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Duality for Multiobjective Variational Problems under Second-Order (Φ, ρ)-Invexity

Vivek Singh[a], I. Ahmad[b], S. K. Gupta[c], S. Al-Homidan[b]

[a]*Department of Mathematics and Statistics, Manipal University Jaipur, Rajasthan - 303007, India*
[b]*Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran - 31261, Saudi Arabia*
[c]*Department of Mathematics, Indian Institute of Technology, Roorkee-247667, India*

Abstract. The purpose of this article is to introduce the concept of second order (Φ, ρ)-invex function for continuous case and apply it to discuss the duality relations for a class of multiobjective variational problem. Weak, strong and strict duality theorems are obtained in order to relate efficient solutions of the primal problem and its second order Mond-Weir type multiobjective variational dual problem using aforesaid assumption. A non-trivial example is also exemplified to show the presence of the proposed class of a function.

1. Introduction

A lot of realistic problems arising practically in different fields of science, business and technology deals with several conflicting objectives, which have to be optimized simultaneously. Such problems are known as vector optimization problems or multiobjective optimization problems. Vector optimization programming has been of great interest since decades as it provides the universal appliance for tremendous applications in different area of mathematics such as functional analysis, statistics, approximation theory, cooperative game theory, optimal control theory *etc*. Vector optimization problems have closely related with convex analysis. Under various types of generalized convexities, optimality conditions and duality results for vector optimization problems have been discussed intensively by many authors [2, 4, 7, 11].

The calculus of variation plays a vital part in various fields of mathematics, science and engineering. It is connected with the optimization of functionals, mapped from a set of functions to the real numbers and are formulated in terms of definite integrals involving functions and their derivatives. The problem of evaluating a piecewise smooth extremal $x = x(t)$ for the integral

$$\int_a^b f(t, x, \dot{x})dt$$

is called a variational problem.

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Corresponding author: I. Ahmad
Email addresses: viveksingh.25jun@gmail.com (Vivek Singh), drizhar@kfupm.edu.sa (I. Ahmad), guptashiv@gmail.com (S. K. Gupta), homidan@kfupm.edu.sa (S. Al-Homidan)

Mond and Hanson [18] were the first who introduced the idea of duality in variational problems. They studied the following primal and dual problems under convexity assumptions:

(P) Minimize $\int_a^b f(t,x,\dot{x})dt$

subject to $Q(t,x,\dot{x}) \geqq 0,\ x(a) = x_0,\ x(b) = x_1,$

(D) Maximize $\int_a^b [f(t,x,\dot{x}) - \lambda(t)Q(t,x,\dot{x})]dt$

subject to $f_x(t,x,\dot{x}) - \lambda(t)Q_x(t,x,\dot{x}) = \frac{d}{dt}[f_{\dot{x}}(t,x,\dot{x}) - \lambda(t)Q_{\dot{x}}(t,x,\dot{x})],$

$\lambda(t) \geqq 0,\ x(a) = x_0,\ x(b) = x_1,$

where $I = [a,b]$ is a real interval, $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $g : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$; $x(t) \in \mathbb{R}^n$ is a piecewise smooth function of t and $\dot{x}(t)$ is the derivative of $x(t)$. The duality related to symmetric nonlinear programming problems were extended to variational optimization problems by Bector et al. [5].

The various generalizations to convexity assumptions like (F,ρ) convexity [20], second-order (F,α,ρ,d) convexity [2], (F,α,ρ,d) type-I convexity [10] , (ρ,b) quasi-invexity [21] etc are there in the literature. Researchers have formulated different variational primal-dual pairs and studied duality relations under different convexity assumptions. For instance, single/multiobjective variational problems [1, 5, 6, 8-10, 12, 14, 16, 18], multiobjective fractional variational models [21], nondifferentiable fractional variational problems [16] and over arbitrary cones [14].

Mishra et al. [16] concentrated their study on symmetric duality for fractional variational problems containing nondifferentiable terms and proved usual duality theorems under invexity. Kailey and Gupta [14] extended the results [16] over arbitrary cones under generalized (F,α,ρ,d)-convexity assumptions. By using the concept of (ϕ,ρ)-invex functions and G-type I functions to the continuous case, Antczak [3, 4] discussed optimality conditions and usual duality theorems for multiobjective variational control models under weaker convexity assumptions.

Motivated by the work, we present the concept of second order (Φ,ρ)-invexity for a multiobjective variational problem with an example. The structure of this article is as follows: After a short introduction, in Section 2, we give some notations and definitions utilized throughout the article. In Section 3, we address Mond-Weir type duality results to multiobjective variational problems under second order (Φ,ρ)-invexity. At last, in Section 4, we present conclusions and future intent of research.

