linearly ordered if and only if $[0, \varphi(A)]$ is linearly ordered. Thus, $A \in H_n^+(\mathbb{R})$ is of rank-one if and only if $\varphi(A)$ is of rank-one.

4. *φ preserves the set of all invertible (i.e. positive definite) matrices.* For every matrix $P \in P_n(\mathbb{R})$ of rank r there exists an orthogonal matrix $Q \in M_n(\mathbb{R})$ such that

$$P = Q \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^t$$

where I_r is the $r \times r$ identity matrix. Let I denote the identity matrix in $M_n(\mathbb{R})$. Since then

$$I - P = Q \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^t$$

it follows by the definition (1) that $P \leq^L I$ for every matrix $P \in P_n(\mathbb{R})$. This implies, $\epsilon P \leq^L \epsilon I$ for every $\varepsilon \geq 0$. Let $\varepsilon > 0$ be arbitrary but fixed. Let us show that then $\varphi(\varepsilon I)$ is invertible. By the transitivity of $\leq^L$, $\alpha P \leq^L \varepsilon I$ for every $P \in P_n(\mathbb{R})$ and any scalar α where $0 \leq \alpha \leq \varepsilon$. Suppose $\varphi(\varepsilon I)$ is not invertible. Then there exists a rank-one $Q \in P_n(\mathbb{R})$ such that $\mathrm{Im}\, Q \not\subseteq \mathrm{Im}\, \varphi(\varepsilon I)$. Since φ is surjective and sends rank-one matrices to rank-one matrices, there exists a rank-one $P \in P_n(\mathbb{R})$ and $\alpha > 0$ such that $\varphi(\alpha P) = Q$. Here $\alpha > \varepsilon$ since φ preserves the order in both directions. From $\varepsilon P \leq^L \alpha P$ we have $\varphi(\varepsilon P) \leq^L \varphi(\alpha P) = Q$. Both εP and Q are of rank-one and therefore $\mathrm{Im}\, \varphi(\varepsilon P) = \mathrm{Im}\, Q$. This is a contradiction since $\varphi(\varepsilon P) \leq^L \varphi(\varepsilon I)$ and therefore $\mathrm{Im}\, \varphi(\varepsilon P) \subseteq \mathrm{Im}\, \varphi(\varepsilon I)$. So, $\varphi(\varepsilon I)$ is invertible for any $\varepsilon > 0$.

Let now $T \in H_n^+(\mathbb{R})$ be an invertible (i.e. positive definite) matrix. By [29, page 93] there exists $\varepsilon > 0$ such that $\varepsilon I \leq^L T$. It follows that $\varphi(\varepsilon I) \leq^L \varphi(T)$ and thus $\mathbb{R}^n = \mathrm{Im}\, \varphi(\varepsilon I) \subseteq \mathrm{Im}\, \varphi(T)$. So, $\varphi(T)$ is invertible. Since φ^{-1} has the same properties as φ, we may conclude that $T \in H_n^+(\mathbb{R})$ is invertible if and only if $\varphi(T)$ is invertible.

5. *φ is linear on the set of all invertible matrices in $H_n^+(\mathbb{R})$.* The interior of the set $H_n^+(\mathbb{R})$ of all positive semidefinite matrices is the set of all invertible (i.e. positive definite) matrices in $H_n^+(\mathbb{R})$ (see [17, page 239]). Since $H_n^+(\mathbb{R})$ is a convex cone which is closed in the real normed vector space $H_n(\mathbb{R})$ and since φ preserves the set of all invertible matrices, we may conclude by Proposition 2.1 that φ is linear (additive and positive homogenous) on the set of all invertible matrices in $H_n^+(\mathbb{R})$.

6. *φ is a linear map.* Let $A, B \in H_n^+(\mathbb{R})$ and let $A_k = A + \frac{1}{k} I$, $B_k = B + \frac{1}{k} I$, $k \in \mathbb{N}$. Then $\{A_k\}$ and $\{B_k\}$ are sequences of positive definite (invertible) matrices in $H_n^+(\mathbb{R})$. Observe that both sequences are monotone decreasing with respect to $\leq^L$ and note that the sequence $\{A_k\}$ converges to A and the sequence $\{B_k\}$ converges to B in the strong operator topology. Also, $\inf_k A_k = A$ and $\inf_k B_k = B$ where inf denotes the infimum of a sequence. We have $A + B = \inf_k (A_k + B_k)$. Since φ preserves the order, it follows that $\varphi(A) = \inf_k \varphi(A_k)$, $\varphi(B) = \inf_k \varphi(B_k)$, and $\varphi(A + B) = \inf_k \varphi(A_k + B_k)$. Therefore, $\{\varphi(A_k)\}$, $\{\varphi(B_k)\}$, and $\{\varphi(A_k + B_k)\}$ are monotone decreasing sequences bounded from below. By [5, Definition 2.8 and Example 2.10] (see also [30, page 263]) there exist the limits (in the strong sense) of these sequences that equal their infima. Thus,

$$\varphi(A) = \lim_{k\to\infty} \varphi(A_k), \quad \varphi(B) = \lim_{k\to\infty} \varphi(B_k), \quad \varphi(A + B) = \lim_{k\to\infty} \varphi(A_k + B_k).$$

Step 5 yields that $\varphi(A_k + B_k) = \varphi(A_k) + \varphi(B_k)$ and hence

$$\varphi(A + B) = \lim_{k\to\infty} \varphi(A_k + B_k) = \lim_{k\to\infty} \varphi(A_k) + \lim_{k\to\infty} \varphi(B_k) = \varphi(A) + \varphi(B),$$

i.e. φ is additive. To show that φ is also (positive) homogenous, let $\lambda \geq 0$ be any scalar. Clearly, $\lambda A = \inf_k (\lambda A_k)$. Again, by the previous step it follows that

$$\varphi(\lambda A) = \lim_{k\to\infty} \varphi(\lambda A_k) = \lambda \lim_{k\to\infty} \varphi(A_k) = \lambda \varphi(A).$$

