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Johnson Pseudo-Connes Amenability of Dual Banach Algebras

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Abstract.
We introduce the notion of Johnson pseudo-Connes amenability for dual Banach algebras. We study the relation between this new notion with the various notions of Connes amenability like Connes amenability, approximate Connes amenability and pseudo Connes amenability. We also investigate some hereditary properties of this new notion. We prove that for a locally compact group G, $M(G)$ is Johnson pseudo-Connes amenable if and only if G is amenable. Also we show that for every non-empty set I, $\mathbb{M}_I(\mathbb{C})$ under this new notion is forced to have a finite index. Finally, we provide some examples of certain dual Banach algebras and we study their Johnson pseudo-Connes amenability.

1. Introduction and Preliminaries

The class of dual Banach algebras were introduced by Runde [11]. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. An $\mathcal{A}$-bimodule E is called dual if there exists a closed submodule E_* of E^* such that $E = (E_*)^*$. The Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule or equivalently the multiplication in $\mathcal{A}$ is separately wk^*-continuous [14]. The measure algebra $M(G)$ of a locally compact group G, the algebra of bounded operators $\mathcal{B}(E)$, for a reflexive Banach space E and the second dual $\mathcal{A}^{**}$ of an Arens regular Banach algebra $\mathcal{A}$ are examples of dual Banach algebras.

For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, the set of all elements $x \in E$ such that the module maps $\mathcal{A} \to E; a \mapsto a\cdot x$ and $a \mapsto x\cdot a$ are wk^*-wk-continuous is denoted by $\sigma wc(E)$ and it is a closed submodule of E. Since $\sigma wc(\mathcal{A}_*) = \mathcal{A}_*$, the adjoint of $\pi_{\mathcal{A}}$ maps $\mathcal{A}_*$ into $\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$, where $\pi_{\mathcal{A}} : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is a bounded $\mathcal{A}$-bimodule morphism defined by $\pi_{\mathcal{A}}(a \otimes b) = ab$ for every $a, b \in \mathcal{A}$. Therefore, $\pi^{**}_{\mathcal{A}}$ drops to an $\mathcal{A}$-bimodule morphism $\pi_{\sigma wc} : (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \to \mathcal{A}$. A dual Banach algebra $\mathcal{A}$ is called Connes amenable if and only if $\mathcal{A}$ has an σwc-virtual diagonal, that is, there exists an element $M \in (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a \cdot M = M \cdot a$ and $a\pi_{\sigma wc}(M) = a$ for every $a \in \mathcal{A}$ [13]. Some new generalizations of Connes amenability like approximate Connes amenability and pseudo-Connes amenability have been introduced by Esslamzadeh *et al.* [2] and Mahmoodi [7]. A unital dual Banach algebra $\mathcal{A}$ is approximate Connes amenable if and only if there exists a net (M_α) in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a \cdot M_\alpha - M_\alpha \cdot a \to 0$ and $\pi_{\sigma wc}(M_\alpha)a \to a$ for every $a \in \mathcal{A}$

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[2, Theorem 3.3]. Also the dual Banach algebra $\mathcal{A}$ is called pseudo-Connes amenable if there exists a net (M_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a \cdot M_\alpha - M_\alpha \cdot a \xrightarrow{wk^*} 0$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ and $\pi_{\sigma wc}(M_\alpha)a \xrightarrow{wk^*} a$ in $\mathcal{A}$ [7, Definition 4.3].

The notion of Johnson pseudo-contractibility for a Banach algebra was introduced by Sahami *et al.*, which is a weaker notion than amenability and pseudo-contractibility but it is stronger than pseudo-amenability [16]. A Banach algebra $\mathcal{A}$ is called Johnson pseudo-contractible, if there exists a not necessarily bounded net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$. They also showed that for a locally compact group G, $M(G)$ is Johnson pseudo-contractible if and only if G is discrete and amenable [16, Proposition 3.3]. They characterized the Johnson pseudo-contractibility of $\ell^1(S)$, where S is a uniformly locally finite inverse semigroup [15, Theorem 2.3]. They showed that for a Brandt semigroup $S = M^0(G, I)$ over a non-empty set I, $\ell^1(S)$ is Johnson pseudo-contractible if and only if G is amenable and I is finite [15, Theorem 2.4].

Motivated by these results, we introduce a new concept of amenability in the category of dual Banach algebras in the parallel with Johnson pseudo-contractibility, which we call it Johnson pseudo-Connes amenability. In fact we show that Johnson pseudo-Connes amenability is weaker notion than Connes amenability and it is a stronger notion than approximate Connes amenability and pseudo-Connes amenability.

Definition 1.1. *A dual Banach algebra $\mathcal{A}$ is called Johnson pseudo-Connes amenable, if there exists a not necessarily bounded net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\langle T, a \cdot m_\alpha\rangle = \langle T, m_\alpha \cdot a\rangle$ and $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$, where $i_{\mathcal{A}_*} : \mathcal{A}_* \hookrightarrow \mathcal{A}^*$ is the canonical embedding.*

It is clear that every Johnson pseudo-contractible dual Banach algebra is Johnson pseudo-Connes amenable.

In this paper we investigate the relation between the notion of Johnson pseudo-Connes amenability with the various notions of Connes amenability. Indeed we show that for a dual Banach algebra $\mathcal{A}$, Johnson pseudo-Connes amenability of $\mathcal{A}$ implies φ-Connes amenability, where φ is a wk^*-continuous character on $\mathcal{A}$. Using this tool we show that for a finite set I and a dual Banach algebra $\mathcal{A}$ with the non-empty wk^*-continuous character space, a class of $I \times I$-upper triangular matrix $UP(I, \mathcal{A})$ is Johnson pseudo-Connes amenable if and only if $\mathcal{A}$ is Johnson pseudo-Connes amenable and I is singleton.