2. Notation and Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. If $p,q \in \mathbb{R}^n$, then

$$p \leqq q \Leftrightarrow p_i \leqq q_i,\ i = 1,2,\cdots,n;$$

$$p \leq q \Leftrightarrow p \leqq q \text{ and } p \neq q;$$

$$p < q \Leftrightarrow p_i < q_i,\ i = 1,2,\cdots,n.$$

Consider a real number $I = [a,b]$. Let X denote the space of all piecewise smooth functions $x : I \to \mathbb{R}^n$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiable operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^b u(s)ds$$

where α is a given boundary value. Therefore $\frac{d}{dt} \equiv D$ except at discontinuities. Let, for each $i \in K = \{1,2,\cdots,k\}$, $f_i(t,x(t),\dot{x}(t))$ and for each $j \in J = \{1,2,\cdots,m\}$, $g_j(t,x(t),\dot{x}(t))$, where $t \in I$ is an independent variable and dot denotes the derivatives with respect to t, be twice continuously differentiable functions.

For convenience, we write $f_i(t,x,\dot{x})$ instead of $f_i(t,x(t),\dot{x}(t))$.The gradient vector of the function f_i with respect to x and $\dot{x}$, respectively, is given by

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \cdots, \frac{\partial f_i}{\partial x_n}\right)^T \text{ and } f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \frac{\partial f_i}{\partial \dot{x}_2}, \cdots, \frac{\partial f_i}{\partial \dot{x}_n}\right)^T$$

where the symbol T the transpose of a vector. In the same approach, the Hessian matrix of f_{ixx} denote a symmetric $n \times n$ matrix. Similarly, $f_{ix\dot{x}}$, $f_{i\dot{x}\dot{x}}$ and the gradients of g_j can also be defined.

Definition 2.1. *Let $\Phi : I \times X \times X \times X \times X \times \mathbb{R}^{n+1} \to \mathbb{R}$. A functional $\Phi(t,x,\dot{x},u,\dot{u};(a,\rho))$ is convex on $\mathbb{R}^{n+1}$, if for any $x,\dot{x},u,\dot{u} \in X$, the following inequality*

$$\Phi(t,x,\dot{x},u,\dot{u};(\lambda(a_1,\rho_1)+(1-\lambda)(a_2,\rho_2)) \leqq \lambda\Phi(t,x,\dot{x},u,\dot{u};(a_1,\rho_1))+(1-\lambda)\Phi(t,x,\dot{x},u,\dot{u};(a_2,\rho_2)),$$

holds for all $a_1,a_2 \in \mathbb{R}^n$, $\rho_1,\rho_2 \in \mathbb{R}$ and for any $\lambda \in [0,1]$.

Now, we introduce the definition of a second order (Φ,ρ)-invex function. Let $\varphi : I \times X \times X \to \mathbb{R}$ be a real valued function. Let $\Phi : I \times X \times X \times X \times X \times \mathbb{R}^{n+1} \to \mathbb{R}$ be convex on $\mathbb{R}^{n+1}$, where $\Phi(t,x,\dot{x},u,\dot{u};(0,\rho)) \geqq 0$ for all $x \in X$ and every $\rho \in \mathbb{R}_+$. Let $C(I,\mathbb{R}^n)$ be the space of continuous n-dimensional vector functions.

Definition 2.2. *The functional $\int_a^b \varphi(t,x,\dot{x})dt$ is called (strictly) second-order (Φ,ρ)-invex at $u(t) \in X$, if there exist a real-valued function Φ, a real number ρ and $\beta \in C(I,\mathbb{R}^n)$ such that for all $x(t) \in X$,*

$$\int_a^b \varphi(t,x,\dot{x})dt - \int_a^b \varphi(t,u,\dot{u})dt + \frac{1}{2}\int_a^b \beta(t)^T\theta\beta(t)dt$$

$$(>) \geqq \int_a^b \Phi(t,x,\dot{x},u,\dot{u};(\varphi_x(t,u,\dot{u}) - D\varphi_{\dot{x}}(t,u,\dot{u}) + \theta\beta(t),\rho))dt, \tag{1}$$

where $\theta(t,x,\dot{x},\ddot{x},\dddot{x}) = \varphi_{xx}(t,x,\dot{x}) - 2D\varphi_{x\dot{x}}(t,x,\dot{x}) + D^2\varphi_{\dot{x}\dot{x}}(t,x,\dot{x}) - D^3\varphi_{\dot{x}\dot{x}}(t,x,\dot{x})$, $t \in I$.
If the functional $\int_a^b \varphi(t,x,\dot{x})dt$ satisfies the inequality (1) at each $x(t) \in X$, then $\int_a^b \varphi(t,x,\dot{x})dt$ is said to be second-order (Φ,ρ)-invex on X.

Remark 2.3. (*i*) *Let $\Phi(t,x,\dot{x},u,\dot{u};(a,\rho)) = F(t,x,\dot{x},u,\dot{u};\alpha a)+\rho d^2(t,x,u)$ where $F : I\times X\times X\times X\times X\times \mathbb{R}^n \to \mathbb{R}$ be a sublinear with respect to six variable, $\alpha : X\times X \to \mathbb{R}_+ \setminus \{0\}$ and $d : I\times X\times X \to \mathbb{R}$ be real valued functions. Then the above definition reduce to a second order (F,α,ρ,d)-convex [13].*

(*ii*) *If $\beta(t) = 0$, then we obtain the definition of (Φ,ρ)-invexity introduced by Antczak [3].*

(*iii*) *Let $\Phi(t,x,\dot{x},u,\dot{u};(a,\rho)) = \eta^T a$, where $\eta : I\times X\times X \to \mathbb{R}^n$. Then the above definition becomes that second order invex [11]. In addition, if $\theta = 0$, then we get the definition of invexity introduced by Mond et al. [17].*

The following example shows that there exists a functional which is second-order (Φ,ρ)-invex but neither (Φ,ρ)-invex nor convex.

