Weak Fuzzy Topology on Vector Spaces

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Abstract. In this paper, we study the concept of weak linear fuzzy topology on a fuzzy topological vector space as a generalization of usual weak topology. We prove that this fuzzy topology consists of all weakly lower semi-continuous fuzzy sets on a given vector space when $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) endowed with its usual fuzzy topology. In the case that the fuzzy topology of $\mathbb{K}$ is different from the usual fuzzy topology, we show that the weak fuzzy topology is not equivalent with the fuzzy topology of weakly lower semi-continuous fuzzy sets.

1. Introduction

The concept of fuzzy topological vector space was firstly introduced by Katsaras and Liu [9]. Katsaras [10] changed the definition of fuzzy topological vector space and he considered the usual fuzzy topology on the corresponding scalar field $\mathbb{K}$ namely, he considered the topology consisting of all lower semi-continuous functions from $\mathbb{K}$ into I, where $I = [0, 1]$. Also, in the studying fuzzy topological vector space Katsaras [11] investigated the idea of fuzzy norm on linear spaces. Since then, many authors like, Felbin [8], Cheng and Mordeson [2], Bag and Samanta [1], Sadeqi and Yaqub Azari [16] started to introduce the notion of fuzzy normed linear space. Later, Xiao and Zhu [18], Fang [7], Daraby et. al. ([3, 4]) redefined the idea of Felbin's [8] definition of fuzzy norm and studied various properties of its topological structure. Das [5] constructed a fuzzy topology generated by fuzzy norm and studied some properties of this topology. Fang [7] investigated a new I-topology $\mathcal{T}^*_{\|.\|}$ on the fuzzy normed linear space.

Yan [19], introduced the concept of L-fuzzy linear topology determined by a family of L-fuzzy linear order-homomorphisms on a vector space. In this paper, we study the weak linear fuzzy topology on a fuzzy topological vector space as a generalization of usual weak topology. Moreover, we prove that the weak fuzzy topology is not equivalent to topology of weakly lower semi-continuous fuzzy sets (This topology consists of all lower semi-continuous functions $f : E \to [0, 1]$, when the vector space E considered with the usual weak topology $\sigma(E, E^*)$). In fact, the weak fuzzy topology on a fuzzy topological vector space is an extension of weak topology in classical functional analysis. The difference between weak topology and weak fuzzy topology leads to some new results on the theory of fuzzy topological vector spaces.

2. Preliminaries

Let X be a non-empty set. A fuzzy set in X is an element of the set I^X of all functions from X into I.

Definition 2.1. ([20]) Let X and Y be any two non-empty sets, $f : X \to Y$ be a mapping and μ be a fuzzy subset of X. Then $f(\mu)$ is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & f^{-1}(y) \neq \emptyset, \\ 0 & else, \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x : f(x) = y\}$. If η is a fuzzy subset of Y, then the fuzzy subset $f^{-1}(\eta)$ of X is defined by $f^{-1}(\eta)(x) = \eta(f(x))$ for all $x \in X$.

Definition 2.2. ([13]) A fuzzy topology on a set X is a subset τ of I^X satisfying the following conditions:

   (*i*) τ contains every constant fuzzy set in X,
   (*ii*) if $\mu_1, \mu_2 \in \tau$, then $\mu_1 \wedge \mu_2 \in \tau$,
   (*iii*) if $\mu_i \in \tau$ for each $i \in A$, then $\sup_{i \in A} \mu_i \in \tau$.

The pair (X, τ) is called a fuzzy topological space.

The elements of τ are called open fuzzy sets in X.

Definition 2.3. ([15]) A fuzzy topological space (X, τ) is said to be fuzzy Hausdorff if for $x, y \in X$ and $x \neq y$ there exist $\eta, \beta \in \tau$ with $\eta(x) = \beta(y) = 1$ and $\eta \wedge \beta = 0$.

A mapping f from a fuzzy topological space X to a fuzzy topological space Y is called fuzzy continuous if $f^{-1}(\mu)$ is open in X for each open fuzzy set μ in Y.

Suppose X is a fuzzy topological space and $x \in X$. A fuzzy set μ in X is called a neighborhood of $x \in X$ if there is an open fuzzy set η with $\eta \leq \mu$ and $\eta(x) = \mu(x) > 0$. Warren [17] has proved that a fuzzy set μ in X is open if and only if μ is a neighborhood of x for each $x \in X$ with $\mu(x) > 0$.

A directed set is a nonempty set T together with a reflexive and transitive binary relation $\leq$ (that is a preorder) such that

$$\forall x, y \in T \quad \exists z \in T \quad s.t \quad x \leq z, \quad y \leq z.$$

Let T be a directed set and X be a topological (or fuzzy topological) space. A net in X is a function from T to X. We often write a net in the form $(x_\alpha)_{\alpha \in T}$, where $\alpha \in T$ is mapped to $x_\alpha \in X$ [12].

Definition 2.4. ([20]) If μ_1 and μ_2 are two fuzzy subsets of a vector space E, then the fuzzy set $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \sup_{x = x_1 + x_2} (\mu_1(x_1) \wedge \mu_2(x_2)).$$

If $t \in \mathbb{K}$, we define the fuzzy sets $\mu_1 \times \mu_2$ and $t\mu$ as follows:

$$(\mu_1 \times \mu_2)(x_1, x_2) = \min\{\mu_1(x_1), \mu_2(x_2)\}$$

and

   (*i*)   for $t \neq 0$, $(t\mu)(x) = \mu(\frac{x}{t})$ for all $x \in E$,

   (*ii*)   for $t = 0$, $(t\mu)(x) = \begin{cases} 0 & x \neq 0, \\ \sup_{y \in E} \mu(y) & x = 0. \end{cases}$

Definition 2.5. ([6]) A fuzzy topology τ on a vector space E is said to be a fuzzy linear topology, if the two mappings

$\quad f : E \times E \to E, (x, y) \to x + y,$
$\quad g : \mathbb{K} \times E \to E, (t, x) \to tx,$

are continuous when $\mathbb{K}$ is equipped with the fuzzy topology induced by the usual topology, $E \times E$ and $\mathbb{K} \times E$ are the corresponding product fuzzy topologies. A vector space E with a fuzzy linear topology τ, denoted by the pair (E, τ) is called fuzzy topological vector space (abbreviated to FTVS).