As an application, we prove that for an arbitrary set I, the Banach algebra of $I \times I$-matrices over $\mathbb{C}$, $M_I(\mathbb{C})$ is Johnson pseudo-Connes amenable if and only if I is finite. Also we show that for a locally compact group G, $M(G)$ is Johnson pseudo-Connes amenable if and only if G is amenable. This result distinguishes our new notion with Johnson pseudo-contractibility. Finally, we provide some examples of certain dual Banach algebras which shows that the difference between this new notion from approximately Connes amenable, pseudo-Connes amenable and φ-Connes amenable.

2. Johnson pseudo-Connes amenability

For a Banach algebra $\mathcal{A}$, the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \qquad (a \otimes b) \cdot c = a \otimes bc \quad (a, b, c \in \mathcal{A}).$$

If E is a Banach $\mathcal{A}$-bimodule, then E^* is also a Banach $\mathcal{A}$-bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \qquad (f \cdot a)(x) = f(a \cdot x) \quad (a \in \mathcal{A}, x \in E, f \in E^*).$$

Remark 2.1. *Let $\mathcal{A}$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. Since $\sigma wc(E^*)$ is a closed $\mathcal{A}$-submodule of E^*, we have a quotient map $q : E^{**} \longrightarrow \sigma wc(E^*)^*$, which is defined by $q(M) = M|_{\sigma wc(E^*)}$ for every $M \in E^{**}$.*

Remark 2.2. *Let $\mathcal{A}$ be a dual Banach algebra. Then for every $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ and $f \in \mathcal{A}_*$ we have*

$$\begin{aligned}\langle f, \pi_{\sigma wc}q(M)\rangle &= \langle \pi^*|_{\mathcal{A}_*}(f), q(M)\rangle = \langle \pi^*|_{\mathcal{A}_*}(f), M\rangle = \langle \pi^*(f), M\rangle \\ &= \langle f, \pi^{**}(M)\rangle = \langle i_{\mathcal{A}_*}(f), \pi^{**}(M)\rangle = \langle f, i^*_{\mathcal{A}_*}\pi^{**}(M)\rangle,\end{aligned}$$

where $q : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is as above. So $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}} = \pi_{\sigma wc}q$.

Lemma 2.3. *Let $\mathcal{A}$ be a dual Banach algebra. If $\mathcal{A}$ is Connes amenable, then $\mathcal{A}$ is Johnson pseudo-Connes amenable.*

Proof. Let $\mathcal{A}$ be a Connes amenable Banach algebra. Then by [13, Theorem 4.8], there is an element $\tilde{M} \in (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that

$$a \cdot \tilde{M} = \tilde{M} \cdot a \quad \text{and} \quad \pi_{\sigma wc}(\tilde{M})a = a \quad (a \in \mathcal{A}).$$

Consider $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $q(M) = \tilde{M}$, where $q : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is the quotient map as in Remark 2.1. By Remark 2.2, for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ we have

$$\langle T, a \cdot M\rangle = \langle T, M \cdot a\rangle \quad \text{and} \quad i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(M)a = a.$$

□

Lemma 2.4. *Let $\mathcal{A}$ be a dual Banach algebra. If $\mathcal{A}$ is Johnson pseudo-Connes amenable, then $\mathcal{A}$ is pseudo-Connes amenable.*

Proof. Since $\mathcal{A}$ is Johnson pseudo-Connes amenable, there exists a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\langle T, a \cdot m_\alpha\rangle = \langle T, m_\alpha \cdot a\rangle$ and $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. By Goldstein's theorem, there is a net $(u^\alpha_\beta)_{\beta\in\Theta}$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $wk^*\text{-}\lim_\beta u^\alpha_\beta = m_\alpha$. Thus for every $a \in \mathcal{A}$

$$wk^*\text{-}\lim_\beta a \cdot u^\alpha_\beta - u^\alpha_\beta \cdot a = a \cdot m_\alpha - m_\alpha \cdot a.$$

Since $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}$ is wk^*-continuous and the multiplication in $\mathcal{A}$ is separately wk^*-continuous [14, Exercise 4.4.1], for every $a \in \mathcal{A}$ we have

$$wk^*\text{-}\lim_\alpha wk^*\text{-}\lim_\beta i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(u^\alpha_\beta)a = wk^*\text{-}\lim_\alpha i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a = a.$$

Let $E = I \times \Theta^I$ be a directed set with product ordering defined by

$$(\alpha, \beta) \leq_E (\alpha', \beta') \Leftrightarrow \alpha \leq_I \alpha' \text{ and } \beta \leq_{\Theta^I} \beta' \qquad (\alpha, \alpha' \in I, \quad \beta, \beta' \in \Theta^I),$$

where Θ^I is the set of all functions from I into Θ and $\beta \leq_{\Theta^I} \beta'$ means that $\beta(d) \leq_\Theta \beta'(d)$ for every $d \in I$. Suppose that $\gamma = (\alpha, \beta_\alpha)$ and $n_\gamma = u^\alpha_{\beta}$. By iterated limit theorem, one can see that $wk^*\text{-}\lim_\gamma a \cdot n_\gamma - n_\gamma \cdot a = 0$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ and by Remark 2.2, $wk^*\text{-}\lim_\gamma \pi_{\sigma wc}q(n_\gamma)a = a$ in $\mathcal{A}$, where $q : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is the quotient map as in Remark 2.1. So $\mathcal{A}$ is pseudo-Connes amenable [7, Definition 4.3]. □

