Filomat 35:2 (2021), 477–483
https://doi.org/10.2298/FIL2102477W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Extensions of Cline's Formula for Some New Generalized Inverses

Zhenying Wu[a], Qingping Zeng[b]

[a]College of Mathematics and Informatics, Fujian Normal University, Fuzhou 350117, P.R. China
[b]College of Computer and Information Sciences, Fujian Agriculture and Forestry University, Fuzhou 350002, P.R. China

Abstract. Let a, b, c, d be elements in a unital associative ring $\mathcal{R}$. In this note, we generalize Cline's formula for some new generalized inverses such as strong Drazin inverse, generalized strong Drazin inverse, Hirano inverse and generalized Hirano inverse to the case when $acd = dbd$ and $dba = aca$. As a particular case, some recent results are recovered.

1. Introduction

Throughout this note, $\mathcal{R}$ denotes an associative ring with unit 1. By $\mathcal{R}^{-1}$ and $\mathcal{R}^{nil}$ we represent the set of all invertible and nilpotent elements of $\mathcal{R}$, respectively. For an element $a \in \mathcal{R}$, the commutant and double commutant of a are defined by $comm(a) = \{x \in \mathcal{R} : ax = xa\}$ and $comm^2(a) = \{x \in \mathcal{R} : xy = yx, \text{for all } y \in comm(a)\}$, respectively. An element $a \in \mathcal{R}$ is said to be quasinilpotent if $1 + ax \in \mathcal{R}^{-1}$ for all $x \in comm(a)$. We use $\mathcal{R}^{qnil}$ to denote the set of all quasinilpotent elements of $\mathcal{R}$.

In 2017, Wang [19] introduced a new generalized inverse, strong Drazin inverse, which is a subclass of the Drazin inverse. An element $a \in \mathcal{R}$ is strongly Drazin invertible (or, s-Drazin invertible) if there exists $b \in \mathcal{R}$ such that

$$b \in comm(a), bab = b \text{ and } a - ab \in \mathcal{R}^{nil}.$$

In this case, b is called a strong Drazin inverse (or, s-Drazin inverse) of a, denoted by $b = a^{sD}$, and the least non-negative integer k for $(a - ab)^k = 0$ is called the strong Drazin index of a, denoted by $\mathrm{i}_{sD}(a)$. Let $\mathcal{R}^{sD}$ be the set of all s-Drazin invertible elements of $\mathcal{R}$. An element $a \in \mathcal{R}$ is called Drazin invertible if we replace the condition $a - ab \in \mathcal{R}^{nil}$ in the definition of the s-Drazin invertible element with $a(1 - ab) \in \mathcal{R}^{nil}$ (see [7]). In this case, b is called a Drazin inverse of a and denoted by $b = a^D$. By $\mathcal{R}^D$ we represent the set of all Drazin invertible elements of $\mathcal{R}$. If $a \in \mathcal{R}^{sD}$, observing that $a(1 - aa^{sD}) = (a - aa^{sD})(1 - aa^{sD})$, then $a \in \mathcal{R}^D$. Thus, the s-Drazin inverse of a is unique if it exists, and it belongs to the double commutant of a. However, invertible elements may not be s-Drazin invertible in general. For example, 2 is invertible but not s-Drazin invertible in complex number field $\mathbb{C}$.

2010 *Mathematics Subject Classification*. Primary 15A09; Secondary 16U99
Keywords. Cline's formula, ring, strong Drazin inverse, generalized strong Drazin inverse, Hirano inverse, generalized Hirano inverse
Received: 15 January 2019; Revised: 29 October 2020; Accepted: 01 February 2021
Communicated by Dragana Cvetković-Ilić
This work was supported by National Natural Science Foundation of China (Grant No. 11971108), Natural Science Foundation of Fujian Province (Grant No. 2020J01569) and the Science and Technology Innovation Fund of Fujian Agriculture and Forestry University (Grant No. CXZX2018036).
Email addresses: zhenyingwu2011@163.com (Zhenying Wu), zqpping2003@163.com (Qingping Zeng)