A net $(x_\alpha)_{\alpha \in T}$ in a fuzzy topological vector space E is fuzzy convergent to x if and only if for each fuzzy neighborhood μ of x and each $0 < \varepsilon < \mu(x)$ there is $\alpha_o \in T$ such that $\mu(x_\alpha) > \varepsilon$ for all $\alpha > \alpha_0$. A net $(x_\alpha)_{\alpha \in T}$ in a fuzzy topological vector space E is fuzzy Cauchy if and only if for each fuzzy neighborhood μ of zero and each $0 < r < \mu(0)$ there is $\alpha_0 \in T$ such that for every $\alpha, \alpha' \in T$ with $\alpha, \alpha' > \alpha_0$, $|\mu(x_\alpha - x_{\alpha'})| > r$. A fuzzy topological vector space E is called complete if and only if each fuzzy Cauchy net in E is convergent [5].

A fuzzy set μ in the vector space E is called balanced if $t\mu \le \mu$ for each scalar t with $|t| \le 1$. As, it is shown in [9], μ is balanced if and only if $\mu(tx) \ge \mu(x)$ for each $x \in E$ and each scalar t with $|t| \le 1$. Also, when μ is balanced, we have $\mu(0) \ge \mu(x)$ for each $x \in E$. The fuzzy set μ is called absorbing if and only if $\sup_{t>0} t\mu = 1$. Then a fuzzy set μ is absorbing whenever $\mu(0) = 1$. We shall say that the fuzzy set μ is convex if and only if for all $t \in I$, $t\mu + (1 - t)\mu \le \mu$ [10].

Definition 2.6. ([10]) Let (E, τ) be a fuzzy topological vector space. The collection $\nu \subset \tau$ of neighborhoods of zero is a local base whenever for each neighborhood μ of zero and each $\theta \in (0, \mu(0))$ there is $\gamma \in \nu$ such that $\gamma \le \mu$ and $\gamma(0) > \theta$.

Definition 2.7. ([11]) A fuzzy seminorm on E is a fuzzy set μ in E which is absolutely convex and absorbing.

We say that a fuzzy set μ, in a vector space E, absorbs a fuzzy set η if $\mu(0) > 0$ and for every $\theta < \mu(0)$ there exists $t > 0$ such that $\theta \wedge (t\eta) \le \mu$. A fuzzy set μ in a fuzzy topological vector space E is called bounded if it is absorbed by every neighborhood of zero.

Definition 2.8. ([11]) A fuzzy topological vector space (E, τ) is called locally convex if it has a neighborhood base at zero consisting of convex fuzzy sets.

A locally convex fuzzy linear space E is called bornological if every absolutely convex fuzzy set in E which absorbs bounded fuzzy sets is a neighborhood of zero or equivalently every fuzzy bounded linear operator from E into any fuzzy topological vector space is fuzzy continuous [11].

In this paper, the concepts of fuzzy real numbers and Felbin-fuzzy norm are considered in the sense of Xiao and Zhu:

Definition 2.9. ([18]) A mapping $\eta : \mathbb{R} \to I$ is called a fuzzy real number, whose α-level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \ge \alpha\}$, if it satisfies two axioms:

(N_1) There exists $r' \in \mathbb{R}$ such that $\eta(r') = 1$.
(N_2) For all $0 < \alpha \le 1$, there exist real numbers $-\infty < \eta_\alpha^- \le \eta_\alpha^+ < +\infty$ such that $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers [3].

Definition 2.10. ([18]) Let E be a vector space over $\mathbb{R}$; L and R (respectively, left norm and right norm) be symmetric and non-decreasing mappings from $I \times I$ into I satisfying $L(0, 0) = 0$ and $R(1, 1) = 1$. Then $\| \cdot \|$ is called a fuzzy norm and $(E, \| \cdot \|, L, R)$ is a fuzzy normed linear space (abbreviated to FNLS) if the mapping $\| \cdot \|$ from E into $F^*(\mathbb{R})$ satisfies the following axioms, where $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ for $x \in E$ and $\alpha \in (0, 1]$:

$(F1)$ $x = 0$ if and only if $\|x\| = \widetilde{0}$,

(F2) $\|rx\| = |r|\|x\|$ for all $x \in E$ and $r \in (-\infty, \infty)$,

(F3) $\forall x, y \in E:$

(F3R) if $s \geq \|x\|_1^-, t \geq \|y\|_1^-$ and $s + t \geq \|x + y\|_1^-$, then
$\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$,

(F3L) if $s \leq \|x\|_1^-, t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$, then
$\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$.

Definition 2.11. ([5]) For fixed $\alpha \in (0, 1]$ and $\varepsilon > 0$, the fuzzy set $\mu_\alpha(x, \varepsilon)$ defined in $(E, \|\cdot\|)$ by

$$\mu_\alpha(x, \varepsilon)(y) = \begin{cases} \alpha & \|y - x\|_\alpha^+ < \varepsilon, \\ 0 & else, \end{cases}$$

is said to be an $\alpha - open$ sphere in $(E, \|\cdot\|)$.

Definition 2.12. ([7]) Any fuzzy set $\mu \in I^E$ is defined to be $\|.\|-$ linearly open if for every $x \in supp(\mu)$ and $\alpha \in (0, \mu(x))$ there exists $\varepsilon > 0$ such that $\mu_\alpha(x, \varepsilon) \leq \mu$.

We set $\mathcal{T}_{\|.\|}^* = \{\mu \in I^E \mid \mu \ is \ \|.\| - linearly \ open\}$. It was proved in [7] that if $(E, \|\cdot\|, L, R)$ is a fuzzy normed space, then $(E, \|\cdot\|)$ is a fuzzy topological vector space.

The Katsaras norm was defined as follows:

Definition 2.13. ([11]) A fuzzy norm on vector space E is an absolutely convex and absorbing fuzzy set ρ with $\inf_{t>0}(t\rho)(x) = 0$, for $x \neq 0$.

If ρ is a Katsaras norm on a vector space E, then the collection

$$B_\rho = \{\theta \wedge (t\rho) \mid t > 0, 0 < \theta \leq 1\},$$

is a base of neighborhoods of zero for a fuzzy linear topology on E. The fuzzy set μ is a fuzzy neighborhood of zero in this fuzzy topology if and only if

$$\exists \theta: \quad 0 < \theta \leq 1, \quad \exists t > 0; \quad \theta \wedge (t\rho) \leq \mu.$$

Also μ is a fuzzy neighborhood of $x \in E$ if and only if

$$(x + \theta \wedge (t\rho)) \leq \mu$$

i.e. for $y \in E$,

$$(x + \theta \wedge (t\rho))(y) = (\theta \wedge (t\rho))(y - x) \leq \mu(y).$$

In the following theorem, we prove that the fuzzy topologies generated by Felbin-fuzzy norm and Katsaras norm are equivalent.