Lemma 2.5. *Let $\mathcal{A}$ be a unital dual Banach algebra. If $\mathcal{A}$ is Johnson pseudo-Connes amenable, then $\mathcal{A}$ is approximately Connes amenable.*

Proof. Since $\mathcal{A}$ is Johnson pseudo-Connes amenable, there exists a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\langle T, a \cdot m_\alpha\rangle = \langle T, m_\alpha \cdot a\rangle$ and $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Let $\tilde{m}_\alpha = q(m_\alpha)$, where $q : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \to (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is the quotient map as in Remark 2.1. So $a \cdot \tilde{m}_\alpha = \tilde{m}_\alpha \cdot a$ for every $a \in \mathcal{A}$ and by Remark 2.2 we have $\pi_{\sigma wc}(\tilde{m}_\alpha)a \to a$. So $\mathcal{A}$ is approximately Connes amenable [2, Theorem 3.3]. □

A dual Banach algebra $\mathcal{A}$ is called Connes biprojective if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$. Shirinkalam and second author [18] showed that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if $\mathcal{A}$ is Connes biprojective and $\mathcal{A}$ has an identity.

Proposition 2.6. *Let $\mathcal{A}$ be a dual Banach algebra with a central approximate identity. If $\mathcal{A}$ is Connes biprojective, then $\mathcal{A}$ is Johnson pseudo-Connes amenable.*

Proof. Let (e_α) be a central approximate identity for $\mathcal{A}$ and let $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ be a bounded $\mathcal{A}$-bimodule morphism such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$. Consider the net $\rho(e_\alpha)$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. So there is a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $q(m_\alpha) = \rho(e_\alpha)$, where $q : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \to (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is the quotient map as in Remark 2.1. For every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ we have

$$\begin{aligned}\langle T, a\cdot m_\alpha\rangle &= \langle T\cdot a, m_\alpha\rangle = \langle T\cdot a, m_\alpha|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}\rangle = \langle T\cdot a, q(m_\alpha)\rangle = \langle T\cdot a, \rho(e_\alpha)\rangle = \langle T, a\cdot\rho(e_\alpha)\rangle\\ &= \langle T, \rho(ae_\alpha)\rangle = \langle T, \rho(e_\alpha a)\rangle = \langle T, \rho(e_\alpha)\cdot a\rangle = \langle a\cdot T, \rho(e_\alpha)\rangle = \langle a\cdot T, q(m_\alpha)\rangle = \langle T, q(m_\alpha)\cdot a\rangle\\ &= \langle T, q(m_\alpha\cdot a)\rangle = \langle T, (m_\alpha\cdot a)|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}\rangle = \langle T, m_\alpha\cdot a\rangle,\end{aligned}$$

and by Remark 2.2

$$i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a = \pi_{\sigma wc}q(m_\alpha)a = \pi_{\sigma wc}\circ\rho(e_\alpha)a = e_\alpha a \to a.$$

So $\mathcal{A}$ is Johnson pseudo-Connes amenable. □

The notion of φ-Connes amenability for a dual Banach algebra $\mathcal{A}$ introduced by Mahmoodi and some characterizations were given [8] and [9], where φ is a wk^*-continuous character on $\mathcal{A}$. We say that $\mathcal{A}$ is φ-Connes amenable if there exists a bounded linear functional m on $\sigma wc(\mathcal{A}^*)$ satisfying $m(\varphi) = 1$ and $m(f\cdot a) = \varphi(a)m(f)$ for every $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$. The set of all wk^*-continuous characters on $\mathcal{A}$ is denoted by $\Delta_{wk^*}(\mathcal{A})$. The following result is an analogue of [16, Proposition 2.4].

Proposition 2.7. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \Delta_{wk^*}(\mathcal{A})$. If $\mathcal{A}$ is Johnson pseudo-Connes amenable, then $\mathcal{A}$ is φ-Connes amenable.*

Proof. Since $\mathcal{A}$ is Johnson pseudo-Connes amenable, there exists a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\langle T, a\cdot m_\alpha\rangle = \langle T, m_\alpha\cdot a\rangle$ and $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Define $\theta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ by $\theta(a\otimes b) = \varphi(b)a$ for every $a, b \in \mathcal{A}$. So for each $a, b, c \in \mathcal{A}$,

$$\theta(a\cdot(b\otimes c)) = \theta(ab\otimes c) = \varphi(c)ab = a\theta(b\otimes c),$$

and

$$\theta((b\otimes c)\cdot a) = \theta(b\otimes ca) = \varphi(ca)b = \varphi(c)\varphi(a)b = \varphi(a)\theta(b\otimes c),$$

and also

$$\langle\theta(b\otimes c), \varphi\rangle = \langle\varphi(c)b, \varphi\rangle = \varphi(b)\varphi(c) = \varphi(bc) = \langle\pi(b\otimes c), \varphi\rangle.$$