Example 2.4. *Let $I = [0,1]$ and $X = [0,1]\times[0,1] \subset \mathbb{R}^2$. Consider the functional $\varphi : I \times X \times X \to \mathbb{R}$ which is defined by*

$$\varphi(t,x,\dot{x}) = x_1^3(t) - \sin x_2(t).$$

Let the functional $\Phi : I\times X\times X\times X\times X\times \mathbb{R}^3 \to \mathbb{R}$ be given by

$$\Phi(t,x,\dot{x},u,\dot{u};(\varphi_x - D\varphi_{\dot{x}} + \theta\beta(t),\rho)) = (\varphi_{x_1} - \varphi_{x_2})|u_1u_2| - \rho|x_1 - x_2|.$$

For $\rho = 1$, $\beta(t) = (\frac{1}{2},1)$ and $u(t) = (0,0)$, we have

$$\psi = \int_0^1 \varphi(t,x,\dot{x})dt - \int_0^1 \varphi(t,u,\dot{u})dt + \frac{1}{2}\int_0^1 \beta(t)^T\theta\beta(t)dt - \int_0^1 \Phi(t,x,\dot{x},u,\dot{u};(\varphi_x(t,u,\dot{u}) - D\varphi_{\dot{x}}(t,u,\dot{u}) + \theta\beta(t),\rho))dt$$

$$= \int_0^1 (x_1^3(t) - \sin x_2(t))dt + \int_0^1 \frac{1}{2}(\frac{3}{2}x_1(t) + \sin x_2(t))dt - \int_0^1 -|x_1(t) - x_2(t)|dt$$

$$= \int_0^1 (x_1^3(t) - \frac{1}{2}\sin x_2(t) + \frac{3}{4}x_1(t) + |x_1(t) - x_2(t)|)dt$$

$\geqq 0, \forall x \in X$, *as it can be seen from Figure 1.*

Hence $\psi \geqq 0$. *This means by Definition 2.2 that the functional* $\int_0^1 \varphi(t,x,\dot{x})dt$ *is second-order* (Φ,ρ)*-invex at*

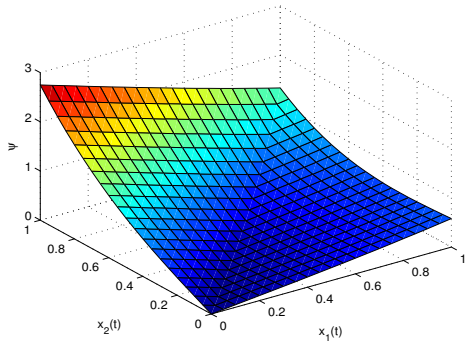

Figure 1: Graph of ψ against $x_1(t)$ and $x_2(t)$

$u(t) = (0,0)$. *Further, note that* $\int_0^1 \varphi(t,x,\dot{x})dt$ *is not* (Φ,ρ)*-invex at* $u(t) = (0,0)$, *since, for* $\rho = 1$ *and* $\beta(t) = (\frac{1}{2},1)$, *we obtain*

$$\psi_1 = \int_0^1 \varphi(t,x,\dot{x})dt - \int_0^1 \varphi(t,u,\dot{u})dt - \int_0^1 \Phi(t,x,\dot{x},u,\dot{u};(\varphi_x(t,u,\dot{u}) - D\varphi_{\dot{x}}(t,u,\dot{u}),\rho))dt$$

$$= \int_0^1 (x_1^3(t) - \sin x_2(t))dt - \int_0^1 -|x_1(t) - x_2(t)|dt$$

$$= \int_0^1 (x_1^3(t) - \sin x_2(t) + |x_1(t) - x_2(t)|)dt$$

$\ngeqq 0, \forall x \in X$, *as it can be seen from Figure 2.*

Further, $\int_0^1 \varphi(t,x,\dot{x})dt$ *is not convex functional can be seen as follows:*

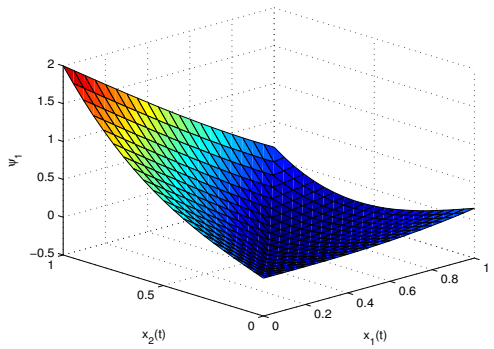

Figure 2: Graph of ψ_1 against $x_1(t)$ and $x_2(t)$

$$\psi_2 = \int_0^1 \varphi(t,x,\dot{x})dt - \int_0^1 \varphi(t,u,\dot{u})dt - \int_0^1 (x-u)\varphi_x(t,u,\dot{u}) - D(x-u)\varphi_{\dot{x}}(t,u,\dot{u})dt$$