Recently, Chen and Sheibani introduced in [3] Hirano inverse. An element $a \in \mathcal{R}$ is called Hirano invertible provided that there exists $b \in \mathcal{R}$ such that

$$b \in comm(a), bab = b \text{ and } a^2 - ab \in \mathcal{R}^{nil}.$$

If such b exists, then it is unique and denoted by $b = a^H$, the Hirano inverse of a, and the minimal non-negative integer k for which $(a^2 - ab)^k = 0$ holds is called the Hirano index $\mathrm{i}_H(a)$ of a. The set of all Hirano invertible elements in $\mathcal{R}$ will be denoted by $\mathcal{R}^H$. As observed in [3], $\mathcal{R}^{sD} \subsetneq \mathcal{R}^H \subsetneq \mathcal{R}^D$. Let $a \in \mathcal{R}$ be s-Drazin invertible. Then a is also Hirano invertible. However, the indices $\mathrm{i}_{sD}(a)$ and $\mathrm{i}_H(a)$ need not be the same in general. For example, take $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{C})$. Since $A^3 = O$, it follows that A is s-Drazin invertible with $A^{sD} = O$ and $\mathrm{i}_{sD}(A) = 3$. But $\mathrm{i}_H(A) = 2$.

The concept of strong Drazin inverse was extended by Mosić [14] in a complex Banach algebra and by Gürgün [8] in a ring. Recall that an element $a \in \mathcal{R}$ is generalized strongly Drazin invertible (or, gs-Drazin invertible) if there exists $b \in \mathcal{R}$ such that

$$b \in comm^2(a), bab = b \text{ and } a - ab \in \mathcal{R}^{qnil}.$$

In this case, b is called a generalized strong Drazin inverse (or, gs-Drazin inverse) of a and denoted by $b = a^{gsD}$. If we use the condition $a(1 - ab) \in \mathcal{R}^{qnil}$ in the above definition instead of $a - ab \in \mathcal{R}^{qnil}$, then $a \in \mathcal{R}$ is called generalized Drazin invertible (see [10]). In the sequel, $\mathcal{R}^{gsD}$ denotes the set of all gs-Drazin invertible elements of $\mathcal{R}$. From [8, Corollary 3.3] it follows that, if $a \in \mathcal{R}^{gsD}$, then a is generalized Drazin invertible. Hence, the gs-Drazin inverse of a is unique if it exists (see [10, Theorem 4.2]).

Another subclass of generalized Drazin inverse, the so-called generalized Hirano inverse, was introduced by Abdolyousefi and Chen [1]. The generalized Hirano inverse is the unique common solution to the equations

$$b \in comm^2(a), bab = b \text{ and } a^2 - ab \in \mathcal{R}^{qnil}.$$

The element b above is unique if it exists and it will be denoted by $b = a^{gH}$. We denote by $\mathcal{R}^{gH}$ the set of all generalized Hirano invertible elements in $\mathcal{R}$.

In 1965, Cline [4] discovered that Drazin invertibility of ab is transferred to that of ba and $(ba)^D = b((ab)^D)^2a$. This equation is now called Cline's formula and it has an important role to express Drazin inverse of the sum of two elements (see [16] for instance). Cline's formula for generalized Drazin inverse was established in [13]. Recently, it has been found that Cline's formula for (generalized) Drazin inverse has suitable analogues under the assumptions

$$aba = aca \tag{1}$$

and

$$\begin{cases} acd = dbd \\ dba = aca \end{cases} \tag{2}$$

see [9, 12, 22, 23]. The cases (1) and (2) were introduced by Corach et al. [6] and Yan et al. [20], respectively. Taking "$a = d$" in (2), it gives (1), while letting "$b = c$" in (1), it goes back to Cline's formula.

The main concern of this note is common properties of the products of elements in a unital associative ring $\mathcal{R}$. We extend Cline's formula for new generalized inverses related to Drazin inverse such as (generalized) strong Drazin inverse and (generalized) Hirano inverse. Precisely, under the assumption (2), we show that

$$ac \in \mathcal{R}^{\bullet} \Longleftrightarrow bd \in \mathcal{R}^{\bullet},$$

and we have

$$(ac)^{\bullet} = d((bd)^{\bullet})^3 bac \text{ and } (bd)^{\bullet} = b((ac)^{\bullet})^2 d,$$

where $\bullet \in \{sD, gsD, H, gH\}$. As a special case, we recover some recent results in [1, 3, 8, 19].

2. Main results

Wang established Cline's formula for s-Drazin inverse (see [19, Theorem 3.1]). In the following, we generalize it to the case when (2) holds.