7. *We will extend the map φ from $H_n^+(\mathbb{R})$ to $H_n(\mathbb{R})$.* Let $A \in H_n(\mathbb{R})$. There exists an orthogonal matrix $Q \in M_n(\mathbb{R})$ such that $A = Q^t DQ$ where D is a diagonal matrix having the eigenvalues of A on the diagonal, i.e. $D = \text{diag}(\lambda_i : 1 \le i \le n)$. Let $D^+ = \text{diag}\Big(\lambda_i^+ : 1 \le i \le n\Big)$ and $D^- = \text{diag}\Big(\lambda_i^- : 1 \le i \le n\Big)$ where $\lambda_i^+ = \max\{\lambda_i, 0\}$ and $\lambda_i^- = \max\{-\lambda_i, 0\}$. Clearly, then $A = Q^t D^+ Q - Q^t D^- Q$. Note that both $Q^t D^+ Q, Q^t D^- Q \in H_n^+(\mathbb{R})$. We call the matrices $Q^t D^+ Q$ and $Q^t D^- Q$ the positive and the negative part of A, respectively. We may now extend the map φ to the map $\widehat{\varphi} : H_n(\mathbb{R}) \to H_n(\mathbb{R})$ in the following way:

$$\widehat{\varphi}(C) = \varphi(C^+) - \varphi(C^-), \quad C \in H_n(\mathbb{R}),$$

where C^+ and C^- are the positive and the negative part of C, respectively. Recall that $\varphi(0) = 0$. Take $C \in H_n^+(\mathbb{R})$ and note that then $C^+ = C$ and $C^- = 0$. So, $\widehat{\varphi}(C) = \varphi(C) - \varphi(0) = \varphi(C)$.

8. *$\widehat{\varphi}$ is a linear map.* Let $A, B \in H_n^+(\mathbb{R})$ and $C = A - B$. So, $C \in H_n(\mathbb{R})$. From $C^+ - C^- = C = A - B$, we have $C^+ + B = A + C^- \in H_n^+(\mathbb{R})$. Recall that φ is additive hence $\varphi(C^+) + \varphi(B) = \varphi(A) + \varphi(C^-)$ and thus

$$\widehat{\varphi}(A - B) = \widehat{\varphi}(C) = \varphi(C^+) - \varphi(C^-) = \varphi(A) - \varphi(B). \tag{5}$$

Let us show that $\widehat{\varphi}$ is additive. Let $C, D \in H_n(\mathbb{R})$. Then by (5)

$$\begin{aligned}\widehat{\varphi}(C + D) &= \widehat{\varphi}(C^+ - C^- + D^+ - D^-) = \widehat{\varphi}((C^+ + D^+) - (C^- + D^-)) \\ &= \varphi(C^+ + D^+) - \varphi(C^- + D^-) = \varphi(C^+) - \varphi(C^-) + \varphi(D^+) - \varphi(D^-) \\ &= \widehat{\varphi}(C) + \widehat{\varphi}(D).\end{aligned}$$

Let us now prove that $\widehat{\varphi}$ is homogenous. Let $C \in H_n(\mathbb{R})$ and let $\lambda \in \mathbb{R}$. Suppose first $\lambda \ge 0$. Then $(\lambda C)^+ = \lambda C^+$ and $(\lambda C)^- = \lambda C^-$ are the positive and the negative part of λC, respectively. Since φ is (positive) homogenous, we have

$$\widehat{\varphi}(\lambda C) = \varphi(\lambda C^+) - \varphi(\lambda C^-) = \lambda\varphi(C^+) - \lambda\varphi(C^-) = \lambda\widehat{\varphi}(C).$$

Let now $\lambda < 0$. Then $(\lambda C)^+ = -\lambda C^-$ and $(\lambda C)^- = -\lambda C^+$. So, $\widehat{\varphi}(\lambda C) = \widehat{\varphi}(-\lambda C^- - (-\lambda C^+))$ and therefore by (5)

$$\widehat{\varphi}(\lambda C) = \varphi(-\lambda C^-) - \varphi(-\lambda C^+) = -\lambda\varphi(C^-) - (-\lambda)\varphi(C^+) = \lambda(\varphi(C^+) - \varphi(C^-)) = \lambda\widehat{\varphi}(C).$$

9. *$\widehat{\varphi}$ preserves the order $\le^L$ in both directions.* Since $\widehat{\varphi}(C) = \varphi(C)$ for every $C \in H_n^+(\mathbb{R})$, we observe that $0 \le^L C$ if and only if $0 \le^L \widehat{\varphi}(C)$. Let $C_1, C_2 \in H_n(\mathbb{R})$. Then $C_1 \le^L C_2$ if and only if $0 \le^L \widehat{\varphi}(C_2 - C_1)$. Since $\widehat{\varphi}$ is linear, this equivalent to $\widehat{\varphi}(C_1) \le^L \widehat{\varphi}(C_2)$.

10. *$\widehat{\varphi}$ is bijective.* Since $\widehat{\varphi}$ preserves the order $\le^L$ in both directions, it is clearly injective (see the first step). To show that $\widehat{\varphi}$ is surjective, let $C \in H_n(\mathbb{R})$. Then we may write $C = C^+ - C^-$ where $C^+, C^- \in H_n^+(\mathbb{R})$. Since φ is surjective, there exist $A, B \in H_n^+(\mathbb{R})$ such that $C^+ = \varphi(A) = \widehat{\varphi}(A)$ and $C^- = \varphi(B) = \widehat{\varphi}(B)$. So,

$$C = C^+ - C^- = \widehat{\varphi}(A) - \widehat{\varphi}(B) = \widehat{\varphi}(A - B),$$

i.e. $\widehat{\varphi}$ is surjective.

11. *$\widehat{\varphi}$ is an adjacency preserving map.* Let us first show that $\widehat{\varphi}$ preserves the set of all rank-one matrices. Let $C \in H_n(\mathbb{R})$ be a rank-one matrix. By the spectral theorem, $C = \alpha P$ where $\alpha \in \mathbb{R}$ is nonzero and $P \in P_n(\mathbb{R})$ is of rank-one. Since $\widehat{\varphi}$ is linear and since $P \in H_n^+(\mathbb{R})$, we have

$$\widehat{\varphi}(C) = \alpha\widehat{\varphi}(P) = \alpha\varphi(P).$$

Recall that φ preserves the set of rank-one matrices. It follows that $\widehat{\varphi}(C)$ is of rank-one. Let now $A, B \in H_n(\mathbb{R})$ with $\text{rank}(A - B) = 1$, i.e. let A and B be adjacent. It follows that $\widehat{\varphi}(A - B)$ is of rank-one. Since $\widehat{\varphi}(A - B) = \widehat{\varphi}(A) - \widehat{\varphi}(B)$, we may conclude that $\widehat{\varphi}(A)$ and $\widehat{\varphi}(B)$ are adjacent.