$$= \int_0^1 (x_1^3(t) - \sin x_2(t) - x_1^2(t)x_2(t)\cos x_2(t))dt$$

$\ngeqq 0, \forall x \in X$, *as it can be seen from Figure 3.*

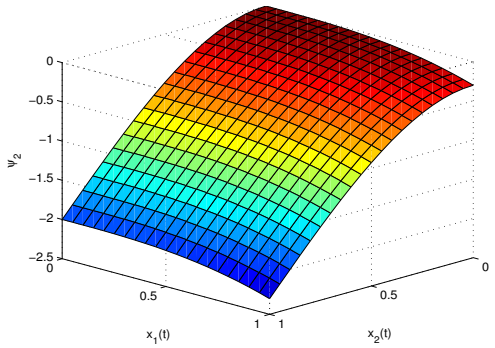

Figure 3: Graph of ψ_2 against $x_1(t)$ and $x_2(t)$

In the present article, we consider the following multiobjective variational problem (MVP):

$$\text{(MVP)} \qquad \text{Minimize} \int_a^b f(t,x,\dot{x})dt$$

$$= \left(\int_a^b f_1(t,x,\dot{x})dt, \cdots, \int_a^b f_k(t,x,\dot{x})dt \right)$$

$$\text{subject to} \quad g(t,x,\dot{x}) \leqq 0, t \in I, \tag{2}$$

$$x(a) = \gamma, x(b) = \delta \tag{3}$$

where $f_i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \in K$ and $g_j : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $j \in J$ are twice continuously differentiable functions. The region where the constraints are satisfied (feasibility region) is given by

$$\Omega = \{x \in X : x(a) = \gamma, x(b) = \delta \text{ and } g(t,x,\dot{x}) \leqq 0, t \in I\}.$$

Definition 2.5. *A point $u \in \Omega$ is an efficient solution of (MVP) if there exists no another $x \in \Omega$ such that*

$$\int_a^b f_i(t,x,\dot{x})dt \leqq \int_a^b f_i(t,u,\dot{u})dt, \forall i \in K_r,$$

and

$$\int_a^b f_r(t,x,\dot{x})dt < \int_a^b f_r(t,u,\dot{u})dt, \text{ for some } r \in K.$$

3. Duality relations

We consider the following second-order Mond-Weir type multiobjective variational dual problem for (MVP) formulated by Gulati and Mehndiratta [9]:

$$\textbf{(VMWD)} \qquad \text{Maximize} \int_a^b (f(t,y,\dot{y}) - \frac{1}{2}\beta(t)^T F\beta(t))dt$$

$$= \left(\int_a^b (f_1(t, y, \dot{y}) - \frac{1}{2}\beta(t)^T F_1 \beta(t))dt, \cdots, \int_a^b (f_k(t, y, \dot{y}) - \frac{1}{2}\beta(t)^T F_k \beta(t))dt \right)$$

subject to

$$\sum_{i=1}^{k} \lambda_i (f_{ix}(t, y, \dot{y}) - D f_{i\dot{x}}(t, y, \dot{y}) + F_i \beta(t)) + g_x(t, y, \dot{y})\xi(t) - D(g_{\dot{x}}(t, y, \dot{y})\xi(t) + G\beta(t) = 0, t \in I \tag{4}$$

$$\xi(t)^T g(t, y, \dot{y}) - \frac{1}{2}\beta(t)^T G\beta(t) \geqq 0, t \in I \tag{5}$$

$$u(a) = \gamma, u(b) = \delta, \tag{6}$$

$$\lambda \geq 0, \xi(t) \geqq 0, t \in I, \tag{7}$$

where $F_i(t, y, \dot{y}, \ddot{y}, \dddot{y}, \ddddot{y}) = f_{ixx}(t, y, \dot{y}) - 2D f_{ix\dot{x}}(t, y, \dot{y}) + D^2 f_{i\dot{x}\dot{x}}(t, y, \dot{y}) - D^3 f_{i\dot{x}\ddot{x}}(t, y, \dot{y}), t \in I$ and $G(t, y, \dot{y}, \ddot{y}, \dddot{y}, \ddddot{y}, \xi(t), \dot{\xi}(t), \ddot{\xi}(t), \dddot{\xi}(t)) = (g_x(t, y, \dot{y})\xi(t))_x - 2D(g_x(t, y, \dot{y})\xi(t))_{\dot{x}} + D^2(g_{\dot{x}}(t, y, \dot{y})\xi(t))_{\dot{x}} - D^3(g_{\dot{x}}(t, y, \dot{y})\xi(t))_{\ddot{x}}, t \in I$.

Now, we prove duality relations between multiobjective variational problem (MVP) and its second-order Mond-Weir type dual model (VMWD).

Theorem 3.1 (Weak Duality). *Let $x(t)$ and $(y(t), \lambda, \xi(t), \beta(t))$ be feasible solutions of* (MVP) *and* (VMWD)*, respectively. Assume that the following conditions hold:*

(*i*) $\int_a^b f_i(t, ., .)dt$, *for* $i = 1, 2, \cdots, k$ *is second-order* (Φ, ρ_i)*-invex at* $y(t)$,

(*ii*) $\int_a^b \sum_{j=1}^{m} \xi_j(t) g_j(t, ., .)dt$ *is second-order* $(\Phi, \check{\rho})$*-invex at* $y(t)$,

(*iii*) $\lambda > 0$ *and* $\sum_{i=1}^{k} \lambda_i \rho_i + \check{\rho} \geqq 0$.