Theorem 2.1. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $ac \in \mathcal{R}^{sD}$ if and only if $bd \in \mathcal{R}^{sD}$. In this case, we have $(ac)^{sD} = d((bd)^{sD})^3 bac$, $(bd)^{sD} = b((ac)^{sD})^2 d$ and $|\mathrm{i}_{sD}(ac) - \mathrm{i}_{sD}(bd)| \le 2$.*

Proof. Suppose that there exist $s \in \mathcal{R}$ and a non-negative integer k such that

$$s(bd) = (bd)s, s(bd)s = s \text{ and } (bd - sbd)^k = 0.$$

Take $t = ds^3bac$. By [22, Theorem 2.1], t is a Drazin inverse of ac, so we have $t(ac) = (ac)t$ and $t(ac)t = t$. Thus, to prove that $ac \in \mathcal{R}^{sD}$, it is sufficient to show $(ac - act)^{k+2} = 0$.

Since $(1 - s)^k (bd)^k = 0$ and

$$\begin{aligned}
&(ac - act)^{k+2} \\
&= (ac - acds^3bac)^{k+2} = (ac - dbds^3bac)^{k+2} = (ac - ds^2bac)^{k+2} \\
&= [(1 - ds^2b)ac]^{k+2} \\
&= (1 - ds^2b)[ac(1 - ds^2b)]^{k+1}(ac) \\
&= (1 - ds^2b)(ac - dbds^2b)^{k+1}(ac) \\
&= (1 - ds^2b)(ac - dsb)^{k+1}(ac) \\
&= (1 - ds^2b)(ac - dsb)^k(dbac - dsbac) \\
&= (1 - ds^2b)(ac - dsb)^k d(1 - s)bac \\
&= (1 - ds^2b)(ac - dsb)^{k-1} d(1 - s)bd(1 - s)bac \\
&= d(1 - s)^{k+2}(bd)^k bac,
\end{aligned}$$

we get $(ac - act)^{k+2} = 0$, as required.

Conversely, suppose that $ac \in \mathcal{R}^{sD}$ and let u be the strong Drazin inverse of ac and m its index. Set $v = bu^2d$. By [22, Theorem 2.1] again, we know that v is a Drazin inverse of bd. Therefore it remains only to show $(bd - bdv)^{m+1} = 0$. Noting $bd - bdv = bd - bdbu^2d = bd - bdbacu^3d = bd - bacacu^3d = b(1 - u)d$ and

$$\begin{aligned}
[b(1 - u)d]^{m+1} &= [b(1 - u)d][b(1 - u)d][b(1 - u)d]^{m-1} \\
&= b(1 - u)(dbd - dbud)[b(1 - u)d]^{m-1} \\
&= b(1 - u)(acd - dbacu^2d)[b(1 - u)d]^{m-1} \\
&= b(1 - u)(acd - acacu^2d)[b(1 - u)d]^{m-1} \\
&= b(1 - u)(ac - acu)d[b(1 - u)d]^{m-1} \\
&= \cdots \\
&= b(1 - u)(ac - acu)^m d \\
&= 0,
\end{aligned}$$

we conclude that $(bd - bdv)^{m+1} = 0$. □

In the sequel, we use $\mathcal{B}(X, Y)$ to denote the set of all bounded linear operators from Banach space X to Banach space Y. Using the technique of block matrices, we obtain the operator case of Theorem 2.1.

Corollary 2.2. *Suppose that $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ACD = DBD$ and $DBA = ACA$. Then AC is s-Drazin invertible if and only if BD is s-Drazin invertible. In this case, we have $(AC)^{sD} = D((BD)^{sD})^3 BAC$ and $(BD)^{sD} = B((AC)^{sD})^2 D$.*

Corollary 2.3. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$ and let $n \geq 2$ be an integer. Then $(ac)^n \in \mathcal{R}^{sD}$ if and only if $(bd)^n \in \mathcal{R}^{sD}$. In this case,*

$$((bd)^n)^{sD} = b[((ac)^n)^{sD}]^2 d(bd)^{n-1} \text{ and } ((ac)^n)^{sD} = d[((bd)^n)^{sD}]^2 b(ac)^{n-1}.$$