We are now in the position to conclude the proof of the theorem. Since $\widehat{\varphi} : H_n(\mathbb{R}) \to H_n(\mathbb{R})$ is a bijective map that preserves adjacency, it follows by Proposition 2.2 that there exists $c \in \{-1, 1\}$ and an invertible $S \in M_n(\mathbb{R})$ such that

$$\widehat{\varphi}(A) = cSAS^t, \quad A \in H_n(\mathbb{R}).$$

Let $A, B \in H_n(\mathbb{R})$, $A \neq B$, and $A \leq^L B$. Then on the one hand by (4), $SAS^t \leq^L SBS^t$. If $c = -1$, we get on the one hand, since $\widehat{\varphi}$ preserves the order $\leq^L$, $-SAS^t \leq^L -SBS^t$. It follows that $SAS^t = SBS^t$ and therefore $A = B$, a contradiction. To conclude, $\widehat{\varphi}(A) = SAS^t$ for every $A \in H_n(\mathbb{R})$ and therefore $\varphi(A) = SAS^t$ for every $A \in H_n^+(\mathbb{R})$. $\square$

Remark 3.2. *The proof of Theorem 3.1 may serve with a few adjustments (e.g. instead of Proposition 2.2 we may use Theorem 1.2 from [15] (see also [13, 14])) as an alternative proof of finite-dimensional (complex) version ($\dim\mathcal{H} < \infty$) of Molnár's result (3).*

Remark 3.3. *Let us present an observation about preservers of the Löwner partial order and linear models. Let $L_1 = (y_1, X_1\beta, \sigma^2 D_1)$ and $L_2 = (y_2, X_2\beta, \sigma^2 D_2)$ be two linear models. Here $X_1 \in M_{n,p}(\mathbb{R})$, $X_2 \in M_{m,p}(\mathbb{R})$, $D_1 \in H_n^+(\mathbb{R})$, and $D_2 \in H_m^+(\mathbb{R})$. We say (see [33]) that L_1 is at least as good as L_2 if for any unbiased estimator $a_2^t y_2$, $a_2 \in M_{m,1}(\mathbb{R})$, of a parameter $k^t\beta$, $k \in M_{p,1}(\mathbb{R})$, there exists an unbiased estimator $a_1^t y_1$, $a_1 \in M_{n,1}(\mathbb{R})$, such that $V(a_1^t y_1) \leq^L V(a_2^t y_2)$ (here $V(a_i^t y_i)$, $i \in \{1, 2\}$, is the variance of $a_i^t y_i$). If this condition is satisfied, we write $L_1 \succeq L_2$. In [32], Stępniak proved that*

$$L_1 \succeq L_2 \quad \textit{if and only if} \quad M_2 \leq^L M_1$$

where $M_i = X_i^t \left(D_i + X_i X_i^t\right)^- X_i$, $i \in \{1, 2\}$. Moreover, Stępniak noted that when $\operatorname{Im} X_i \subseteq \operatorname{Im} D_i$, $i \in \{1, 2\}$, we may replace $X_i^t \left(D_i + X_i X_i^t\right)^- X_i$ with $X_i^t D_i^- X_i$. When $D_i = X_i$, $i \in \{1, 2\}$, these matrices may be further simplified to $M_i = X_i^t D_i^- X_i = D_i^t D_i^- D_i = D_i D_i^- D_i = D_i$. For such models $L_1 = (y_1, D_1\beta, \sigma^2 D_1)$ and $L_2 = (y_2, D_2\beta, \sigma^2 D_2)$ we thus have

$$L_1 \succeq L_2 \quad \textit{if and only if} \quad D_2 \leq^L D_1. \tag{6}$$

Let $n > 1$. For a random $n \times 1$ vector of observed quanitities y_i, an unspecified $n \times 1$ vector β_i, and an unspecified nonnegative scalar σ_i^2, let $\mathcal{L}_i$ be the set of all linear models $L_i = (y_i, D\beta_i, \sigma_i^2 D)$ where $D \in H_n^+(\mathbb{R})$ may vary from model to model. Define a map $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ with $\psi((y_1, D\beta_1, \sigma_1^2 D)) = (y_2, \varphi(D)\beta_2, \sigma_2^2 \varphi(D))$ where $\varphi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ is a surjective map. Suppose

$$L_{1_a} \succeq L_{1_b} \quad \textit{if and only if} \quad \psi(L_{1_a}) \succeq \psi(L_{1_b})$$

for every $L_{1_a}, L_{1_b} \in \mathcal{L}_1$. This assumption may be reformulated as $D_{1_b} \leq^L D_{1_a}$ if and only if $\varphi(D_{1_b}) \leq^L \varphi(D_{1_a})$, $D_{1_a}, D_{1_b} \in H_n^+(\mathbb{R})$, and therefore Theorem 3.1 completely determines the form of any such a map ψ.

4. Preservers of the minus partial order

Let $A, B \in M_n(\mathbb{F})$. It is known (see e.g. [18, page 149]) that

$$A \leq^- B \quad \text{if and only if} \quad \operatorname{Im} B = \operatorname{Im} A \oplus \operatorname{Im}(B - A) \quad \text{if and only if} \quad RAL \leq^- RBL \tag{7}$$

for any invertible $R, L \in M_n(\mathbb{F})$. Let $A, B \in M_n(\mathbb{C})$. If there exists an invertible matrix $S \in M_n(\mathbb{C})$ such that

a) $B = SAS^t$, then we say that A and B are congruent;

b) $B = SAS^*$, then we say that A and B are *congruent.

By Sylvester's law of inertia (see [12, page 282]) two (Hermitian) matrices $A, B \in H_n(\mathbb{C})$ are *congruent if and only if they have the same inertia, i.e. they have the same number of positive eigenvalues and the same number of negative eigenvalues. Two (real symmetric) matrices $A, B \in H_n(\mathbb{R})$ are *congruent via a complex matrix if and only if they are congruent via a real matrix [12, page 283]. So, Sylvester's law for the real case states that $A, B \in H_n(\mathbb{R})$ are congruent via an invertible $S \in M_n(\mathbb{R})$ (i.e. $B = SAS^t$) if and only if A and B have the same number of positive eigenvalues and the same number of negative eigenvalues. Note that congruent (respectively, *congruent) matrices have the same rank [12, page 281].