Then, the following can not hold simultaneously :

$$\int_a^b f_i(t, x, \dot{x})dt \leqq \int_a^b \{f_i(t, y, \dot{y}) - \frac{1}{2}\beta(t)^T F_i \beta(t)\}dt, \forall i \in K_r \tag{8}$$

and

$$\int_a^b f_r(t, x, \dot{x})dt < \int_a^b \{f_r(t, y, \dot{y}) - \frac{1}{2}\beta(t)^T F_r \beta(t)\}dt, \text{ for some } r \in K. \tag{9}$$

Proof. We prove by contradiction. Suppose, to the contrary that (8) and (9) hold. Then in view of $\lambda > 0$, we have

$$\int_a^b \sum_{i=1}^{k} \lambda_i f_i(t, x, \dot{x})dt < \int_a^b \sum_{i=1}^{k} \lambda_i \{f_i(t, y, \dot{y}) - \frac{1}{2}\beta(t)^T F_i \beta(t)\}dt \tag{10}$$

By the hypotheses (*i*) and (*ii*) together with Definition 2.2, we obtain

$$\int_a^b f_i(t, x, \dot{x})dt - \int_a^b f_i(t, y, \dot{y})dt + \frac{1}{2}\int_a^b \beta(t)^T F_i \beta(t)dt$$

$$\geqq \int_a^b \Phi(t, x, \dot{x}, y, \dot{y}; (f_{ix}(t, y, \dot{y}) - D f_{i\dot{x}}(t, y, \dot{y}) + F_i \beta(t), \rho_i))dt \tag{11}$$

and

$$\int_a^b \sum_{j=1}^{m} \xi_j(t) g_j(t,x,\dot{x})dt - \int_a^b \sum_{j=1}^{m} \xi_j(t) g_j(t,y,\dot{y})dt + \frac{1}{2}\int_a^b \beta(t)^T G\beta(t)dt$$

$$\geqq \int_a^b \Phi(t,x,\dot{x},y,\dot{y};(\sum_{j=1}^{m} \xi_j(t) g_{jx}(t,y,\dot{y}) - D(\sum_{j=1}^{m} \xi_j(t) g_{j\dot{x}}(t,y,\dot{y})) + G\beta(t),\check{\rho}))dt. \tag{12}$$

Take $A = \sum_{i=1}^{k} \lambda_i + 1$. It is easy to see that $A > 0$.

Now, multiplying inequality (11) by $\frac{\lambda_i}{A} \geq 0$, $i = 1, 2, \cdots, k$ and inequality (12) by $\frac{1}{A} > 0$, then summing the resultant inequalities, we get

$$\begin{aligned}
&\left[\int_a^b \sum_{i=1}^{k} \frac{\lambda_i}{A} f_i(t,x,\dot{x})dt - \int_a^b \sum_{i=1}^{k} \frac{\lambda_i}{A}\{f_i(t,y,\dot{y}) - \frac{1}{2}\beta(t)^T F_i\beta(t)\}dt\right] \\
&+ \left[\int_a^b \sum_{j=1}^{m} \frac{\xi_j}{A}(t) g_j(t,x,\dot{x})dt - \int_a^b \sum_{j=1}^{m} \frac{\xi_j}{A}(t) g_j(t,y,\dot{y})dt + \frac{1}{2A}\int_a^b \beta(t)^T G\beta(t)dt\right] \\
&\geqq \int_a^b \sum_{i=1}^{k} \frac{\lambda_i}{A} \Phi\left(t,x,\dot{x},y,\dot{y};(f_{ix}(t,y,\dot{y}) - Df_{i\dot{x}}(t,y,\dot{y}) + F_i\beta(t),\rho_i)\right)dt \\
&+ \int_a^b \frac{1}{A}\Phi\left(t,x,\dot{x},y,\dot{y};\left(\sum_{j=1}^{m} \xi_j(t) g_{jx}(t,y,\dot{y}) - D(\sum_{j=1}^{m} \xi_j(t) g_{j\dot{x}}(t,y,\dot{y})) + G\beta(t),\check{\rho}\right)\right)dt.
\end{aligned}$$

On using the feasibility of $x(t)$ and $(y(t), \lambda, \xi(t), \beta(t))$ of (MVP) and (VMWD), respectively, the inequality above yields

$$\begin{aligned}
&\frac{1}{A}\int_a^b \sum_{i=1}^{k} \lambda_i \left[f_i(t,x,\dot{x})dt - f_i(t,y,\dot{y}) + \frac{1}{2}\beta(t)^T F_i\beta(t)\right]dt \\
&\geqq \int_a^b \sum_{i=1}^{k} \frac{\lambda_i}{A}\Phi(t,x,\dot{x},y,\dot{y};(f_{ix}(t,y,\dot{y}) - Df_{i\dot{x}}(t,y,\dot{y}) + F_i\beta(t),\rho_i))dt \\
&+ \int_a^b \frac{1}{A}\Phi\left(t,x,\dot{x},y,\dot{y};\left(\sum_{j=1}^{m} \xi_j(t) g_{jx}(t,y,\dot{y}) - D(\sum_{j=1}^{m} \xi_j(t) g_{j\dot{x}}(t,y,\dot{y})) + G\beta(t),\check{\rho}\right)\right)dt.
\end{aligned}$$