Proof. Let $b' = b(db)^{n-1}$ and $c' = c(ac)^{n-1}$. Then $ac'd = db'd$ and $db'a = ac'a$. From Theorem 2.1, it follows that $(ac)^n = ac' \in \mathcal{R}^{sD}$ if and only if $(bd)^n = b'd \in \mathcal{R}^{sD}$, and we have

$$((bd)^n)^{sD} = (b'd)^{sD} = b'((ac')^{sD})^2 d = b[((ac)^n)^{sD}]^2 d(bd)^{n-1}$$

and

$$((ac)^n)^{sD} = (ac')^{sD} = d((b'd)^{sD})^3 b'ac' = d[((bd)^n)^{sD}]^2 b(ac)^{n-1},$$

as required. □

Jacobson's lemma states that $1 - ab \in \mathcal{R}^{-1}$ if and only if $1 - ba \in \mathcal{R}^{-1}$. In recent years, numerous mathematicians paid much attention to Jacobson's lemma for (generalized) Drazin inverse (see [2, 5, 11, 17, 18, 24]). Finding proper counterparts of Jacobson's lemma for (generalized) Drazin inverse under the condition (2), the interested readers should refer to [15, 21].

In [19, Lemma 3.3], Wang proved that if $a \in \mathcal{R}^{sD}$ with $\mathrm{i}_{sD}(a) = k$, then $1 - a \in \mathcal{R}^{sD}$ with $\mathrm{i}_{sD}(1 - a) = k$ and $(1 - a)^{sD} = \sum_{i=0}^{k-1} a^i(1 - aa^{sD})$. This result establishes a bridge from Cline's formula for s-Drazin inverse to Jacobson's Lemma for s-Drazin inverse.

Corollary 2.4. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$.*

(1) *If $1 - ac \in \mathcal{R}^{sD}$ with $\mathrm{i}_{sD}(1 - ac) = k$, then $1 - bd \in \mathcal{R}^{sD}$ and*

$$(1 - bd)^{sD} = \sum_{i=0}^{k} (bd)^i - b[\sum_{i=0}^{k} \sum_{j=0}^{k-1} (bd)^i (1 - ac)^j][1 - (1 - ac)(1 - ac)^{sD}]d.$$

(2) *If $1 - bd \in \mathcal{R}^{sD}$ with $\mathrm{i}_{sD}(1 - bd) = k$, then $1 - ac \in \mathcal{R}^{sD}$ and*

$$(1 - ac)^{sD} = \sum_{i=0}^{k+1} (ac)^i - d[\sum_{i=0}^{k+1} (ac)^i][\sum_{j=0}^{k-1} (1 - bd)^j]^2 [1 - (1 - bd)(1 - bd)^{sD}]bac.$$

Proof. (1) By [19, Lemma 3.3], $ac \in \mathcal{R}^{sD}$ and

$$(ac)^{sD} = [\sum_{j=0}^{k-1} (1 - ac)^j][1 - (1 - ac)(1 - ac)^{sD}].$$

Applying Theorem 2.1, we can get $bd \in \mathcal{R}^{sD}$ and $(bd)^{sD} = b((ac)^{sD})^2 d$ and $\mathrm{i}_{sD}(bd) \leq k + 1$. According to [19, Lemma 3.3] again, it follows that $1 - bd \in \mathcal{R}^{sD}$ and

$$\begin{aligned}
(1 - bd)^{sD} &= [\sum_{i=0}^{k} (bd)^i][1 - (bd)(bd)^{sD}] = [\sum_{i=0}^{k} (bd)^i][1 - b(ac)^{sD} d] \\
&= [\sum_{i=0}^{k} (bd)^i][1 - b[\sum_{j=0}^{k-1} (1 - ac)^j][1 - (1 - ac)(1 - ac)^{sD}]d] \\
&= \sum_{i=0}^{k} (bd)^i - b[\sum_{i=0}^{k} \sum_{j=0}^{k-1} (bd)^i (1 - ac)^j][1 - (1 - ac)(1 - ac)^{sD}]d.
\end{aligned}$$

(2) The proof is similar to that of (1). □

Cline's formula for gs-Drazin inverse was established in [8, Theorem 4.14] under the assumption (1). Next, we generalize it to the case when (2) holds. The following auxiliary lemma is needed.

Lemma 2.5. ([22, Lemma 2.6]) *Suppose that* $a, b, c, d \in \mathcal{R}$ *satisfy* $acd = dbd$ *and* $dba = aca$. *Then* $ac \in \mathcal{R}^{qnil}$ *if and only if* $bd \in \mathcal{R}^{qnil}$.