Theorem 2.14. *If $(E, \|\cdot\|)$ is a Felbin-fuzzy normed space, then there is a Katsaras norm ρ on E such that the fuzzy topologies generated by $\|\cdot\|$ and ρ are equivalent.*

Proof. Let $\|\cdot\|$ be a Felbin-fuzzy norm on E. We define the fuzzy set ρ on E as follows:

$$\rho(x) = \begin{cases} 1 & \|x\|_\alpha^+ < 1, \\ 0 & else. \end{cases}$$

The fuzzy set ρ is convex, since for $t \in (0, 1]$,

$$(t\rho)(x) + ((1-t)\rho)(x) = \sup_{x=x_1+x_2} ((t\rho)(x_1) \wedge ((1-t)\rho)(x_2))$$

$$= \sup_{x=x_1+x_2} \left(\begin{cases} 1 & \|x_1\|_\alpha^+ < t \\ 0 & else \end{cases} \wedge \begin{cases} 1 & \|x_2\|_\alpha^+ < 1-t \\ 0 & else \end{cases} \right)$$

$$\leq \begin{cases} 1 & \|x\|_\alpha^+ < t + (1-t) = 1 \\ 0 & else \end{cases}$$

$$= \rho(x).$$

Also ρ is balanced since for $t \neq 0$ with $|t| \leq 1$,

$$(t\rho)(x) = \rho(\frac{x}{t})$$

$$= \begin{cases} 1 & \|x\|_\alpha^+ < |t| \\ 0 & else \end{cases}$$

$$\leq \begin{cases} 1 & \|x\|_\alpha^+ < 1 \\ 0 & else \end{cases}$$

$$= \rho(x).$$

Also, we have $\sup_{t>0}(t\rho)(x) = 1$ and $\inf_{t>0}(t\rho)(x) = 0$. Therefore ρ is a Katsaras norm. Now, we prove that the fuzzy topologies generated by $\|\cdot\|$ and ρ are equivalent. For $0 < \alpha \leq 1$ and $\varepsilon > 0$ we have

$$(\alpha \wedge (\varepsilon\rho))(y) = \alpha \wedge \rho(\frac{y}{\varepsilon})$$

$$= \alpha \wedge \begin{cases} 1 & \|y\|_\alpha^+ < \varepsilon \\ 0 & else \end{cases}$$

$$= \begin{cases} \alpha & \|y\|_\alpha^+ < \varepsilon \\ 0 & else \end{cases}$$

$$= \mu_\alpha(0, \varepsilon)(y).$$

This shows that these two fuzzy topologies have a same base at zero, therefore these are equivalent. □

Example 2.15. Consider the Felbin-fuzzy normed space $(\mathbb{R}^n, \|\cdot\|)$, where $\|\cdot\| : \mathbb{R}^n \to F^*(\mathbb{R})$ is defined by:

$$\|(x_1, x_2, \cdots, x_n)\|(t) = \begin{cases} 1 & t = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \\ 0 & else. \end{cases}$$

Then, we have

$$\|(x_1, x_2, \cdots, x_n)\|_\alpha^+ = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Now, consider the Katsaras norm

$$\rho(x) = \begin{cases} 1 & \Sigma_{i=1}^n x_i^2 < 1, \\ 0 & else. \end{cases}$$

Therefore by Theorem 2.14 the fuzzy topologies generated by ρ and $\|\cdot\|$ are equivalent and for $0 < \alpha \leq 1$ and $\varepsilon > 0$, we have $\mu_\alpha(0, \varepsilon)(y) = (\alpha \wedge \varepsilon\rho)(y)$.

3. Weak Fuzzy Topology

Let X be a nonempty set, $\{(Y_\alpha, \Gamma_\alpha)\}_{\alpha \in T}$ be a family of fuzzy topological spaces and let for each $\alpha \in T$, $f_\alpha : X \to Y_\alpha$ be a function. The weak fuzzy topology on X generated by the family $\mathcal{F} = \{f_\alpha\}_{\alpha \in T}$ is the coarsest or weakest fuzzy topology on X that makes all the functions f_α fuzzy continuous. This fuzzy topology is generated by the fuzzy sets $\{f_\alpha^{-1}(\mu) : \alpha \in T, \ \mu \in \Gamma_\alpha\}$, where the fuzzy sets $f_\alpha^{-1}(\mu)$ are defined as follows:

$$f_\alpha^{-1}(\mu)(x) = \mu(f_\alpha(x)) \quad \forall x \in X.$$

Another subbase for the weak fuzzy topology consists of the fuzzy sets

$$\{f_\alpha^{-1}(\mu) : \alpha \in T, \ \mu \in S_\alpha\},$$

where S_α is a subbase for Γ_α. We denote the weak fuzzy topology on X by $\sigma_f(X, \mathcal{F})$. A base for the weak fuzzy topology can be constructed as the collection of the fuzzy sets of the form $\bigwedge_{k=1}^n f_{\alpha_k}^{-1}(\mu_{\alpha_k})$, where $\mu_{\alpha_k} \in \Gamma_{\alpha_k}$, $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset T$ [13].

The following lemma has an important role in the study of weak fuzzy topologies.

Lemma 3.1. *A net $(x_j)_{j \in T}$ converges to x in the fuzzy topology $\sigma_f(X, \mathcal{F})$ if and only if $(f_\alpha(x_j))_{j \in T}$ converges to $f_\alpha(x)$ for each $\alpha \in T$.*