By the convexity of $\Phi(t,x,\dot{x},y,\dot{y};(.,.))$ on $\mathbb{R}^{n+1}$, the above inequality becomes

$$\begin{aligned}
&\frac{1}{A}\int_a^b \sum_{i=1}^{k} \lambda_i \left[f_i(t,x,\dot{x})dt - f_i(t,y,\dot{y}) + \frac{1}{2}\beta(t)^T F_i\beta(t)\right]dt \\
&\qquad \geqq \int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\frac{1}{A}\left(\sum_{i=1}^{k} \lambda_i[f_{ix}(t,y,\dot{y}) - Df_{i\dot{x}}(t,y,\dot{y}) + F_i\beta(t)]\right.\right. \\
&\qquad\quad \left.\left. + \sum_{j=1}^{m} \xi_j(t) g_{jx}(t,y,\dot{y}) - D(\sum_{j=1}^{m} \xi_j(t) g_{j\dot{x}}(t,y,\dot{y})) + G\beta(t), \sum_{i=1}^{k} \lambda_i\rho_i + \check{\rho}\right)\right)dt.
\end{aligned}$$

The above inequality together with the first dual constraint (4), the hypothesis (*iii*) and the fact $\Phi(t, x, \dot{x}, y, \dot{y}; (0, \rho)) \geqq 0$, for every $\rho \in \mathbb{R}_+$ reduces to

$$\frac{1}{A}\int_a^b \sum_{i=1}^{k} \lambda_i \left[f_i(t, x, \dot{x})dt - f_i(t, y, \dot{y}) + \frac{1}{2}\beta(t)^T F_i \beta(t) \right] dt \geqq 0.$$

Since $A > 0$, the above inequality implies that

$$\int_a^b \sum_{i=1}^{k} \lambda_i f_i(t, x, \dot{x})dt \leqq \int_a^b \sum_{i=1}^{k} \lambda_i \{f_i(t, y, \dot{y}) - \frac{1}{2}\beta(t)^T F_i \beta(t)\}dt,$$

which contradicts (10). □

Theorem 3.2 (Strong duality). *Let $\bar{x}(t)$ be a normal and an efficient solution in problem* (MVP). *Then, there exist $\bar{\lambda} \in \mathbb{R}^k$ and the piecewise smooth functions $\bar{\xi}(.) : I \to \mathbb{R}^m$ such that $(\bar{x}(t), \bar{\lambda}, \bar{\xi}(t), \bar{\beta}(t) = 0)$ is a feasible solution of* (VMWD) *and the corresponding objective values of* (MVP) *and* (VMWD) *are equal. Further, if the hypotheses of Theorem 3.1 holds, then $(\bar{x}(t), \bar{\lambda}, \bar{\xi}(t), \bar{\beta}(t) = 0)$ is an efficient solution of* (VMWD).

Proof. By the assumption, $\bar{x}(t)$ is a normal and an efficint solution of (MVP), hence by Theorem 4 [9], there exist $\bar{\lambda} \in \mathbb{R}^k$ and the piecewise smooth functions $\bar{\xi} : I \to \mathbb{R}^m$ such that the following conditions are satisfied:

$$\sum_{i=1}^{k} \bar{\lambda}\{f_{ix}(t, \bar{x}, \dot{\bar{x}}) - Df_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}})\} + \bar{\xi}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}) - D(\bar{\xi}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) = 0, t \in I \tag{13}$$

$$\bar{\xi}(t)^T g(t, \bar{x}, \dot{\bar{x}}) = 0, t \in I \tag{14}$$

$$\bar{\lambda} \geq 0, \tag{15}$$

$$\bar{\xi}(t) \geqq 0, t \in I. \tag{16}$$

On utilizing the hypothesis $\bar{\beta}(t) = 0$ in equations (13) and (14), we obtain

$$\sum_{i=1}^{k} \bar{\lambda}\{f_{ix}(t, \bar{x}, \dot{\bar{x}}) - Df_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + F_i\bar{\beta}(t)\} + \bar{\xi}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}) - D(\bar{\xi}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + G\bar{\beta}(t) = 0, t \in I, \tag{17}$$

and

$$\bar{\xi}(t)^T g(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T G\bar{\beta}(t) = 0, t \in I. \tag{18}$$