The next theorem gives a characerization of the minus partial order on the cone of all positive semidefinite matrices. Observe first that if A is an $n \times n$ zero matrix, then $A \leq^- B$ for every $B \in M_n(\mathbb{F})$ (see e.g. (7)).

Theorem 4.1. *Let $A, B \in H_n^+(\mathbb{F})$ and $A \neq 0$. Then $A \leq^- B$ if and only if there exists an invertible matrix $S \in M_n$ such that*

$$A = S\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}S^* \quad \text{and} \quad B = S\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}S^*$$

where I_r and I_s are $r \times r$ and $s \times s$, $s \leq n$, identity matrices, respectively, and $r < s$ if $A \neq B$, and $r = s$, otherwise. (Obviously, in case when $s = n$, the zeros on the right-hand side of the formula for B are absent.)

Proof. To simplify notation we will use the term *congruent for both *congruent complex matrices (via an invertible complex matrix) and congruent real matrices (via a real invertible matrix). Of course, $S^* = S^t$ when $S \in M_n(\mathbb{R})$.

Let $A \in H_n^+(\mathbb{F})$, $A \neq 0$. Suppose $A \leq^- B$ for some $B \in H_n^+(\mathbb{F})$. By (2), $\operatorname{rank}(B - A) = \operatorname{rank}(B) - \operatorname{rank}(A)$. Let $C = B - A$. So, $\operatorname{rank}(C) + \operatorname{rank}(A) = \operatorname{rank}(A + C)$. Observe that $A + C$ is positive semidefinite (because B is). All the eigenvalues of the matrix $A + C$ are thus nonnegative and therefore by Sylvester's law of inertia it follows that there exists an invertible matrix $V \in M_n(\mathbb{F})$ such that

$$V(A + C)V^* = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}$$

where I_s is an $s \times s$, $s \leq n$, identity matrix. Let

$$Q = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = VAV^*, \quad \text{and} \quad C_1 = VCV^*. \tag{8}$$

Since *congruent matrices have the same rank, it follows that $\operatorname{rank}(A + C) = \operatorname{rank}(Q)$, $\operatorname{rank}(A) = \operatorname{rank}(A_1)$, $\operatorname{rank}(C) = \operatorname{rank}(C_1)$, and therefore

$$\operatorname{rank}(Q) = \operatorname{rank}(A_1) + \operatorname{rank}(C_1). \tag{9}$$

Observe that

$$\operatorname{Im} Q = \operatorname{Im}(V(A + C)V^*) = \operatorname{Im}(VAV^* + VCV^*) \subseteq \operatorname{Im}(VAV^*) + \operatorname{Im}(VCV^*). \tag{10}$$

By (9) and (10) we have $\operatorname{Im} Q = \operatorname{Im} A_1 + \operatorname{Im} C_1$. Also, if $\operatorname{Im} A_1 \cap \operatorname{Im} C_1 \neq \{0\}$, then $\operatorname{rank}(A_1) + \operatorname{rank}(C_1) > \operatorname{rank}(Q)$, a contradiction. Thus,

$$\operatorname{Im} Q = \operatorname{Im} A_1 \oplus \operatorname{Im} C_1. \tag{11}$$

Let $x \in \operatorname{Ker} Q$, i.e. $Qx = 0$. From $Q = A_1 + C_1$, we have $0 = Qx = A_1x + C_1x$. Since $0 = 0 + 0$, it follows by (11) that $A_1x = 0$ and $C_1x = 0$. So, $A_1(\operatorname{Ker} Q) = \{0\}$ and $C_1(\operatorname{Ker} Q) = \{0\}$. The matrix Q is clearly a self-adjoined idempotent, i.e. $Q^* = Q = Q^2$. So,

$$\mathbb{F}^n = \operatorname{Im} Q \oplus \operatorname{Ker} Q$$

where $(\operatorname{Im} Q)^\perp = \operatorname{Ker} Q$.

Consider the representation of a linear operator $D : \mathbb{F}^n \to \mathbb{F}^n$ with respect to the decomposition $\mathbb{F}^n = \operatorname{Im} Q \oplus \operatorname{Ker} Q$:

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

where $D_1 : \operatorname{Im} Q \to \operatorname{Im} Q$, $D_2 : \operatorname{Ker} Q \to \operatorname{Im} Q$, $D_3 : \operatorname{Im} Q \to \operatorname{Ker} Q$, and $D_4 : \operatorname{Ker} Q \to \operatorname{Ker} Q$ are linear operators. Since we may consider A_1 and C_1 as operators from $\operatorname{Im} Q \oplus \operatorname{Ker} Q$ to itself, we may conclude that with respect to this decomposition

$$A_1 = \begin{bmatrix} \widetilde{A_1} & 0 \\ \widetilde{A_2} & 0 \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} \widetilde{C_1} & 0 \\ \widetilde{C_2} & 0 \end{bmatrix}.$$

Observe that $A_1^* = (VAV^*)^* = VA^*V^* = VAV^* = A_1$. Similarly, $C_1^* = C_1$ and hence it follows that $\widetilde{A_2} = 0$ and $\widetilde{C_2} = 0$, i.e.

$$A_1 = \begin{bmatrix} \widetilde{A_1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} \widetilde{C_1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\operatorname{rank}(Q) = s$ (see (8)), it follows by (11) that

$$\mathbb{F}^s = \operatorname{Im}\widetilde{A_1} \oplus \operatorname{Im}\widetilde{C_1}. \tag{12}$$

Note that $Qx = x$ for every $x \in \operatorname{Im} Q$. Let $x \in \operatorname{Im}\widetilde{A_1}$. On the one hand $x = \widetilde{A_1}x + \widetilde{C_1}x$ and on the other hand $x = x + 0$. By (12) it follows $x = \widetilde{A_1}x$ and $0 = \widetilde{C_1}x$. Let now $x \in \operatorname{Im}\widetilde{C_1}$. Similarly, then $x = \widetilde{A_1}x + \widetilde{C_1}x$ and $x = 0 + x$ and therefore $0 = \widetilde{A_1}x$ and $\widetilde{C_1}x = x$. So, $\widetilde{A_1}$ acts as the identity operator on $\operatorname{Im}\widetilde{A_1}$ and as the zero operator on $\operatorname{Im}\widetilde{C_1}$, and similarly, $\widetilde{C_1}$ acts as the identity operator on $\operatorname{Im}\widetilde{C_1}$ and as the zero operator on $\operatorname{Im}\widetilde{A_1}$. This yields by (12) that $\operatorname{Im}\widetilde{A_1} = \operatorname{Ker}\widetilde{C_1}$ and $\operatorname{Ker}\widetilde{A_1} = \operatorname{Im}\widetilde{C_1}$. It follows that $\widetilde{A_1}$ and $\widetilde{C_1}$ are pairwise orthogonal idempotent operators on $\mathbb{F}^s$, and therefore $\widetilde{A_1}$ and $\widetilde{C_1}$ are simultaneously diagonalizable (see e.g. [16]). Recall that both $\widetilde{A_1}$ and $\widetilde{C_1}$ are self-adjoined. It follows that there exists a unitary (i.e. an orthogonal in the real case) matrix $U \in M_s(\mathbb{F})$ such that