Proof. If we consider the fuzzy topology $\sigma_f(X, \mathcal{F})$ on X, then the functions f_α are fuzzy continuous. This shows that if $(x_j)_{j \in T}$ converges to x in the fuzzy topology $\sigma_f(X, \mathcal{F})$, then $(f_\alpha(x_j))_{j \in T}$ converges to $f_\alpha(x)$ for each $\alpha \in T$. Conversely, let $V = \bigwedge_{k=1}^n f_{\alpha_k}^{-1}(\mu_{\alpha_k})$ be a weak fuzzy neighborhood of x, where $\mu_{\alpha_k} \in \Gamma_{\alpha_k}$. For each k, if $(f_{\alpha_k}(x_j))_{j \in T}$ converges to $f_{\alpha_k}(x)$, then for each $0 < r_k < \mu_{\alpha_k}(f_{\alpha_k}(x))$ there is j_k such that for each $j > j_k$ we have $\mu_{\alpha_k}(f_{\alpha_k}(x_j)) > r_k$. Therefor for $j > max\{j_1, j_2, \cdots, j_n\} = j'$, we have

$$f_{\alpha_k}^{-1}(\mu_{i_k})(x_j) > max\{r_1, r_2, \cdots, r_n\} = r.$$

Then $V(x_j) > r$ for each $j > j'$. This proves our claim. $\quad\square$

Remark 3.2. On $\mathbb{K}$, if we consider the fuzzy topology is obtained from the usual topology of $\mathbb{K}$ that is the family of fuzzy sets $\mu : \mathbb{K} \to I$, which are lower semi-continuous, then endowed with this fuzzy topology $\mathbb{K}$ is a fuzzy topological vector space. This fuzzy topology is compatible with the fuzzy norm

$$\rho(x) = \begin{cases} 1 & |x| < 1, \\ 0 & else. \end{cases}$$

For this, we prove that the collection $B_\rho = \{\theta \wedge (t\rho) \mid t > 0, 0 < \theta \leq 1\}$ is a base of fuzzy neighborhoods for zero. Let μ be a fuzzy neighborhood of zero in $\mathbb{K}$. Then μ is lower semi-continuous and $\mu(0) > 0$. Let $0 < r < \mu(0)$. Then $\mu^{-1}(r, 1]$ is open in $\mathbb{K}$ and $0 \in \mu^{-1}(r, 1]$. Then there is $m > 0$ such that

$$\{x \in \mathbb{K} : |x| < m\} \subseteq \mu^{-1}(r, 1].$$

Now, we claim that $r \wedge (m\rho) \leq \mu$. Indeed, for $x \in \mathbb{K}$, we have

$$r \wedge (m\rho)(x) = r \wedge \rho(\frac{x}{m}) = \rho(x) = \begin{cases} r & |x| < m, \\ 0 & else. \end{cases}$$

Then, we have $r \wedge (m\rho)(x) \leq \mu(x)$. Also this fuzzy topology is compatible with the Flebin-fuzzy norm $\|.\| : \mathbb{K} \to F^*(\mathbb{R})$, defined for $t \in \mathbb{K}$ as follows:

$$\|x\|(t) = \begin{cases} 1 & t = |x|, \\ 0 & else, \end{cases}$$

with $L = \min$ and $R = \max$. But this fuzzy topology is not the only Hausdorff linear fuzzy topology on $\mathbb{K}$. For example it was shown in [8] that the fuzzy norm

$$\|x\|^*(t) = \begin{cases} 1 - \frac{t}{|x|} & 0 \leq t \leq |x|, x \neq 0, \\ 0 & else, \end{cases}$$

is not equivalent with the fuzzy norm

$$\|x\|(t) = \begin{cases} 1 & t = |x|, \\ 0 & else. \end{cases}$$

Then, we can consider more than one linear fuzzy topology on $\mathbb{K}$.

Definition 3.3. Let (E, τ) be a fuzzy topological vector space and Γ be locally convex Hausdorff linear fuzzy topology on $\mathbb{K}$. We denote the family of all fuzzy continuous linear functionals $f : (E, \tau) \to (\mathbb{K}, \Gamma)$ by E'_Γ and call it the $\Gamma-$ fuzzy dual of E. Also, we denote the algebraic dual of E by E^*.

In [14], the concept of dual pair defined as follows:
We call (E, E') a dual pair, whenever E and E' are two vector spaces over the same $\mathbb{K}$ scalar field and $\langle x, x' \rangle$ is a bilinear form on E and E' satisfying the following conditions:

(D) For each $x \neq 0$ in E, there is $x' \in E'$ such that $\langle x, x' \rangle \neq 0$.

(D') For each $x' \neq 0$ in E' there is $x \in E$ such that $\langle x, x' \rangle \neq 0$.

Let (E, E') be a dual pair. Then each $x' \in E'$ is a linear functional on E by taking:

$$x'(x) = \langle x, x' \rangle \quad x \in E.$$

Consider on $\mathbb{K}$ a locally convex Hausdorff linear fuzzy topology Γ. We denote the weak fuzzy topology on E generated by mapping $x' : E \to \mathbb{K}$ by $\sigma_f^\Gamma(E, E')$.

We note that if Γ_1 and Γ_2 are two fuzzy topologies on $\mathbb{K}$ and $\Gamma_1 \subset \Gamma_2$, then $\sigma_f^{\Gamma_1}(E, E') \subset \sigma_f^{\Gamma_2}(E, E')$.

Theorem 3.4. *Let (E, E') be a dual pair. Then E endowed with $\sigma_f^\Gamma(E, E')$ is a locally convex fuzzy topological vector space.*

Proof. We prove that the addition and scalar multiplication are fuzzy continuous. For $x, y, z \in E$, let $z = x + y$. Also suppose $V = \bigwedge_{i=1}^n f_i^{-1}(\mu_i)$ is a weak fuzzy neighborhood of z, where $f_i \in E'$ and μ_i is a fuzzy neighborhood in $\mathbb{K}$. Then $V(z) > 0$. This shows that $\mu_i(f_i(z)) = f_i^{-1}(\mu_i)(z) > 0$ for each $i = 1, 2, \cdots, n$. Then for each $i = 1, 2, \cdots, n$, μ_i is a fuzzy neighborhood of $f_i(z)$ in $\mathbb{K}$. Now since we have $f_i(x) + f_i(y) = f_i(z)$ and $\mathbb{K}$ is a fuzzy topological vector space, for each $i = 1, 2, \cdots, n$, there are fuzzy neighborhoods ω_i and ψ_i of $f_i(x)$ and $f_i(y)$ (respectively) such that if $\omega_i(a) > 0$ and $\psi_i(b) > 0$, then $\mu_i(a + b) > 0$ and $\omega_i + \psi_i \leq \mu_i$. Now we set

$$U = \bigwedge_{i=1}^n f_i^{-1}(\omega_i) \text{ and } O = \bigwedge_{i=1}^n f_i^{-1}(\psi_i).$$

Then, we have

$$U(x) = \bigwedge_{i=1}^n f_i^{-1}(\omega_i)(x) = \bigwedge_{i=1}^n \omega_i(f_i(x)) > 0,$$

and

$$O(y) = \bigwedge_{i=1}^n f_i^{-1}(\psi_i)(y) = \bigwedge_{i=1}^n \psi_i(f_i(y)) > 0.$$