Thus for every $a \in \mathcal{A}$ and $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$

$$\theta(a\cdot u) = a\theta(u), \quad \theta(u\cdot a) = \varphi(a)\theta(u), \tag{1}$$

and

$$\langle\theta(u), \varphi\rangle = \langle\pi(u), \varphi\rangle. \tag{2}$$

By Goldstein's Theorem for every $F \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ there is a bounded net (u_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $wk^*\text{-}\lim_\alpha u_\alpha = F$. Since $\theta^{**} : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \to \mathcal{A}^{**}$ is a wk^*-continuous map, by (1) for every $a \in \mathcal{A}$ we have

$$\theta^{**}(a\cdot F) = wk^*\text{-}\lim_\alpha \theta^{**}(a\cdot u_\alpha) = wk^*\text{-}\lim_\alpha a\cdot\theta^{**}(u_\alpha) = a\cdot\theta^{**}(F), \tag{3}$$

and

$$\theta^{**}(F\cdot a) = wk^*\text{-}\lim_\alpha \theta^{**}(u_\alpha\cdot a) = wk^*\text{-}\lim_\alpha \varphi(a)\theta^{**}(u_\alpha) = \varphi(a)\theta^{**}(F), \tag{4}$$

and (2) implies that

$$\langle \varphi, \theta^{**}(F) \rangle = \lim_{\alpha} \langle \varphi, \theta^{**}(u_\alpha) \rangle = \lim_{\alpha} \langle \varphi, \pi^{**}_{\mathcal{A}}(u_\alpha) \rangle = \langle \varphi, \pi^{**}_{\mathcal{A}}(F) \rangle. \tag{5}$$

For every $a \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$ by (1) we have

$$\langle u, a \cdot \theta^*(f) \rangle = \langle u \cdot a, \theta^*(f) \rangle = \langle \theta(u \cdot a), f \rangle = \varphi(a)\langle \theta(u), f \rangle = \varphi(a)\langle u, \theta^*(f) \rangle,$$

and

$$\langle u, \theta^*(f) \cdot a \rangle = \langle a \cdot u, \theta^*(f) \rangle = \langle \theta(a \cdot u), f \rangle = \langle a\theta(u), f \rangle = \langle \theta(u), f \cdot a \rangle = \langle u, \theta^*(f \cdot a) \rangle.$$

So

$$a \cdot \theta^*(f) = \varphi(a)\theta^*(f), \quad \theta^*(f) \cdot a = \theta^*(f \cdot a) \quad (a \in \mathcal{A}, f \in \mathcal{A}^*). \tag{6}$$

Since φ is wk^*-continuous, (6) implies that

$$\theta^*(\sigma wc(\mathcal{A}^*)) \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*. \tag{7}$$

Consider the quotient map $q : \mathcal{A}^{**} \to (\sigma wc(\mathcal{A}^*))^*$. By (3), (4) and (7) for every $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$ we have

$$\begin{aligned}\langle f \cdot a, q \circ \theta^{**}(m_\alpha) \rangle &= \langle f \cdot a, \theta^{**}(m_\alpha)|_{\sigma wc(\mathcal{A}^*)} \rangle = \langle f \cdot a, \theta^{**}(m_\alpha) \rangle = \langle f, a \cdot \theta^{**}(m_\alpha) \rangle \\ &= \langle f, \theta^{**}(a \cdot m_\alpha) \rangle = \langle \theta^*(f), a \cdot m_\alpha \rangle = \langle \theta^*(f), m_\alpha \cdot a \rangle = \langle f, \theta^{**}(m_\alpha \cdot a) \rangle \\ &= \langle f, \varphi(a)\theta^{**}(m_\alpha) \rangle = \varphi(a)\langle f, \theta^{**}(m_\alpha)|_{\sigma wc(\mathcal{A}^*)} \rangle = \varphi(a)\langle f, q \circ \theta^{**}(m_\alpha) \rangle.\end{aligned}$$

Since φ is wk^*-continuous, $\varphi \in \mathcal{A}_*$. Note that $\mathcal{A}_* \subseteq \sigma wc(\mathcal{A}^*)$. Using (5) we have

$$\begin{aligned}\langle \varphi, q \circ \theta^{**}(m_\alpha) \rangle &= \langle \varphi, \theta^{**}(m_\alpha)|_{\sigma wc(\mathcal{A}^*)} \rangle = \langle \varphi, \theta^{**}(m_\alpha) \rangle = \langle \varphi, \pi^{**}_{\mathcal{A}}(m_\alpha) \rangle \\ &= \langle i_{\mathcal{A}_*}(\varphi), \pi^{**}_{\mathcal{A}}(m_\alpha) \rangle = \langle \varphi, i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha) \rangle.\end{aligned} \tag{8}$$

Since $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$ and φ is continuous,

$$\lim_{\alpha} \langle \varphi, i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha) \rangle \varphi(a) = \lim_{\alpha} \langle \varphi, i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a \rangle = \varphi(a).$$

Equation (8) implies that $\lim_{\alpha} \langle \varphi, q \circ \theta^{**}(m_\alpha) \rangle = 1$ in $\mathbb{C}$. For sufficiently large α, $\langle \varphi, q \circ \theta^{**}(m_\alpha) \rangle$ stays away from zero. Replacing $q \circ \theta^{**}(m_\alpha)$ by $\dfrac{q \circ \theta^{**}(m_\alpha)}{\langle \varphi, q \circ \theta^{**}(m_\alpha) \rangle}$, we may assume that

$$\langle f \cdot a, q \circ \theta^{**}(m_\alpha) \rangle = \varphi(a)\langle f, q \circ \theta^{**}(m_\alpha) \rangle, \qquad \langle \varphi, q \circ \theta^{**}(m_\alpha) \rangle = 1,$$

for every $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$. So $\mathcal{A}$ is φ-Connes amenable. □

Proposition 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras. Suppose that $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous epimorphism which is also wk^*-continuous. If $\mathcal{A}$ is Johnson pseudo-Connes amenable, then $\mathcal{B}$ is Johnson pseudo-Connes amenable.*