Consequently (15)-(18) imply that $(\bar{x}(t), \bar{\lambda}, \bar{\xi}(t), \bar{\beta}(t) = 0)$ is feasible solution of (VMWD) and the objective values of (MVP) and (VMWD) are equal. Now, assume on the contrary that $(\bar{x}(t), \bar{\lambda}, \bar{\xi}(t), \bar{\beta}(t) = 0)$ is not an efficient solution for (VMWD), then there exists a point $(\tilde{y}(t), \tilde{\lambda}, \tilde{\xi}(t), \tilde{\beta}(t) = 0)$ feasible of (VMWD) such that

$$\int_a^b \{f_i(t, \tilde{y}, \dot{\tilde{y}}) - \frac{1}{2}\tilde{\beta}(t)^T F_i \tilde{\beta}(t)\}dt \geqq \int_a^b \{f_i(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T F_i \bar{\beta}(t)\}dt, \forall i \in K_r \text{ and}$$

$$\int_a^b f_r(t, \tilde{y}, \dot{\tilde{y}}) - \frac{1}{2}\tilde{\beta}(t)^T F_r \tilde{\beta}(t)\}dt > \int_a^b \{f_r(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T F_r \bar{\beta}(t)\}dt, \text{ for some } r \in K.$$

Since $\bar{\beta}(t) = 0, t \in I$, we get

$$\int_a^b \{f_i(t, \tilde{y}, \dot{\tilde{y}}) - \frac{1}{2}\tilde{\beta}(t)^T F_i \tilde{\beta}(t)\}dt \geqq \int_a^b f_i(t, \bar{x}, \dot{\bar{x}})dt, \forall i \in K_r,$$

$$\int_a^b f_r(t, \tilde{y}, \dot{\tilde{y}}) - \frac{1}{2}\tilde{\beta}(t)^T F_r \tilde{\beta}(t)\}dt > \int_a^b f_r(t, \bar{x}, \dot{\bar{x}})dt, \text{ for some } r \in K,$$

which contradicts the Theorem 3.1. Hence, $(\bar{x}(t), \bar{\lambda}, \bar{\xi}(t), \bar{\beta}(t) = 0)$ is an efficient solution of (VMWD). □

Theorem 3.3 (Strict converse duality). *Let $\bar{x}(t)$ and $(\bar{y}(t), \bar{\lambda}, \bar{\xi}(t), \bar{\beta}(t))$ be efficient solutions of* (MVP) *and* (VMWD), *respectively, such that*

$$\int_a^b \sum_{i=1}^{k} \bar{\lambda}_i f_i(t, \bar{x}, \dot{\bar{x}})dt = \int_a^b \sum_{i=1}^{k} \bar{\lambda}_i \{f_i(t, \bar{y}, \dot{\bar{y}}) - \frac{1}{2}\bar{\beta}(t)^T F_i(t, \bar{y}, \dot{\bar{y}}, \ddot{\bar{y}}, \dddot{\bar{y}}, \ddddot{\bar{y}})\bar{\beta}(t)\}dt \tag{19}$$

Further, assume that the following conditions hold:

(*i*) $\int_a^b f_i(t, ., .)dt$, *for* $i = 1, 2, \cdots, k$ *is strictly second-order* (Φ, ρ_i)*-invex at* $\bar{y}(t)$,

(*ii*) $\int_a^b \sum_{j=1}^{m} \bar{\xi}_j(t) g_j(t, ., .)dt$ *is second-order* $(\Phi, \check{\rho})$*-invex at* $\bar{y}(t)$,

(*iii*) $\sum_{i=1}^{k} \lambda_i \rho_i + \check{\rho} \geqq 0$.

Then, $\bar{x}(t) = \bar{y}(t)$.

Proof. Suppose, contrary to the result, that $\bar{x}(t_o) \neq \bar{y}(t_0)$ for some $t_0 \in I$.
Now, from the hypotheses (*i*) and (*ii*), we have

$$\int_a^b f_i(t_0, \bar{x}, \dot{\bar{x}})dt - \int_a^b f_i(t_0, \bar{y}, \dot{\bar{y}})dt + \frac{1}{2}\int_a^b \bar{\beta}(t_0)^T F_i \bar{\beta}(t_0)dt$$

$$> \int_a^b \Phi(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; (f_{ix}(t_0, \bar{y}, \dot{\bar{y}}) - Df_{i\dot{x}}(t_0, \bar{y}, \dot{\bar{y}}) + F_i\bar{\beta}(t_0), \rho_i))dt, \tag{20}$$

and

$$\int_a^b \sum_{j=1}^{m} \bar{\xi}_j(t_0) g_j(t_0, \bar{x}, \dot{\bar{x}})dt - \int_a^b \sum_{j=1}^{m} \bar{\xi}_j(t_0) g_j(t_0, \bar{y}, \dot{\bar{y}})dt + \frac{1}{2}\int_a^b \bar{\beta}(t_0)^T G\bar{\beta}(t_0)dt$$

$$\geqq \int_a^b \Phi(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; (\sum_{j=1}^{m} \bar{\xi}_j(t_0) g_{jx}(t_0, \bar{y}, \dot{\bar{y}}) - D(\sum_{j=1}^{m} \bar{\xi}_j(t_0) g_{j\dot{x}}(t_0, \bar{y}, \dot{\bar{y}})) + G\bar{\beta}(t_0), \check{\rho}))dt. \tag{21}$$

Take $\bar{A} = \sum_{i=1}^{k} \bar{\lambda}_i + 1$. It is easy to see that $\bar{A} > 0$.