Therefore U and O are weak fuzzy neighborhoods of x and y, respectively. Now, let for $m, n \in E$, $U(m) > 0$ and $O(n) > 0$. Then for each $i = 1, 2, \cdots, n$, we have $\omega_i(f_i(m)) > 0$ and $\psi_i(f_i(n)) > 0$ and therefore $\mu_i(f_i(m + n)) > 0$. Thus

$$V(m + n) = \bigwedge_{i=1}^{n} f_i^{-1}(\mu_i)(m + n)$$

$$= \bigwedge_{i=1}^{n} \mu_i(f_i(m + n)) > 0.$$

Also, since $\omega_i + \psi_i \leq \mu_i$ for all $i = 1, 2, \cdots, n$, we have

$$(U + O)(x) = \sup_{x=x_1+x_2} \bigwedge_{i=1}^{n} (\omega_i(f_i(x_1)) \wedge \psi_i(f_i(x_2)))$$

$$= \bigwedge_{i=1}^{n} \sup_{x=x_1+x_2} (\omega_i(f_i(x_1)) \wedge \psi_i(f_i(x_2)))$$

$$= \bigwedge_{i=1}^{n} (\omega_i + \psi_i)(f_i(x))$$

$$\leq \bigwedge_{i=1}^{n} \mu_i(f_i(x))$$

$$= V(x). \tag{1}$$

Since there are fuzzy neighborhood U and O such that $U + O \leq V$, therefore the addition is fuzzy continuous with respect to $\sigma_f^{\Gamma}(E, E')$.

Now, we show that the scalar multiplication is fuzzy continuous. For $t \in \mathbb{K}$ and $x \in E$, let $M = \bigwedge_{i=1}^{n} f_i^{-1}(\gamma_i)$ is a weak fuzzy neighborhood of tx. Then for each $i = 1, 2, \cdots, n$, we have

$$\gamma_i(t f_i(x)) = \gamma_i(f_i(tx)) = f_i^{-1}(\mu_i)(tx) > 0.$$

Then μ_i is fuzzy neighborhood of $t f_i(x)$ in $\mathbb{K}$. Since $\mathbb{K}$ is a fuzzy topological vector space, there is fuzzy neighborhood η_i and ϑ_i of t and $f_i(x)$ in $\mathbb{K}$ such that if for $a, b \in \mathbb{K}$, $\eta_i(a) > 0$ and $\vartheta_i(b) > 0$ then $\mu_i(ab) > 0$ and $a\vartheta_i \leq \mu_i$. Now, we set

$$S = \bigwedge_{i=1}^{n} f_i^{-1}(\vartheta_i).$$

We have

$$S(x) = \bigwedge_{i=1}^{n} f_i^{-1}(\vartheta_i)(x) = \bigwedge_{i=1}^{n} \vartheta_i(f_i(x)) > 0.$$

Then S is a weak fuzzy neighborhood of x. Now, let for $a \in \mathbb{K}$, $\eta_i(a) > 0$ and for $y \in S$, $S(y) > 0$, then $\vartheta_i(f_i(y)) > 0$ for each $i = 1, 2, \cdots, n$. The above arguments show that $f_i^{-1}(\mu_i)(ay) = \mu_i(a f_i(y)) > 0$ for each i. Then $V(ay) = \bigwedge_{i=1}^{n} f_i^{-1}(\mu_i)(ay) > 0$. Also for $a \neq 0$ with $\eta_i(a) > 0$, we have

$$(aS)(y) = \bigwedge_{i=1}^{n} f_i^{-1}(\vartheta_i)(\frac{y}{a})$$

$$= \bigwedge_{i=1}^{n} (\vartheta_i(f_i(\frac{y}{a})))$$

$$= \bigwedge_{i=1}^{n} (a\vartheta_i(f_i(y)) \leq \mu_i(y).$$

Then the scalar multiplication is fuzzy continuous. Now, we prove that $\sigma_f^\Gamma(E, E')$ is locally convex. Since $(\mathbb{K}, \Gamma)$ is locally convex, Γ has a base of convex fuzzy neighborhoods. Let $\mu \in \Gamma$ be convex. It is enough to show that $x'^{-1}(\mu)$ is convex for $x' \in E'$. For $t \in I$ and $x, y \in E$ we have

$$
\begin{aligned}
x'^{-1}(\mu)(tx + (1 - t)y) &= \mu(x'(tx + (1 - t)y)) \\
&= \mu(tx'(x) + (1 - t)x'(y)) \\
&\le t\mu(x'(x)) + (1 - t)\mu(x'(y)) \\
&= tx'^{-1}(\mu)(x) + (1 - t)x'^{-1}(\mu)(y).
\end{aligned}
$$

Therefore $x'^{-1}(\mu)$ is convex. $\square$

Remark 3.5. We note that since in a fuzzy topological vector space the addition and scalar multiplication are fuzzy continuous, one can use local basis instead of basis for a linear fuzzy topology.

Let (E, E') be a dual pair and S be a local base for the fuzzy linear topology μ on $\mathbb{K}$. Then the fuzzy sets $\bigwedge_{i=1}^n x_i^{-1}(\mu_i)$, where $x_1, x_2, \cdots, x_n \in E'$ and $\mu_1, \mu_2, \cdots, \mu_n \in S$ consist a local base for the topology $\sigma_f^\Gamma(E, E')$. If we consider the usual linear fuzzy topology on $\mathbb{K}$ which is compatible with the fuzzy norm

$$
\rho(x) = \begin{cases} 1 & |x| < 1, \\ 0 & else, \end{cases}
$$

then, the collection

$$
B_\rho = \{\theta \wedge (t\rho) \mid t > 0, 0 < \theta \le 1\},
$$

is a base of fuzzy neighborhood of zero in $\mathbb{K}$. This shows that the collection

$$
B_\rho = \{\bigwedge_{i=1}^n f_i^{-1}(\theta_i \wedge (t_i\rho)) \mid t_i > 0, 0 < \theta_i \le 1, f_i \in E'\},
$$

is a base of fuzzy neighborhood of zero for $\sigma_f^\Gamma(E, E')$.