Proof. Since $\mathcal{A}$ is Johnson pseudo-Connes amenable, there exists a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that

$$\langle T, a \cdot m_\alpha \rangle = \langle T, m_\alpha \cdot a \rangle \quad \text{and} \quad i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a,$$

for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Define $\theta \otimes \theta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{B}\hat{\otimes}\mathcal{B}$ by $\theta \otimes \theta(x \otimes y) = \theta(x) \otimes \theta(y)$, for every $x, y \in \mathcal{A}$. So $\theta \otimes \theta$ is a bounded linear map. For every $a, b, c \in \mathcal{A}$ we have

$$\begin{aligned}(\theta \otimes \theta)(a \cdot (b \otimes c)) &= \theta(ab) \otimes \theta(c) = \theta(a)\theta(b) \otimes \theta(c) \\ &= \theta(a) \cdot (\theta(b) \otimes \theta(c)) = \theta(a) \cdot (\theta \otimes \theta(b \otimes c)).\end{aligned}$$

By similarity for the right action, for every $a \in \mathcal{A}$ and $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$ we have

$$\theta(a) \cdot (\theta \otimes \theta)(u) = (\theta \otimes \theta)(a \cdot u), \qquad (\theta \otimes \theta)(u) \cdot \theta(a) = (\theta \otimes \theta)(u \cdot a). \tag{9}$$

By Goldstein's Theorem for every $F \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ there is a bounded net (u_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $wk^*\text{-}\lim_\alpha u_\alpha = F$. Since $(\theta \otimes \theta)^{**}$ is a wk^*-continuous map, (9) implies that for every $a \in \mathcal{A}$

$$\theta(a) \cdot (\theta \otimes \theta)^{**}(F) = wk^*\text{-}\lim_\alpha \theta(a) \cdot (\theta \otimes \theta)^{**}(u_\alpha) = wk^*\text{-}\lim_\alpha (\theta \otimes \theta)^{**}(a \cdot u_\alpha) = (\theta \otimes \theta)^{**}(a \cdot F), \tag{10}$$

and

$$(\theta \otimes \theta)^{**}(F) \cdot \theta(a) = wk^*\text{-}\lim_\alpha (\theta \otimes \theta)^{**}(u_\alpha) \cdot \theta(a) = wk^*\text{-}\lim_\alpha (\theta \otimes \theta)^{**}(u_\alpha \cdot a) = (\theta \otimes \theta)^{**}(F \cdot a). \tag{11}$$

Using (9) for every $a \in \mathcal{A}$, $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$ and $f \in (\mathcal{B}\hat{\otimes}\mathcal{B})^*$ we have

$$\begin{aligned}\langle u, a \cdot (\theta \otimes \theta)^*(f)\rangle &= \langle u \cdot a, (\theta \otimes \theta)^*(f)\rangle = \langle \theta \otimes \theta(u \cdot a), f\rangle \\ &= \langle (\theta \otimes \theta)(u), \theta(a) \cdot f\rangle = \langle u, (\theta \otimes \theta)^*(\theta(a) \cdot f)\rangle.\end{aligned}$$

So by similarity for the right action, we have

$$a \cdot (\theta \otimes \theta)^*(f) = (\theta \otimes \theta)^*(\theta(a) \cdot f), \qquad (\theta \otimes \theta)^*(f) \cdot a = (\theta \otimes \theta)^*(f \cdot \theta(a)). \tag{12}$$

Since θ is a wk^*-continuous map, (12) implies that

$$(\theta \otimes \theta)^*(\sigma wc(\mathcal{B}\hat{\otimes}\mathcal{B})^*) \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*. \tag{13}$$

By (10), (11) and (13) for every $a \in \mathcal{A}$ and $U \in \sigma wc(\mathcal{B}\hat{\otimes}\mathcal{B})^*$ we have

$$\begin{aligned}\langle U, \theta(a) \cdot (\theta \otimes \theta)^{**}(m_\alpha)\rangle &= \langle U, (\theta \otimes \theta)^{**}(a \cdot m_\alpha)\rangle = \langle (\theta \otimes \theta)^*(U), a \cdot m_\alpha\rangle = \langle (\theta \otimes \theta)^*(U), m_\alpha \cdot a\rangle \\ &= \langle U, (\theta \otimes \theta)^{**}(m_\alpha \cdot a)\rangle = \langle U, (\theta \otimes \theta)^{**}(m_\alpha) \cdot \theta(a)\rangle.\end{aligned}$$

One can see that $\theta^*(\mathcal{B}_*) \subseteq \mathcal{A}_*$. For every $\psi \in \mathcal{A}^{**}$ and $h \in \mathcal{B}_*$ we have

$$\langle i^*_{\mathcal{B}_*}\theta^{**}(\psi), h\rangle = \langle \theta^{**}(\psi), i_{\mathcal{B}_*}(h)\rangle = \langle \theta^{**}(\psi), h\rangle = \langle \psi, \theta^*(h)\rangle = \langle \psi, i_{\mathcal{A}_*}\theta^*(h)\rangle = \langle \theta i^*_{\mathcal{A}_*}(\psi), h\rangle,$$

thus

$$i^*_{\mathcal{B}_*}\theta^{**} = \theta i^*_{\mathcal{A}_*}. \tag{14}$$

Since $\pi_{\mathcal{B}} \circ \theta \otimes \theta = \theta \circ \pi_{\mathcal{A}}$, (14) implies that for every $a \in \mathcal{A}$

$$\lim_\alpha (i^*_{\mathcal{B}_*}\pi^{**}_{\mathcal{B}}(\theta \otimes \theta)^{**}(m_\alpha))\theta(a) = \lim_\alpha (\theta i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha))\theta(a) = \theta(\lim_\alpha i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a) = \theta(a).$$