$$U\widetilde{A_1}U^* = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad U\widetilde{C_1}U^* = \begin{bmatrix} 0 & 0 \\ 0 & I_{s-r} \end{bmatrix}$$

where I_r and I_{s-r} are $r \times r$ and $(s-r) \times (s-r)$ identity matrices, respectively. Let

$$Z = \begin{bmatrix} U & 0 \\ 0 & I_{n-s} \end{bmatrix}.$$

Note that $Z \in M_n(\mathbb{F})$ is invertible. Then

$$ZA_1Z^* = \begin{bmatrix} U & 0 \\ 0 & I_{n-s} \end{bmatrix}\begin{bmatrix} \widetilde{A_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} U^* & 0 \\ 0 & I_{n-s} \end{bmatrix} = \begin{bmatrix} U\widetilde{A_1}U^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly,

$$Z\widetilde{C_1}Z^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{s-r} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $S = (ZV)^{-1}$. Then by (8),

$$A = V^{-1}A_1(V^*)^{-1} = V^{-1}Z^{-1}\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(Z^*)^{-1}(V^*)^{-1} = S\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}S^*.$$

Similarly,

$$C = S\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{s-r} & 0 \\ 0 & 0 & 0 \end{bmatrix}S^*$$

and therefore

$$B = A + C = S\begin{bmatrix} I_r & 0 & 0 \\ 0 & I_{s-r} & 0 \\ 0 & 0 & 0 \end{bmatrix}S^*.$$

So,

$$A = S\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}S^* \quad \text{and} \quad B = S\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}S^*$$

where $r \leq s$. Clearly, if $A \neq B$, then $r < s$, and $r = s$, otherwise.

Conversely, let $A = S\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}S^*$ and $B = S\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}S^*$ where $r \leq s$. It follows that

$$B - A = S\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{s-r} & 0 \\ 0 & 0 & 0 \end{bmatrix}S^*.$$

Since congruence preserves rank, we have $\text{rank}(B - A) = \text{rank}(B) - \text{rank}(A)$ and therefore $A \leq^- B$. $\square$

As an example of an application of the minus partial order in statistics we present the following two corollaries to Theorem 4.1. The first result is a direct corollary to Theorem 4.1 and the main result in [4, page 366].

Corollary 4.2. *Consider a linear model $(y, X\beta, \sigma^2 D)$. Then the statistics Ly with $V(Ly) \neq V(y)$ is BLUE of $X\beta$ if and only if the following conditions hold:*

(i) $LX = X$;

(ii) $\text{Im}(LD) \subseteq \text{Im}\, X$;

(iii) There exist an invertible matrix $S \in M_n(\mathbb{R})$ such that

$$V(Ly) = S\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}S^t \quad \textit{and} \quad V(y) = S\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}S^t$$

where I_r is a $r \times r$ identity matrix, and I_s is a $s \times s$ identity matrix with $r < s \leq n$.

Note that for a positive semidefinite matrix $A \in M_n(\mathbb{R})$, the matrix $W^t A W \in M_m(\mathbb{R})$ is still positive semidefinite for any matrix $W \in M_{n,m}(\mathbb{F})$. The following result thus follows directly from Theorem 4.1 and [2, Theorem 1].

Corollary 4.3. *Let $A = \sum_{i=1}^{k} A_i$ where $A_i \in M_n(\mathbb{R})$ are positive semidefinite matrices, $i = 1, 2, \ldots, k$. Let the $n \times 1$ random vector x follow a multivariate normal distribution with the mean μ and the variance-covariance matrix V. Let $W = (V : \mu)$ be a $n \times (n+1)$ partitioned matrix. Consider the quadratic forms $Q = x^t A x$ and $Q_i = x^t A_i x$, $i = 1, 2, \ldots, k$. Then the following statements are equivalent.*

(i) Q_i, $i = 1, 2, \ldots, k$, are mutually independent and distributed as chi-squared variables;

(ii) Q is distributed as a chi-squared variable and there exist invertible matrices $S_i \in M_{n+1}(\mathbb{R})$ such that

$$W^t A_i W = S_i\begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix}S_i^t \quad \textit{and} \quad W^t A W = S_i\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}S_i^t$$

for every $i = 1, 2, \ldots, k$, where I_{r_i} are $r_i \times r_i$ identity matrices, and I_s is a $s \times s$ identity matrix with $r_i \leq s \leq n+1$. (Here $I_{r_i} = 0$ if $W^t A_i W = 0$ for some $i \in \{1, 2, \ldots, k\}$.)

With our final result we will describe the form of all additive, surjective maps on $H_n^+(\mathbb{R})$, $n \geq 3$, that preserve the minus partial order in both directions. Denote by E_{ij} the $n \times n$ matrix with all entries equal to zero except the (i, j)-entry which is equal to one. Let $E_k = E_{11} + E_{22} + \ldots + E_{kk}$. For $A, B \in M_n(\mathbb{R})$ we will write $A <^- B$ when $A \leq^- B$ and $A \neq B$. We will denote by $x \otimes y^t$ a rank one linear operator on $\mathbb{R}^n$ defined with $(x \otimes y^t)z = \langle z, y\rangle x$ for every $z \in \mathbb{R}^n$ (here $\langle z, y\rangle = y^t z$). Note that every rank-one linear operator on $\mathbb{R}^n$ may be written in this form. Moreover, $P = x \otimes x^t$ for some $x \in \mathbb{R}^n$ with $\|x\| = 1$ if and only if $P \in P_n(\mathbb{R})$ and P is of rank-one. Indeed, let first $x \in \mathbb{R}^n$ with $\|x\| = 1$. Then $(x \otimes x^t)^t = x \otimes x^t$ and for every $z \in \mathbb{R}^n$ we have $(x \otimes x^t)^2 z = \langle z, x\rangle \langle x, x\rangle x = \langle z, x\rangle x = (x \otimes x^t)z$. Conversely, let $P \in P_n(\mathbb{R})$ be of rank-one. Then $P = x \otimes y^t$ for some nonzero $x, y \in \mathbb{R}^n$. By transferring the appropriate scalar to the second factor we may assume without loss of generality that $\|x\| = 1$. Since and $P^2 = P = P^t$, we have $x \otimes y^t = (x \otimes y^t)^t = y \otimes x^t$ which implies $y = \mu x$ for some nonzero $\mu \in \mathbb{R}$, and $(x \otimes y^t)^2 = x \otimes y^t$ then yields $\mu = 1$, i.e. $P = x \otimes x^t$ with $\|x\| = 1$.