Lemma 3.6. ([14]) *If $f_0, f_1, \cdots, f_n$ are linear functionals on a vector space E, then either f_0 is a linear combinations of $f_1, f_2, \cdots, f_n$, or there is $a \in E$ such that $f_0(a) = 1$ and*

$$
f_1(a) = f_2(a) = \cdots = f_n(a) = 0.
$$

Proof. See [14], Chapter II, Section 3. $\square$

Proposition 3.7. *If (E, E') is a dual pair, then the fuzzy dual of E under $\sigma_f^\Gamma(E, E')$ is E'.*

Proof. Let f be a linear form on E fuzzy continuous under $\sigma_f^\Gamma(E, E')$ and let S be a base in $\mathbb{K}$. For each $\mu \in S$ with $\mu(0) > \mu(1)$, there is $x_1', x_2', \cdots, x_n' \in E'$ such that

$$
\bigwedge_{i=1}^n x_i'^{-1}(\mu) \le f^{-1}(\mu)
$$

and

$$
\bigwedge_{i=1}^n x_i'^{-1}(\mu)(0) = f^{-1}(\mu)(0) > 0
$$

therefor for $x \in E$, $\mu(x_i'(x)) \le \mu(f(x))$. Now by Lemma 3.6, either f is a linear combination of $x_1', x_2', \cdots, x_n'$ or there is some $a \in E$ such that $f(a) = 1$ but $x_i'(a) = 0$ for all $i = 1, 2, \cdots, n$. This shows that $\mu(0) \le \mu(1)$ and this is a contradiction. Then

$$
f = \sum \lambda_i x_i' \in E'.
$$

On the other hand, if $x' \in E'$ then it is fuzzy continuous under $\sigma_f^\Gamma(E, E')$ by the definition of weak fuzzy topology. $\square$

Theorem 3.8. *Let E be a vector space and E^* be its algebraic dual. Then E^* is complete under the weak fuzzy topology $\sigma_f^\Gamma(E^*, E)$.*

Proof. We consider E^* endowed with the weak fuzzy topology $\sigma_f^\Gamma(E^*, E)$. Suppose $(f_j)_{j \in T}$ is a fuzzy Cauchy net in E^*. Then for each $a \in E$ the net $(f_j(a))_{j \in T}$ is a fuzzy Cauchy net in $\mathbb{K}$. Since $\mathbb{K}$ is complete under its usual fuzzy topology, for each $a \in E$ there is $f(a) \in \mathbb{K}$ such that $(f_j(a))_{j \in T}$ converges to $f(a)$. Since each f_j is linear, then f is a linear functional on E. Now Lemma 3.1 shows that $(f_j)_{j \in T}$ converges to f in the fuzzy topology $\sigma_f^\Gamma(E^*, E)$. Then E^* is complete under the fuzzy topology $\sigma_f^\Gamma(E^*, E)$. $\square$

Definition 3.9. Let (E, E') be a dual pair. The locally convex fuzzy linear topology on E is called *dual pair topology* if the dual of E under τ is E'. Also, the locally convex linear topology τ' on E' is called a *dual pair topology* if the dual of E' under τ' is E.

In a special case if (E, τ) is a Hausdorff fuzzy topological vector space, then (E, E'_Γ) is a dual pair and τ is a fuzzy topology of dual pair. Also, $\sigma_f^\Gamma(E, E')$ is the coarsest fuzzy topology of dual pair (E, E').

Theorem 3.10. *If (E, E') is a dual pair, then $(E, \sigma_f^\Gamma(E, E'))$ is fuzzy Hausdorff.*

Proof. Let $x, y \in E$ and $x \neq y$. Then by condition (D) there is $x' \in E'$ such that $x'(x - y) \neq 0$ i.e. $x'(x) \neq x'(y)$. Since $\mathbb{R}$ is fuzzy Hausdorff there is fuzzy open sets β, η in $\mathbb{R}$ such that $\beta \wedge \eta = 0$ and $\beta(x'(x)) = \eta(x'(y)) = 1$. Now we consider the fuzzy open sets $x'^{-1}(\eta), x'^{-1}(\beta)$ in E. These fuzzy sets are the fuzzy neighborhoods of x and y and we have

$$(x'^{-1}(\eta) \wedge x'^{-1}(\beta))(a) = min\{\eta(x'(a)), \beta(x'(a))\} = 0$$

and

$$x'^{-1}(\eta)(y) = \eta(x'(y)) = 1$$

similarly,

$$x'^{-1}(\beta)(x) = \beta(x'(x)) = 1.$$

Now the proof is complete. $\square$

Corollary 3.11. *Let (E, E') be a dual pair and τ be a fuzzy topology of dual pair on E. Then E is fuzzy Hausdorff endowed with τ.*

Proof. Clearly, τ is finer than $\sigma_f^\Gamma(E, E')$. Now Theorem 3.10 shows that E is fuzzy Hausdorff endowed with τ. $\square$

Proposition 3.12. *If (E, τ) is finite dimensional Hausdorff fuzzy topological vector space, then $E'_\Gamma = E^*$.*

Proof. Let $B = \{e_1, e_2, \cdots, e_n\}$ be a base for E. Then there is a base $B' = \{e_1^*, e_2^*, \cdots, e_n^*\}$ for E^* such that $e_i^*(e_i) = 1$ for each $i = 1, 2, \cdots, n$ and $e_i^*(e_j) = 0$ for $i \neq j$. Then E and E^* are algebraical isomorphic. Since (E, E'_Γ) is a dual pair, we have $E \subseteq (E'_\Gamma)^* \simeq E'_\Gamma$ and $E'_\Gamma \subseteq E^* \simeq E$. Then $E'_\Gamma = E^*$. $\square$

Remark 3.13. Let E be finite dimensional vector space. We know that there is only a unique Hausdorff linear topology on E. But this is not true in the fuzzy case, i.e. the Hausdorff linear fuzzy topology on E is not unique. For example consider the vector space $\mathbb{R}$. It was shown in [8], that the fuzzy norm

$$\|x\|^*(t) = \begin{cases} 1 - \dfrac{t}{|x|} & 0 \leq t \leq |x|, x \neq 0 \\ 0 & \text{else}, \end{cases}$$

is not equivalent with the fuzzy norm

$$\|x\|(t) = \begin{cases} 1 & t = |x| \\ 0 & \text{else}. \end{cases}$$

Therefore, the fuzzy topologies generated with these norms are not equivalent.

Corollary 3.14. *Let (E, τ) be a finite dimensional Hausdorff fuzzy topological vector space. Then we have $\sigma_f^\Gamma(E, E_\Gamma') = \sigma_f^\Gamma(E, E^*)$. In fact this is concluded form Proposition 3.12.*

It is well known that between all the linear topologies on a finite dimensional vector space E there is a strongest one, namely the unique Hausdorff linear topology on E. We denote this topology by τ. Let $\omega(\tau)$ ($\omega(\tau)$ was firstly used by R. Lowen [13]) be the collection of all lower semi-continuous functions from E into I. Then $\omega(\tau)$ is a linear fuzzy topology on E.