So $\mathcal{B}$ is Johnson pseudo-Connes amenable. □

Corollary 2.9. *Let $\mathcal{A}$ be a dual Banach algebra and let I be a wk^*-closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is Johnson pseudo-Connes amenable, then $\mathcal{A}/I$ is Johnson pseudo-Connes amenable.*

Proof. Since the quotient map $Q : \mathcal{A} \to \mathcal{A}/I$ is a wk^*-continuous map, by Proposition 2.8 the dual Banach algebra $\mathcal{A}/I$ is Johnson pseudo-Connes amenable. □

Lemma 2.10. *Let $\mathcal{A}$ be a Johnson pseudo-Connes amenable dual Banach algebra. Then $\mathcal{A}$ possess a central approximate identity.*

Proof. Suppose that $\mathcal{A}$ is Johnson pseudo-Connes amenable. Then there is a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\langle T, a \cdot m_\alpha\rangle = \langle T, m_\alpha \cdot a\rangle$ and $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Let $M_\alpha = i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}}(m_\alpha)$. By Remark 2.2 we have $i^*_{\mathcal{A}_*}\pi^{**}_{\mathcal{A}} = \pi_{\sigma wc}q$, where $q : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is the quotient map as in Remark 2.1. Since $\pi_{\sigma wc}$ is an $\mathcal{A}$-bimodule morphism, for every $a \in \mathcal{A}$ we have

$$\begin{aligned} aM_\alpha &= a\pi_{\sigma wc}q(m_\alpha) = \pi_{\sigma wc}q(a \cdot m_\alpha) = \pi_{\sigma wc}(a \cdot m_\alpha|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}) \\ &= \pi_{\sigma wc}(m_\alpha \cdot a|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}) = \pi_{\sigma wc}q(m_\alpha \cdot a) = \pi_{\sigma wc}q(m_\alpha)a \\ &= M_\alpha a. \end{aligned}$$

So (M_α) is a net in $\mathcal{A}$ satisfies $aM_\alpha = M_\alpha a$ and $M_\alpha a \to a$. So $\mathcal{A}$ has a central approximate identity. □

3. Some Applications and Examples

Proposition 3.1. *Let G be a locally compact group. The measure algebra $M(G)$ is Johnson pseudo-Connes amenable if and only if G is amenable.*

Proof. Let $M(G)$ be Johnson pseudo-Connes amenable. Since $M(G)$ is unital, by Lemma 2.5, $M(G)$ is approximate Connes amenable. Hence G is amenable [2, Theorem 5.2].

Conversely if G is amenable, then $M(G)$ is Connes amenable [12, Theorem 5.4]. Thus by Lemma 2.3, $M(G)$ is Johnson pseudo-Connes amenable. □

Let $\mathcal{A}$ be a Banach algebra, I and J be arbitrary nonempty index sets and let P be a $J \times I$ matrix over $\mathcal{A}$ such that $||P||_\infty = \sup\{||P_{j,i}|| : j \in J, i \in I\} \leq 1$. The set of all $I \times J$ matrices over $\mathcal{A}$ with finite ℓ^1-norm and product $XY = XPY$ is a Banach algebra, which is denoted by $LM(\mathcal{A}, P)$ and it is called the ℓ^1-Munn $I \times J$ matrix algebra over $\mathcal{A}$ with sandwich matrix P or briefly the ℓ^1-Munn algebra [1].

Suppose that $\mathcal{A}$ is a unital dual Banach algebra. Shojaee *et al.* showed that the ℓ^1-Munn algebra $LM(\mathcal{A}, P) = \ell^1(I \times J, \mathcal{A})$ is a dual Banach algebra with respect to the predual $c_0(I \times J, \mathcal{A}_*)$, where $I = J$ [19].

The Banach algebra of $I \times I$-matrices over $\mathbb{C}$, with finite ℓ^1-norm and matrix multiplication is denoted by $\mathbb{M}_I(\mathbb{C})$, where I is an arbitrary set. So $\mathbb{M}_I(\mathbb{C})$ is a dual ℓ^1-Munn algebra over $\mathbb{C}$ with sandwich matrix $P = id$.

Theorem 3.2. *Let I be a non-empty set. Then $\mathbb{M}_I(\mathbb{C})$ is Johnson pseudo-Connes amenable if and only if I is finite.*

Proof. Let $\mathcal{A} = \mathbb{M}_I(\mathbb{C})$ be Johnson pseudo-Connes amenable. By Lemma 2.10, $\mathcal{A}$ has a central approximate identity. Applying [3, Theorem 2.2], gives that I must be finite.

Conversely, if I is finite, then $\mathbb{M}_I(\mathbb{C})$ is Connes amenable [6, Theorem 3.7]. So by Lemma (2.3), $\mathbb{M}_I(\mathbb{C})$ is Johnson pseudo-Connes amenable. □

Let $\mathcal{A}$ be a dual Banach algebra and let I be a totally ordered set. Then the set of all $I \times I$-upper triangular matrices with the usual matrix operations and the norm $\| [a_{i,j}]_{i,j\in I} \| = \sum_{i,j \in I} \| a_{i,j} \| < \infty$, becomes a Banach algebra and it is denoted by

$$UP(I, \mathcal{A}) = \{\begin{bmatrix} a_{i,j} \end{bmatrix}_{i,j\in I}; a_{i,j} \in A \text{ and } a_{i,j} = 0 \text{ for every } i > j\}.$$