Theorem 4.4. *Let $n \geq 3$ be an integer. Then $\varphi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ is a surjective, additive map that preserves the minus order $\leq^-$ in both directions if and only if there exists an invertible matrix $S \in M_n(\mathbb{R})$ such that*

$$\varphi(A) = SAS^t$$

for every $A \in H_n^+(\mathbb{R})$.

Proof. Let $\varphi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ be of the form $\varphi(A) = SAS^t$, $A \in H_n^+(\mathbb{R})$, where $S \in M_n(\mathbb{R})$ is an invertible matrix. Then φ preserves by (7) the order $\leq^-$ in both directions and is clearly surjective and additive.

Conversely, let $\varphi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ be a surjective, additive map that preserves the order $\leq^-$ in both directions. We will again split the proof into several steps.

1. *φ is bijective and $\varphi(0) = 0$.* Since $\leq^-$ is a partial order and since φ preserves this order in both directions, the proof that φ is bijective and that $\varphi(0) = 0$ may be the same as in the first two steps of Theorem 3.1.

2. *φ preserves the rank, i.e.* $\operatorname{rank}(A) = \operatorname{rank}(\varphi(A))$ *for every $A \in H_n^+(\mathbb{R})$.* Let $A \in H_n^+(\mathbb{R})$ with $\operatorname{rank}(A) = k$. By Sylvester's law of inertia there exists an invertible matrix $R \in M_n(\mathbb{R})$ such that $E_k = RAR^t$. Clearly (see (2)),

$$0 <^- E_1 <^- E_2 <^- \ldots <^- E_n = I.$$

Since congruence preserves rank, we have by (7)

$$0 <^- R^{-1}E_1(R^{-1})^t <^- R^{-1}E_2(R^{-1})^t <^- \ldots <^- R^{-1}E_k(R^{-1})^t <^- \ldots <^- R^{-1}E_n(R^{-1})^t.$$

From $(R^{-1})^t = (R^t)^{-1}$ and since φ preserves the order $\leq^-$ and is injective, we obtain

$$0 <^- \varphi(R^{-1}E_1(R^t)^{-1}) <^- \varphi(R^{-1}E_2(R^t)^{-1}) <^- \ldots < \varphi(A) <^- \ldots <^- \varphi(R^{-1}(R^t)^{-1}). \tag{13}$$

Let $C, D \in M_n(\mathbb{R})$ with $C <^- D$ and $\operatorname{rank}(C) = \operatorname{rank}(D)$. Then by (2), $\operatorname{rank}(D - C) = 0$ and therefore $D = C$, a contradiction. So, if $C <^- D$, then $\operatorname{rank}(C) < \operatorname{rank}(D)$.

Every succeeding matrix in (13) has the rank that is strictly greater then its predecessor. Since $\operatorname{rank} \varphi(R^{-1}(R^t)^{-1}) \leq n$, it follows that $\operatorname{rank} \varphi(R^{-1}(R^t)^{-1}) = n$ and therefore $\operatorname{rank}(\varphi(A)) = k$.

3. *We may without loss of generality assume that $\varphi(I) = I$.* By the previous step, $\varphi(I) = B$ where $B \in H_n^+(\mathbb{R})$ is an invertible (positive definite) matrix. It follows that there exists a positive definite matrix $\sqrt{B} \in H_n^+(\mathbb{R})$ such that $\varphi(I) = \sqrt{B}\sqrt{B}$. Let $\psi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ be defined with

$$\psi(A) = \left(\sqrt{B}\right)^{-1} \varphi(A) \left(\sqrt{B}\right)^{-1}.$$

Then ψ is a bijective map that preserves the order $\leq^-$ in both directions. Also, $\psi(I) = I$. We will thus from now on assume that

$$\varphi(I) = I.$$

4. *There exists a bijective, linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that for every $P \in P_n(\mathbb{R})$ the matrix $\varphi(P)$ is the orthogonal projection matrix on $T(\operatorname{Im} P)$, i.e.*

$$\varphi(P) = P_{T(\operatorname{Im} P)}.$$

Let $P \in M_n(\mathbb{R})$ be an idempotent matrix, i.e. $P^2 = P$. Then $\mathbb{R}^n = \operatorname{Im} P \oplus \operatorname{Ker} P = \operatorname{Im} P \oplus \operatorname{Im}(I - P)$ and therefore by (7), $P \leq^- I$. Moreover, if $Q \in M_n(\mathbb{R})$ is an idempotent matrix and if $A \leq^- Q$ for $A \in M_n(\mathbb{R})$, then by e.g. [22, Lemma 2.9], $A^2 = A$. Thus for $P \in M_n(\mathbb{R})$ we have

$$P \leq^- I \quad \text{if and only if} \quad P^2 = P.$$

Let now $P \in P_n(\mathbb{R})$, i.e. P is a symmetric and idempotent matrix. It follows that $P \leq^- I$ and therefore $\varphi(P) \leq^- \varphi(I) = I$. So, $\varphi(P)$ is an idempotent matrix and by the definition of the map φ also symmetric, i.e. $\varphi(P) \in P_n(\mathbb{R})$. Since φ^{-1} has the same properties as φ, we may conclude that

$$P \in P_n(\mathbb{R}) \quad \text{if and only if} \quad \varphi(P) \in P_n(\mathbb{R}),$$

i.e. φ preserves the set of all orthogonal projection matrices. Recall that we may identify subspaces of $\mathbb{R}^n$ with elements of $P_n(\mathbb{R})$. Let $C(\mathbb{R}^n)$ be the lattice of all subspaces of $\mathbb{R}^n$. It follows that the map φ induces a lattice automorphisms, i.e. a bijective map $\tau : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ such that

$$M \subseteq N \quad \text{if and only if} \quad \tau(M) \subseteq \tau(N)$$

for all $M, N \in C(\mathbb{R}^n)$. In [21, page 246] (see also [8, pages 820 and 823] or [28, page 82]) Mackey proved that for $n \geq 3$ every such a map is induced by an invertible linear operator, i.e. there exists an invertible linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\tau(M) = T(M)$ for every $M \in C(\mathbb{R}^n)$. For the map φ it follows that

$$\varphi(P) = P_{T(\operatorname{Im} P)} \tag{14}$$

for every $P = P_{\operatorname{Im} P} \in P_n(\mathbb{R})$.