Let $\sigma(E, E')$ be the usual weak topology.

Theorem 3.15. *Let (E, E') be a dual pair. If we consider the usual fuzzy topology $\omega(\tau)$ on $\mathbb{K}$, then*

$$\sigma_f^{\omega(\tau)}(E, E') = \omega(\sigma(E, E')).$$

Proof. It is enough to show that for $f_1, f_2, \cdots, f_n \in E'$ and $\mu_1, \mu_2, \cdots, \mu_n \in \omega(\tau)$ the fuzzy set $\bigwedge_{i=1}^n f_i^{-1}(\mu_i)$ is lower semi-continuous on E with respect to the topology $\sigma(E, E')$. Since $\mu_i \in \omega(\tau)$ for $i = 1, 2, \cdots, n$, then $\mu_i : \mathbb{K} \to I$ are lower semi-continuous. Also

$$f_i : (E, \sigma(E, E')) \to \mathbb{K}$$

is continuous and then is lower semi-continuous. Then $\mu_i \circ f_i$ is lower semi-continuous for each $i = 1, 2, \cdots, n$. Then

$$\bigwedge_{i=1}^n f_i^{-1}(\mu_i) = \bigwedge_{i=1}^n \mu_i \circ f_i$$

is lower semi-continuous on $(E, \sigma(E, E'))$ for each $i = 1, 2, \cdots, n$. $\square$

If we consider on $\mathbb{K}$ any Hausdorff linear fuzzy topology different from the topology $\omega(\tau)$, then the fuzzy topologies $\sigma_f^{\omega(\tau)}(E, E')$ and $\omega(\sigma(E, E'))$ are not equivalent. We illustrate this fact by an example.

Example 3.16. Consider the vector space $\mathbf{C}(\Omega)$, the collection of all real-valued continuous functions on the open set $\Omega \subseteq \mathbb{R}^n$. For each $x \in \Omega$, we set the evaluation function $\delta_x : \mathbf{C}(\Omega) \to \mathbb{R} : \delta_x(f) = f(x)$. Then δ_x is a linear functional on $\mathbf{C}(\Omega)$. Let $V = span\{\delta_x : x \in \Omega\}$. Then V is a real vector space and $(\mathbf{C}(\Omega), V)$ is a dual pair. On $\mathbb{R}$ the fuzzy norms

$$\|x\|^*(t) = \begin{cases} 1 - \frac{t}{|x|} & 0 \le t \le |x|, x \ne 0, \\ 0 & else, \end{cases}$$

and

$$\|x\|(t) = \begin{cases} 1 & t = |x|, \\ 0 & else, \end{cases}$$

are not equivalent (See [8], Remark 3.1). The fuzzy norm $\|.\|$ induces the fuzzy topology $\omega(\tau)$ which is the finest fuzzy topology on $\mathbb{R}$ (See [10], Theorem 3.18). The fuzzy norm $\|.\|^*$ induces the fuzzy topology $\mathcal{T}_{\|.\|}^*$ on $\mathbb{R}$ which is weaker than $\omega(\tau)$. Then there is $\mu \in \omega(\tau)$ such that $\mu \notin \mathcal{T}_{\|.\|}^*$. Now by Theorem 3.15, we have

$$\bigwedge_{i=1}^n \delta_{x_i}^{-1}(\mu) \in \omega(\sigma(\mathbf{C}(\Omega), V))$$

for $x_1, x_2, \cdots, x_n \in \Omega$. But if we consider $\mathcal{T}_{\|.\|}^*$ on $\mathbb{R}$, then we have

$$\bigwedge_{i=1}^n \delta_{x_i}^{-1}(\mu) \notin \sigma_f^{\mathcal{T}_{\|.\|}^*}(\mathbf{C}(\Omega), V)$$

for $x_1, x_2, \cdots, x_n \in \Omega$, since $\mu \notin \mathcal{T}_{\|.\|}^*$. This show that $\sigma_f^{\mathcal{T}_{\|.\|}^*}(\mathbf{C}(\Omega), V)$ is strictly weaker than $\omega(\sigma(\mathbf{C}(\Omega), V))$.

Theorem 3.15 and Example 3.16 show that the weak fuzzy topology is an extension of weak topology. Let $T : E \to F$ be a linear operator between two vector spaces. Every $y^* \in F^*$ gives rise to a real function T^*y^* on E defined pointwise via the formula

$$T^*y^*(x) = y^* \circ T(x) = y^*(T(x)),$$

for $x \in E$. Clearly T^*y^* is linear and so belongs to E^*. The operator T^* is called the *algebraic adjoint* of T.

Definition 3.17. Let (E, τ) and (F, ξ) be fuzzy topological vector spaces. The linear operator T is called weakly fuzzy continuous whenever

$$T : (E, \sigma_f^\Gamma(E, E_\Gamma')) \to (F, \sigma_f^\Gamma(F, F_\Gamma')),$$

is fuzzy continuous.

Lemma 3.18. *Let (E, τ) be a fuzzy topological vector space. Then the net $(x_i)_{i \in S}$ converges to x in the fuzzy topology $\sigma_f^\Gamma(E, E')$ if and only if the net $(x_i - x)_{i \in S}$ converges to 0.*

Proof. Let $(x_i)_{i \in S}$ converges to x in the fuzzy topology $\sigma_f^\Gamma(E, E')$. Then for each $T \in E'$, we have $(T(x_i))_{i \in S}$ converges to $T(x)$ in the fuzzy topology $\sigma_f^\Gamma(E, E')$. Then for each fuzzy neighborhood μ of zero and each $0 < r < \mu(0)$ there is $i_0 \in S$ such that for every $i > i_0$, $(T(x) + \mu)(T(x_i)) > r$. This shows that $\mu(T(x_i) - T(x)) > r$. Since T is linear, we have $\mu(T(x_i - x)) > r$. It follows that $(x_i - x)_{i \in S}$ converges to 0. The converse is similar. $\square$

The next result offers a very simple criterion for deciding whether a linear operator is weakly continuous. You only have to check that its adjoint carries fuzzy continuous functionals into fuzzy continuous functionals.