Theorem 3.3. *Let $\mathcal{A}$ be a dual Banach algebra, $\varphi \in \Delta_{wk^*}(\mathcal{A})$ and let I be a finite set. Then $UP(I, \mathcal{A})$ is Johnson pseudo-Connes amenable if and only if $\mathcal{A}$ is Johnson pseudo-Connes amenable and $|I| = 1$.*

Proof. Let $UP(I, \mathcal{A})$ be Johnson pseudo-Connes amenable. Assume that $I = \{i_1, ..., i_n\}$ and φ in $\Delta_{wk^*}(\mathcal{A})$. We define a map $\psi : UP(I, \mathcal{A}) \longrightarrow \mathbb{C}$ by $\begin{bmatrix} a_{i,j} \end{bmatrix}_{i,j\in I} \longmapsto \varphi(a_{i_n,i_n})$ for every $\begin{bmatrix} a_{i,j} \end{bmatrix}_{i,j\in I} \in UP(I, \mathcal{A})$. Since φ is wk^*-continuous, $\psi \in \Delta_{wk^*}(UP(I, \mathcal{A}))$. By Proposition 2.7, $UP(I, \mathcal{A})$ is ψ-Connes amenable. By similar argument as in [17, Theorem 3.2], we have $|I| = 1$.

Converse is clear. □

A Banach algebra $\mathcal{A}$ is called pseudo-amenable (pseudo-contractible) if there exists a not necessarily bounded net (u_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a \cdot u_\alpha - u_\alpha \cdot a \to 0$ $(a \cdot u_\alpha = u_\alpha \cdot a)$ and $\pi_{\mathcal{A}}(u_\alpha)a \to a$ for every $a \in \mathcal{A}$, respectively. For more details see [5].

Example 3.4. *We give a Banach algebra which is Johnson pseudo-Connes amenable but it is not approximately Connes amenable. The dual Banach sequence algebra ℓ^1 with pointwise multiplication is pseudo-contractible [5, page 2]. By [16, Lemma 2.2], ℓ^1 is Johnson pseudo-contractible. It follows that ℓ^1 is Johnson pseudo-Connes amenable but [7, Theorem 3.2] implies that ℓ^1 is not wk^*-approximately Connes amenable (for the definition see [7, Definition 2.1]). Therefore ℓ^1 is not approximately Connes amenable.*

Remark 3.5. *Note that the dual Banach sequence algebra ℓ^1 with pointwise multiplication has no unit. So the previous example does not contradict Lemma 2.5.*

A Banach algebra $\mathcal{A}$ is called biflat if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\pi_{\mathcal{A}}^{**} \circ \rho$ is the canonical embedding of $\mathcal{A}$ into $\mathcal{A}^{**}$, see [14].

Example 3.6. *We give a Banach algebra which is pseudo-Connes amenable but it is not Johnson pseudo-contractible. The dual Banach algebra $\mathbb{M}_I(\mathbb{C})$ as in Theorem 3.2 is biflat for every infinite set I [10, Proposition 2.7]. Using [4, Proposition 3.6], $\mathbb{M}_I(\mathbb{C})$ has an approximate identity. Now [4, Proposition 3.5] implies that $M_I(\mathbb{C})$ is pseudo-amenable. So $M_I(\mathbb{C})$ is pseudo-Connes amenable for every infinite set I [7, page 11], but Theorem 3.2 implies that $M_I(\mathbb{C})$ is not Johnson pseudo-Connes amenable, where I is an infinite set.*

Example 3.7. *Consider the Banach algebra ℓ^1 of all sequences $a = (a_n)$ of complex numbers with*

$$||a|| = \sum_{n=1}^{\infty} |a_n| < \infty,$$

and the following product

$$(a * b)(n) = \begin{cases} a(1)b(1) & if \quad n = 1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & if \quad n > 1 \end{cases}$$

for every $a, b \in \ell^1$. It is easy to see that $\Delta(\ell^1) = \{\varphi_1\} \cup \{\varphi_1 + \varphi_n : n \geq 2\}$, where $\varphi_n(a) = a(n)$ for every $a \in \ell^1$. We claim that $(\ell^1, *)$ is a dual Banach algebra with respect to c_0. It is clear that c_0 is a closed subspace of ℓ^∞. We show that c_0 is an ℓ^1-module with dual actions. For every $a \in \ell^1$ and $\lambda \in c_0$ we have

$$a \cdot \lambda(n) = \begin{cases} \sum_{k=1}^{\infty} a(k)\lambda(k) & if \quad n = 1 \\ (a(1) + a(n))\lambda(n) & if \quad n > 1. \end{cases}$$

Since λ vanishes at infinity and $\sup_n |a(n)| < \infty$, one can see that $a \cdot \lambda$ vanishes at infinity. So $a \cdot \lambda \in c_0$ and similarity for the right action. We claim that ℓ^1 is not Johnson pseudo-Connes amenable. Suppose conversely that ℓ^1 is Johnson pseudo-Connes amenable. Since φ_1 is wk-continuous, by Proposition 2.7, ℓ^1 is φ_1-Connes amenable. Using [17, Proposition 3.1] and by similar argument as in [17, Theorem 3.2] there is a bounded net (m_α) in ℓ^1 that satisfies*

$$a * m_\alpha - \varphi_1(a)m_\alpha \xrightarrow{wk^*} 0 \quad and \quad \varphi_1(m_\alpha) \longrightarrow 1 \qquad (a \in \ell^1). \tag{15}$$

Choose $a = \delta_n$ in ℓ^1, where $n \geq 2$. So $\varphi_1(\delta_n) = 0$, using (15) we have $\delta_n * m_\alpha \xrightarrow{wk^*} 0$ in ℓ^1. One can see that $\delta_n * m_\alpha = (m_\alpha(1) + m_\alpha(n))\delta_n$. Consider δ_n as an element in c_0, where $n \geq 2$. So

$$\lim_\alpha \langle \delta_n, \delta_n * m_\alpha \rangle = \lim_\alpha m_\alpha(1) + m_\alpha(n) = 0.$$

Since $\lim_\alpha m_\alpha(1) = 1$ and $\lim_\alpha m_\alpha(n) = -1$ for every $n \geq 2$, we have $\sup_\alpha ||m_\alpha|| = \infty$, which contradicts the boundedness of the net (m_α).