5. *We may without loss of generality assume that $\varphi(P) = P$ for every $P \in P_n(\mathbb{R})$.* Let $x \in \mathbb{R}^n$ with $\|x\| = 1$. Recall that then $x \otimes x^t \in P_n(\mathbb{R})$ is of rank-one. So, by steps 2 and 4 there exists $a \in \mathbb{R}^n$ with $\|a\| = 1$ such that

$$\varphi(x \otimes x^t) = a \otimes a^t.$$

Let $y \in \mathbb{R}^n$, $\left\|y\right\| = 1$, and $\langle x, y\rangle = 0$. We have $\varphi(y \otimes y^t) = b \otimes b^t$ for some $b \in \mathbb{R}^n$, $\|b\| = 1$. Note that $x \otimes x^t + y \otimes y^t \in P_n(\mathbb{R})$ and that it is of rank-two. It follows that $\varphi(x \otimes x^t + y \otimes y^t)$ is a rank-two orthogonal projection matrix. Since φ is additive, we obtain

$$\varphi(x \otimes x^t + y \otimes y^t) = a \otimes a^t + b \otimes b^t.$$

Since this is a rank-two matrix, we may conclude that a and b are linearly independent vectors. Moreover, from

$$\left(a \otimes a^t + b \otimes b^t\right)^2 = a \otimes a^t + b \otimes b^t$$

and since $\|a\| = \|b\| = 1$ we get

$$\langle z, a\rangle a + \langle z, b\rangle b + \langle z, a\rangle \langle a, b\rangle b + \langle z, b\rangle \langle b, a\rangle a = \langle z, a\rangle a + \langle z, b\rangle b$$

and thus $\langle z, a\rangle \langle a, b\rangle b = -\langle z, b\rangle \langle b, a\rangle a$ for every $z \in \mathbb{R}^n$. Let $z = a$ and assume that $\langle a, b\rangle \neq 0$. Then $b = -\langle b, a\rangle a$, i.e. a and b are linearly dependent, a contradiction. It follows that

$$\langle a, b\rangle = 0.$$

On the one hand, $\operatorname{Im}\varphi(x \otimes x^t)$ $=\operatorname{Lin}\{a\}$ and on the other hand by (14) $\operatorname{Im}\varphi(x \otimes x^t) = T(\operatorname{Lin}\{x\})$ $=\operatorname{Lin}\{Tx\}$. It follows that a and Tx are linearly dependent, i.e. $a = \mu Tx$ for some $\mu \in \mathbb{R}\backslash\{0\}$. Similarly, there exists $\nu \in \mathbb{R}\backslash\{0\}$ such that $b = \nu Ty$. This yields

$$0 = \langle \mu Tx, \nu Ty \rangle = \mu\nu \langle Tx, Ty \rangle = \mu\nu \left\langle T^tTx, y \right\rangle$$

and therefore $\langle T^tTx, y\rangle = 0$. This equation holds for every $y \in \mathbb{R}^n$ with $\left\|y\right\| = 1$ and $\langle x, y\rangle = 0$. Since $\langle T^tTx, y\rangle = \|x\| \left\|y\right\| \left\langle T^tT\frac{x}{\|x\|}, \frac{y}{\left\|y\right\|}\right\rangle$, we may conclude that for any fixed $x \in \mathbb{R}^n$ we have $\langle T^tTx, y\rangle = 0$ for every $y \in \mathbb{R}^n$ with $\langle x, y\rangle = 0$. So, T^tTx is a scalar multiple of x, i.e. T^tT and I are locally linearly dependent. It is known that for linear operators of rank at least 2, local linear dependence implies (global) linear dependence (see e.g. [26, page 1869]). Note that $T^tT \in H_n^+(\mathbb{R})$. Therefore,

$$T^tT = \alpha I$$

for some scalar $\alpha > 0$. Let now $Q = \frac{1}{\sqrt{\alpha}}T$. It follows that $Q^tQ = \frac{1}{\alpha}T^tT = I$. So, Q is a linear isometry and since it is also invertible (and thus surjective), it is also coisometry ($QQ^t = I$). For any $P \in P_n(\mathbb{R})$ we thus have $\varphi(P) = P_{Q(\operatorname{Im} P)}$ where Q is an orthogonal operator, i.e. it may be represented with an (orthogonal) matrix Q where $QQ^t = Q^tQ = I$. Therefore, for every $P \in P_n(\mathbb{R})$

$$\operatorname{Im}\varphi(P) = Q(\operatorname{Im} P) = QP(\mathbb{R}^n) = QPQ^t(\mathbb{R}^n) = \operatorname{Im} QPQ^t.$$

Since clearly $QPQ^t \in P_n(\mathbb{R})$, we may conclude that

$$\varphi(P) = QPQ^t$$

for every $P \in P_n(\mathbb{R})$.

Let $\psi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ be defined with

$$\psi(A) = Q^t\varphi(A)Q.$$

Then ψ still preserves the order $\leq^-$ and is bijective. Moreover $\psi(P) = P$ for every $P \in P_n(\mathbb{R})$. We will thus from on assume that

$$\varphi(P) = P$$

for every $P \in P_n(\mathbb{R})$.