Theorem 2.6. *Suppose that* $a, b, c, d \in \mathcal{R}$ *satisfy* $acd = dbd$ *and* $dba = aca$. *Then* $ac \in \mathcal{R}^{gsD}$ *if and only if* $bd \in \mathcal{R}^{gsD}$. *In this case, we have* $(ac)^{gsD} = d((bd)^{gsD})^3bac$ *and* $(bd)^{gsD} = b((ac)^{gsD})^2d$.

Proof. Suppose that $bd \in \mathcal{R}^{gsD}$ and let s be the generalized strong Drazin inverse of bd. Take $t = ds^3bac$. From [22, Theorem 2.7] it follows that, t is a generalized Drazin inverse of ac. Hence, in order to show that $ac \in \mathcal{R}^{gsD}$, it is sufficient to prove $(ac - act) \in \mathcal{R}^{qnil}$. Let $a' = (1 - ds^2b)a$ and $b' = (1 - s)b$. Then $b'd \in \mathcal{R}^{qnil}$. Moreover, a direct calculation shows that $a'cd = db'd$, $db'a' = a'ca'$ and $ac - act = a'c$. Then by Lemma 2.5, we deduce that $ac - act \in \mathcal{R}^{qnil}$, which yields that $ac \in \mathcal{R}^{gsD}$.

By similar arguments as above, one can show that if $ac \in \mathcal{R}^{gsD}$, then $bd \in \mathcal{R}^{gsD}$ and $(bd)^{gsD} = b((ac)^{gsD})^2d$. □

From the proof of [19, Lemma 3.3], we can also deduce that $a \in \mathcal{R}^{gsD}$ if and only if $1 - a \in \mathcal{R}^{gsD}$. Hence, as an immediate consequence of Theorem 2.6, we arrive at the following result.

Corollary 2.7. *Suppose that* $a, b, c, d \in \mathcal{R}$ *satisfy* $acd = dbd$ *and* $dba = aca$. *Then* $1 - ac \in \mathcal{R}^{gsD}$ *if and only if* $1 - bd \in \mathcal{R}^{gsD}$.

In the following, as an extension of [3, Theorem 4.1], we establish Cline's formula for Hirano inverse under the condition (2).

Theorem 2.8. *Suppose that* $a, b, c, d \in \mathcal{R}$ *satisfy* $acd = dbd$ *and* $dba = aca$. *Then* $ac \in \mathcal{R}^H$ *if and only if* $bd \in \mathcal{R}^H$. *In this case, we have* $(ac)^H = d((bd)^H)^3bac$, $(bd)^H = b((ac)^H)^2d$ *and* $|\mathrm{i}_H(ac) - \mathrm{i}_H(bd)| \leq 1$.

Proof. Suppose that there exist $s \in \mathcal{R}$ and a non-negative integer k such that

$$s(bd) = (bd)s, s(bd)s = s \text{ and } ((bd)^2 - sbd)^k = 0.$$

Take $t = ds^3bac$. By [22, Theorem 2.1], t is a Drazin inverse of ac, so we have $t(ac) = (ac)t$ and $t(ac)t = t$. Thus, to prove that $ac \in \mathcal{R}^H$, it is sufficient to show $((ac)^2 - act)^{k+1} = 0$.

Since $(bd - s)^k(bd)^k = 0$ and

$$\begin{aligned}
&((ac)^2 - act)^{k+1} \\
&= ((ac)^2 - acds^3bac)^{k+1} = ((ac)^2 - dbds^3bac)^{k+1} \\
&= ((ac)^2 - ds^2bac)^{k+1} = [(ac - ds^2b)ac]^{k+1} \\
&= (ac - ds^2b)[ac(ac - ds^2b)]^k(ac) \\
&= (ac - ds^2b)(dbac - dsb)^k(ac) \\
&= (ac - ds^2b)(dbac - dsb)^{k-1}d(bd - s)bac \\
&= (ac - ds^2b)(dbac - dsb)^{k-2}d(bd - s)bd(bd - s)bac \\
&= d(1 - s^2)(bd - s)^k(bd)^kbac,
\end{aligned}$$

we get $((ac)^2 - act)^{k+1} = 0$, as required.

Conversely, suppose that $ac \in \mathcal{R}^H$ and let u be the Hirano inverse of ac and m its index. Set $v = bu^2d$. By [22, Theorem 2.1] again, we know that v is a Drazin inverse of bd. Therefore it remains only to show $((bd)^2 - bdv)^{m+1} = 0$. Noting $(bd)^2 - bdv = bacd - bdbu^2d = bacd - bdbacu^3d = bacd - bacacu^3d = b(ac - u)d$ and $[b(ac - u)d]^{m+1} = b(ac - u)((ac)^2 - acu)^m d = 0$, we conclude that $(bd - bdv)^{m+1} = 0$. □

At last, Cline's formula for generalized Hirano inverse is established under the assumption (2), extending [1, Theorem 4.1].