Combining (19)- (20), and multiplying the obtained inequality by $\frac{\bar{\lambda}_i}{\bar{A}} \geq 0$, $i = 1, 2, \cdots, k$, then taking summation over i, we get

$$\int_a^b \frac{\bar{\lambda}_i}{\bar{A}} \Phi(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; (f_{ix}(t_0, \bar{y}, \dot{\bar{y}}) - Df_{i\dot{x}}(t_0, \bar{y}, \dot{\bar{y}}) + F_i\bar{\beta}(t_0), \rho_i))dt < 0. \tag{22}$$

By feasibility of $\bar{x}(t)$ in (MVP) and (7), inequality (21) becomes

$$-\int_a^b \sum_{j=1}^{m} \bar{\xi}_j(t_0) g_j(t_0, \bar{y}, \dot{\bar{y}})dt + \frac{1}{2}\int_a^b \bar{\beta}(t_0)^T G\bar{\beta}(t_0)dt$$

$$\geqq \int_a^b \Phi(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; (\sum_{j=1}^{m} \bar{\xi}_j(t_0) g_{jx}(t_0, \bar{y}, \dot{\bar{y}}) - D(\sum_{j=1}^{m} \bar{\xi}_j(t_0) g_{j\dot{x}}(t_0, \bar{y}, \dot{\bar{y}})) + G\bar{\beta}(t_0), \check{\rho}))dt.$$

Further, using the feasibility $(\bar{y}(t), \bar{\lambda}, \bar{\xi}(t), \bar{\beta}(t))$ in (VMWD), and then multiplying by $\bar{A} > 0$, the above inequality yields

$$\int_a^b \frac{1}{\bar{A}} \Phi(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; (\sum_{j=1}^{m} \bar{\xi}_j(t_0) g_{jx}(t_0, \bar{y}, \dot{\bar{y}}) - D(\sum_{j=1}^{m} \bar{\xi}_j(t_0) g_{j\dot{x}}(t_0, \bar{y}, \dot{\bar{y}})) + G\bar{\beta}(t_0), \check{\rho}))dt \leqq 0. \tag{23}$$

On adding (22) and (23), we get

$$\int_a^b \frac{\bar{\lambda}}{\bar{A}} \Phi(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; (f_{ix}(t_0, \bar{y}, \dot{\bar{y}}) - D f_{i\dot{x}}(t_0, \bar{y}, \dot{\bar{y}}) + F_i \bar{\beta}(t_0), \rho_i))dt$$

$$+ \int_a^b \frac{1}{\bar{A}} \Phi(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; (\sum_{j=1}^{m} \bar{\xi}_j(t_0) g_{jx}(t_0, \bar{y}, \dot{\bar{y}}) - D(\sum_{j=1}^{m} \bar{\xi}_j(t_0) g_{j\dot{x}}(t_0, \bar{y}, \dot{\bar{y}})) + G\bar{\beta}(t_0), \check{\rho}))dt < 0.$$

The above inequality along with convexity of $\Phi(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; (.,.))$ on $\mathbb{R}^{n+1}$, reduces to

$$\int_a^b \Phi\Bigg(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; \frac{1}{\bar{A}}\Bigg(\sum_{i=1}^{k} \bar{\lambda}_i [f_{ix}(t_0, \bar{y}, \dot{\bar{y}}) - D f_{i\dot{x}}(t_0, \bar{y}, \dot{\bar{y}}) + F_i \beta(t)]$$
$$+ \sum_{j=1}^{m} \xi_j(t) g_{jx}(t_0, \bar{y}, \dot{\bar{y}}) - D(\sum_{j=1}^{m} \xi_j(t) g_{j\dot{x}}(t_0, \bar{y}, \dot{\bar{y}})) + G\beta(t), \sum_{i=1}^{k} \lambda_i \rho_i + \check{\rho}\Bigg)\Bigg) dt < 0.$$

Hence, the first constraint of (VMWP) implies

$$\int_a^b \Phi\Bigg(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; \frac{1}{\bar{A}}\Bigg(0, \sum_{i=1}^{k} \lambda_i \rho_i + \check{\rho}\Bigg)\Bigg) dt < 0. \tag{24}$$

On the other hand, by the hypothesis (*iii*), we obtain

$$\sum_{i=1}^{k} \lambda_i \rho_i + \check{\rho} \geqq 0.$$

The above inequality together with the fact $\Phi(t, x, \dot{x}, y, \dot{y}; (0, \rho)) \geqq 0$ for every $\rho \in \mathbb{R}_+$ becomes

$$\int_a^b \Phi\Bigg(t_0, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}; \frac{1}{\bar{A}}\Bigg(0, \sum_{i=1}^{k} \lambda_i \rho_i + \check{\rho}\Bigg)\Bigg) dt \geqq 0,$$

which contradicts to (24). Hence, $\bar{x}(t) = \bar{y}(t)$. This completes the proof. □

4. Conclusions

We have introduced the idea of a second order (Φ, ρ)-invex function, which includes several generalized convexity concepts in optimization theory as special cases. We have utilized the introduced class of functions to multiobjective variational problem and its second order dual problem. Moreover, a non-trivial example has been demonstrated to show the existence of such a function. This work can be further extended to study for higher order multiobjective variational problems and nondifferentiable multiobjective variational problems involving support functions, which will be some potential future directions.

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