6. $\varphi(\lambda P) = \lambda\varphi(P)$ *for every* $P \in P_n(\mathbb{R})$ *of rank-one and every* $\lambda \in [0, \infty)$. Let $P \in P_n(\mathbb{R})$ be of rank-one and let $\lambda > 0$. Since φ preserves the rank, there exists by the spectral theorem $Q \in P_n(\mathbb{R})$ of rank-one and $\mu > 0$ such that

$$\varphi(\lambda P) = \mu Q.$$

Suppose $P \neq Q$. Then $P + \alpha Q$ is of rank-two for every scalar $\alpha > 0$. Since φ is additive, we obtain

$$\begin{aligned}\varphi(P + \lambda P) &= \varphi(P) + \varphi(\lambda P)\\ &= P + \mu Q.\end{aligned}$$

So, on the one hand $\varphi(P + \lambda P)$ is of rank-two but on the other hand $(1 + \lambda)P$ is of rank-one and therefore, since φ preserves the rank, $\varphi(P + \lambda P) = \varphi((1 + \lambda)P)$ is of rank-one, a contradiction. It follows that $P = Q$ and therefore there exists a function $f_P : [0, \infty) \to [0, \infty)$ such that

$$\varphi(\lambda P) = f_P(\lambda)P.$$

Since $\varphi(P) = P$ and $\varphi(0) = 0$, we have $f_P(1) = 1$ and $f_P(0) = 0$. From

$$f_P(\lambda + \mu)P = \varphi((\lambda + \mu)P) = \varphi(\lambda P) + \varphi(\mu P) = f_P(\lambda)P + f_P(\mu)P$$

we may conclude that f_p is additive, i.e. $f_P(\lambda + \mu) = f_P(\lambda) + f_P(\mu)$ for every $\lambda, \mu \in [0, \infty)$. Let r be an arbitrary (but fixed) positive integer. Since f_p is additive, it follows that

$$1 = f_P(1) = f_P\left(r\frac{1}{r}\right) = rf_P\left(\frac{1}{r}\right)$$

and thus $f_P\left(\frac{1}{r}\right) = \frac{1}{r}$. Let now $\frac{q}{r}$ be any (but fixed) nonnegative rational number (here q and r are nonnegative and positive integers, respectively). Then, again by the additivity of f_p,

$$f_P\left(\frac{q}{r}\right) = qf_P\left(\frac{1}{r}\right) = \frac{q}{r}. \tag{15}$$

Note that f_p is monotone increasing. Namely, for $\lambda, \mu \in [0, \infty)$ with $\lambda \leq \mu$ we have $\mu = \lambda + \nu$ for some $\nu \geq 0$. Thus, $f_P(\lambda) \leq f_P(\lambda) + f_P(\nu) = f_P(\mu)$.

Let $\lambda \in (0, \infty)$ be arbitrary. Then λ is a limit of a monotone increasing sequence $\{s_i\}$ of nonnegative rational numbers and a limit of a monotone decreasing sequence $\{z_i\}$ of positive rational numbers. Since for every $i \in \mathbb{N}$, we have by (15), $f_P(s_i) = s_i$ and $f_p(z_i) = z_i$, it follows by the monotonicity of f_P that

$$f_P(\lambda) = \lambda$$

for every $\lambda \in (0, \infty)$. Recall that $f_P(0) = 0$. It follows that

$$\varphi(\lambda P) = \lambda\varphi(P) \tag{16}$$

for every rank-one $P \in P_n(\mathbb{R})$ and every $\lambda \in [0, \infty)$.

We are now in position to conclude the proof of the theorem. Let $A \in H_n^+(\mathbb{R})$ be arbitrary. By the spectral theorem there exist pairwise orthogonal rank-one (idempotent and symmetric) matrices $P_1, P_2, \ldots, P_k \in P_n(\mathbb{R})$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in [0, \infty)$ such that

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k.$$

By (16) and since φ is additive, we may conclude that

$$\varphi(A) = A$$

for every $A \in H_n^+(\mathbb{R})$. To sum up, taking into account our assumptions, a surjective, additive map $\varphi : H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$, $n \geq 3$, that preserves the minus order $\leq^-$ in both directions is of the following form:

$$\varphi(A) = SAS^t$$

for every $A \in H_n^+(\mathbb{R})$ where $S \in M_n(\mathbb{R})$ is an invertible matrix. □

Remark 4.5. *We believe that the same result holds also without the additivity assumption and it would be interesting to find a proof of this conjecture. Also, we expect that a surjective map $\varphi : H_2^+(\mathbb{R}) \to H_2^+(\mathbb{R})$ that preserves the minus order in both directions has the form $\varphi(A) = SAS^t$ for every $A \in H_2^+(\mathbb{R})$ where $S \in M_2(\mathbb{R})$ is an invertible matrix.*

5. Concluding remarks

Many other partial orders may be defined on $M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. The star partial order $\leq^*$ is defined in the following way (see [9]): For $A, B \in M_n(\mathbb{F})$ we write

$$A \leq^* B \quad \text{when} \quad A^*A = A^*B \text{ and } AA^* = BA^*.$$

It is known (see e.g. [24]) that $A \leq^* B$ implies $A \leq^- B$. Two partial orders that are "related" to the minus and the star partial orders are the left-star and the-right star partial orders [3]. For $A, B \in M_n(\mathbb{F})$ we say that A is below B with respect to the left-star partial order and write

$$A \mathrel{*\!\leq} B \quad \text{when} \quad A^*A = A^*B \text{ and } \operatorname{Im} A \subseteq \operatorname{Im} B.$$

Similarly, we define the right-star partial order: For $A, B \in M_n(\mathbb{F})$ we write

$$A \mathrel{\leq\!*} B \quad \text{when} \quad AA^* = AB^* \text{ and } \operatorname{Im} A^* \subseteq \operatorname{Im} B^*.$$

It is known (see [24]) that for $A, B \in M_n(\mathbb{F})$, $A \leq^* B$ implies both $A \mathrel{*\!\leq} B$ and $A \mathrel{\leq\!*} B$ and each $A \mathrel{*\!\leq} B$ and $A \mathrel{\leq\!*} B$ implies $A \leq^- B$. The converse implications do not hold in general. Note that the left-star partial order has applications in the theory of linear models (see [24, Theorem 15.3.7, Corollary 15.3.8]).

Let $A, B \in H_n^+(\mathbb{F})$. Since then $A^*A = A^*B$ if and only if $(A^*A)^* = (A^*B)^*$ if and only if $A^2 = BA$ which is equivalent to $AA^* = BA^*$, we may conclude that the star, the left-star, and the right-star partial orders are the same partial order on $H_n^+(\mathbb{F})$. Maps on $M_n(\mathbb{F})$ preserving these orders have already been studied (see [10, 19]). It would be interesting to describe (surjective) maps that preserve the star order (in both directions) on the set $H_n^+(\mathbb{F})$ of all real or complex positive semidefinite matrices.

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