Theorem 3.19. *Let (E, E') and (F, F') be dual pairs and let $T : E \to F$ be a linear operator, where E and F are endowed with their weak fuzzy topologies. Then T is weakly fuzzy continuous if and only if the algebraic adjoint T^* satisfies $T^*(F_\Gamma') \subseteq E_\Gamma'$.*

Proof. Firstly, suppose that T is weakly fuzzy continuous. Since each $y' \in F_\Gamma'$ is weakly fuzzy continuous, then $y' \circ T$ is weakly fuzzy continuous linear operator on E. This shows that $y' \circ T \in E_\Gamma'$. Since $T^*(y') = y' \circ T$, then $T^*(y') \in E_\Gamma'$.

Conversely, let $T^*(F_\Gamma') \subseteq E_\Gamma'$ and $(x_i)_{i \in S}$ be a net in E such that $(x_i)_{i \in S}$ converges to x in the fuzzy topology $\sigma_f(E, E_\Gamma')$. This shows that for each $f \in E_\Gamma'$, $(f(x_i))_{i \in S}$ converges to $f(x)$. Now since $T^*(F_\Gamma') \subseteq E_\Gamma'$, then for each $y' \in F_\Gamma'$, we have $(T^*(y')(x_i))_{i \in S}$ converges to $T^*(y')(x_i)$. Therefore $(y'(T(x_i)))_{i \in S}$ converges to $y'(T(x))$ for each $y' \in F_\Gamma'$. This shows that $(T(x_i))_{i \in S}$ converges to $T(x)$ in the fuzzy topology $\sigma_f(F, F_\Gamma')$. Therefore T is weakly fuzzy continuous. $\square$

Theorem 3.20. *Let E and F be Hausdorff locally convex fuzzy topological vector spaces and $T : E \to F$ be a fuzzy continuous linear operator. Then T is weakly fuzzy continuous.*

Proof. Suppose $T : E \to F$ is fuzzy continuous. Then for each $y' \in F_\Gamma'$, $T^*(y') = y' \circ f$ is a fuzzy continuous linear operator on E and then $T^*(y') \in E_\Gamma'$. Therefore $T^*(F_\Gamma') \subseteq E_\Gamma'$. Now, Theorem 3.19 shows that T is weakly fuzzy continuous. $\square$

Theorem 3.20 shows that every fuzzy continuous linear operator is weakly fuzzy continuous. In the following theorem, we consider some conditions under which every weakly fuzzy continuous linear operator in fuzzy continuous.

Theorem 3.21. *Let E be a bornological fuzzy topological vector space and F be a fuzzy topological vector space such that every weakly bounded fuzzy sets in F is bounded. Then every weakly fuzzy continuous linear operator $T : E \to F$ is fuzzy continuous.*

Proof. Since E is bornological, it is enough show that T is fuzzy bounded. Let ψ be a bounded fuzzy set in E. Then ψ is weakly fuzzy bounded. Since T is weakly fuzzy continuous, $T(\psi)$ is a weakly fuzzy bounded set in F. Now the assumptions of theorem shows that $T(\psi)$ is a bounded fuzzy set in F. Then T is fuzzy bounded. $\square$

Corollary 3.22. *Every linear functional on a bornological fuzzy linear space is weakly fuzzy continuous if and only if it is fuzzy continuous. Therefore the weak fuzzy topology on $\mathbb{K}$ is identical with the original topology.*

Corollary 3.23. *Every linear functional on a seminormed fuzzy linear space is weakly fuzzy continuous if and only if it is fuzzy continuous, hence every seminormed fuzzy linear space is bornological.*

Example 3.24. Let E be vector space and $\mathcal{F}_0$ be the collection of all absolutely convex and absorbent fuzzy sets in E. Then $\mathcal{F}_0$ satisfies the conditions of Theorem 4.2 from [10]. Then by Theorem 4.2 from [10], there exists a linear fuzzy topology τ_β on E such that $\mathcal{F}_0$ coincides with the family of all fuzzy neighborhoods of zero. In fact, τ_β is the topology which is created by the collection of all absolutely convex and absorbing fuzzy sets as the neighborhoods of zero. The fuzzy topological space (E, τ_β) is fuzzy Hausdorff. Indeed for $0 \neq a \in E$, we consider the set $\left\{\frac{1}{2}a\right\}$ and extend it for a base β for the vector space E. Let A is the absolutely convex envelop of A. Now, χ_E is absolutely convex and absorbing in the fuzzy sense. Then it is a fuzzy neighborhood of zero in τ_β, but we have $\chi_E(a) = 0$, since $a \notin E$.

The weakly fuzzy bounded subsets of a fuzzy topological vector space have an important role in the constructing of fuzzy topologies on the dual space.

Theorem 3.25. *Let (E, E') be a dual pair. Then the fuzzy set μ in E is weakly fuzzy bounded if and only if*

$$\varphi_\mu(x') = \sup_{y \in \mathbb{K}} x'(\mu)(y),$$

is a fuzzy seminorm on E'.

Proof. Let μ be weakly fuzzy bounded. Then for each $x' \in E'$, $x'(\mu)$ is fuzzy bounded set in $\mathbb{K}$. This shows that φ_μ is well defined. Since μ is convex and x' is linear then φ_μ is convex. We have

$$\varphi_\mu(0) = \sup_{y \in \mathbb{K}} 0(\mu)(y) = \sup_{x \in E} \mu(x) = 1.$$

This shows that φ_μ is absorbing. For $t \in \mathbb{K}$ with $|t| \leq 1$, we have

$$\begin{aligned}
\varphi_\mu(tx') &= \sup_{y \in \mathbb{K}} (tx')(\mu)(y) \\
&= \sup_{y \in \mathbb{K}} \sup_{x \in (tx')^{-1}(y)} \mu(x) \\
&= \sup_{y \in \mathbb{K}} \sup_{x \in x'^{-1}(\frac{y}{t})} \mu(x) \\
&= \sup_{y \in \mathbb{K}} \sup_{x \in x'^{-1}(y)} (\mu)(x) \\
&= \varphi_\mu(x').
\end{aligned}$$

This shows that φ_μ is balanced. The converse is clear. $\square$

4. Conclusion

We proved that when $\Gamma = \omega(\tau)$, we have $\sigma_f^\Gamma(E, E') = \omega(\sigma(E, E'))$. Then, one can consider $\sigma(E, E')$ as an special case of $\sigma_f^\Gamma(E, E')$. Also, the difference between weak topology and weak fuzzy topology leads to new results on the theory of fuzzy topological vector spaces.

The weak fuzzy topology is very useful in constructing topologies on dual spaces. In the future, we will concentrate for proving the Mackey-Arens and Banach-Alaoglu theorems in the fuzzy topological vector spaces using weak fuzzy topology.

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