Example 3.8. *We give a Banach algebra which is ϕ-Connes amenable but it is not Johnson pseudo-Connes amenable. Set $\mathcal{A} = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{pmatrix}$. With the usual matrix multiplication and ℓ^1-norm, $\mathcal{A}$ is a Banach algebra. Suppose in contradiction that $\mathcal{A}$ is Johnson pseudo-Connes amenable. Since $\mathbb{C}$ is a dual Banach algebra, $\mathcal{A}$ is a dual Banach algebra. So by Lemma 2.10 there exists a net (M_α) in $\mathcal{A}$ satisfies $aM_\alpha = M_\alpha a$ and $M_\alpha a \to a$. Define the map $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ by*

$$\phi \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = a, \qquad (a, b \in \mathbb{C}).$$

It is clear that ϕ is a wk-continuous linear and multiplicative map and also for every $X, Y \in \mathcal{A}$ we have $YX = \phi(Y)X$. Choose $X \in \mathcal{A}$ such that $\phi(X) = 1$. So $X = X \lim_\alpha M_\alpha = \lim_\alpha M_\alpha$. One can see that X is a unit for $\mathcal{A}$. So* *for every $Y \in \mathcal{A}$ we have $Y = YX = \phi(Y)X$. So $\dim \mathcal{A} = 1$, which is a contradiction. So $\mathcal{A}$ is not Johnson pseudo-Connes amenable but we show that $\mathcal{A}$ is ϕ-Connes amenable. By similar argument as in [17, Example 5.2], let $u = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}\hat{\otimes}\mathcal{A}$. Since $\mathcal{A}\hat{\otimes}\mathcal{A}$ embeds in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, we may assume that u is in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. Now for every $a, b \in \mathbb{C}$, we have*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \cdot u = \phi\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} u \quad \textit{and} \quad \langle u, \phi \otimes \phi \rangle = \phi\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \phi\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 1.$$

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References

[1] G. H. Esslamzadeh, Double centralizer algebras of certain Banach algebras, Monatsh. Math. 142 (2004), 193–203.
[2] G. H. Esslamzadeh, B. Shojaee and A. Mahmoodi, Approximate Connes-amenability of dual Banach algebras, Bull. Belg. Math. Soc. Simon stevin. 19 (2012), 193–213.
[3] M. Essmaili, M. Rostami and A. R. Medghalchi, Pseudo-contractibility and pseudo-amenability of semigroup algebras, Arch. Math. 97 (2) (2011), 167–177.
[4] M. Essmaili, M. Rostami and A. Pourabbas, Pseudo-amenability of certain semigroup algebras, Semigroup Forum. 82 (2011), 478–484.
[5] F. Ghahramani and Y. Zhang, Pseudo-amenable and pseudo-contractible Banach algebras, Math. Proc. Camb. Philos. Soc. 142 (2007), 111–123.
[6] A. Mahmoodi, Beurling and matrix algebras, (approximate) Connes-amenability, U.P.B. Sci. Bull. Series A. 78. Iss. 1 (2016), 157–170.
[7] A. Mahmoodi, Connes amenability-like properties, Studia. Math. 220(2014), 55–72.
[8] A. Mahmoodi, On φ-Connes amenability for dual Banach algebras, J. Linear Topol. Algebra. 03 (2014), 211–217.
[9] M. Ramezanpour, Character Connes amenability of dual Banach algebras, Czech Math. J. 68 (2018), 243–255.
[10] P. Ramsden, Biflatness of semigroup algebras, Semigroup Forum 79 (2009), 515–530.
[11] V. Runde, Amenability for dual Banach algebras, Studia. Math. 148 (2001), 47–66.
[12] V. Runde, Connes-amenability and normal, virtual diagonals for measure algebras I, J. London Math. Soc. 67 (2003), 643–656.
[13] V. Runde, Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule, Math. Scand. 95 (2004), 124–144.
[14] V. Runde, Lectures on amenability, Lecture Notes in Mathematics, Vol. 1774, Springer-Verlag, Berlin, 2002.
[15] A. Sahami, and A. Pourabbas, Johnson pseudo-contractibility of certain semigroup algebras, Semigroup Forum. 97 (2018), 203–213.
[16] A. Sahami, and A. Pourabbas, Johnson pseudo-contractibility of various classes of Banach algebras, Bull. Belg. Math. Soc. Simon Stevin. 25 (2018), 171–182.
[17] S. F. Shariati, A. Pourabbas and A. Sahami, On Connes amenability of upper triangular matrix algebras, U.P.B. Sci. Bull. Series A. 80. Iss. 2 (2018), 145–152.
[18] A. Shirinkalam and A. Pourabbas, Connes-biprojective dual Banach algebras, U.P.B. Sci. Bull. Series A. 78 Iss. 3 (2016), 174–184.
[19] B. Shojaee, G. H. Esslamzadeh and A. Pourabbas, First order cohomology of ℓ^1-Munn algebras and certain semigroup algebras, Bull. Iranian Math. Soc. 35 (2009), 211–219.