Theorem 2.9. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $ac \in \mathcal{R}^{gH}$ if and only if $bd \in \mathcal{R}^{gH}$. In this case, we have $(ac)^{gH} = d((bd)^{gH})^3 bac$ and $(bd)^{gH} = b((ac)^{gH})^2 d$.*

Proof. Suppose that $bd \in \mathcal{R}^{gH}$ and let s be the generalized Hirano inverse of bd. Take $t = ds^3bac$. From [22, Theorem 2.7] it follows that, t is a generalized Drazin inverse of ac. Hence, in order to show that $ac \in \mathcal{R}^{gH}$, it is sufficient to prove $(ac)^2 - act \in \mathcal{R}^{qnil}$. Let $a' = d(1 - s^2)ba$ and $b' = bac - sb$. Then $b'd \in \mathcal{R}^{qnil}$. Moreover, a direct calculation shows that $a'cd = db'd$, $db'a' = a'ca'$ and $(ac)^2 - act = a'c$. Then by Lemma 2.5, we deduce that $(ac)^2 - act \in \mathcal{R}^{qnil}$, which yields that $ac \in \mathcal{R}^{gH}$.

By similar arguments as above, one can show that if $ac \in \mathcal{R}^{gH}$, then $bd \in \mathcal{R}^{gH}$ and $(bd)^{gH} = b((ac)^{gH})^2 d$. □

Remark that similar results of Corollaries 2.2 and 2.3 hold also for generalized strong Drazin inverse, Hirano inverse and generalized Hirano inverse. We conclude this note by a numerical example to demonstrate Theorems 2.1, 2.6, 2.8 and 2.9.

Example 2.10. *Consider the matrices $A, B, C, D \in M_6(\mathbb{C})$ as following:*

$$A = D = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -4 & 2 & -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

A direct calculation shows that $ACD = DBD$ and $DBA = ACA$. Moreover,

$$AC = \begin{pmatrix} 1 & 0 & 8 & 0 & 0 & 0 \\ 2 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \text{ and } BD = \begin{pmatrix} -2 & 1 & 0 & 1 & 1 & 2 \\ -6 & 3 & -1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & 6 \\ -4 & 2 & -1 & 4 & 4 & 8 \\ 2 & -1 & 0 & -3 & -3 & -6 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 & 0 & 8 \\ 2 & 0 & 16 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix}$ are idempotents, it is clear that AC is strong Drazin invertible (resp. generalized strongly Drazin invertible, Hirano invertible, generalized Hirano invertible), and $(AC)^\bullet = AC$, where $\bullet \in \{sD, gsD, H, gH\}$. Hence, by Theorem 2.1 (resp. Theorem 2.6, Theorem 2.8, Theorem 2.9), we obtain the exact value of $(BD)^\bullet$,

$$(BD)^\bullet = B((AC)^\bullet)^2 D = \begin{pmatrix} -2 & 1 & -1 & 1 & 1 & 2 \\ -6 & 3 & -3 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 6 \\ -4 & 2 & -2 & 4 & 4 & 8 \\ 2 & -1 & 1 & -3 & -3 & -6 \end{pmatrix}.$$

Acknowledgement The authors are grateful to the referees for their valuable comments and suggestions.

References

[1] M.S. Abdolyousefi, H.Y. Chen, Generalized Hirano inverses in rings, Comm. Algebra 47(7) (2019) 2967–2978.
[2] N. Castro-González, C. Mendes-Araújo, P. Patricio, Generalized inverses of a sum in rings, Bull. Aust. Math. Soc. 82(1) (2010) 156–164.
[3] H.Y. Chen, M. Sheibani, On Hirano inverses in rings, Turk. J. Math. 43(4) (2019) 2049–2057.
[4] R.E. Cline, An application of representation for the generalized inverse of a matrix, MRC Technical Report 592, 1965.
[5] D. Cvetković-Ilić, R.E. Harte, On Jacobson's lemma and Drazin invertibility, Appl. Math. Lett. 23(4) (2010) 417–420.
[6] G. Corach, B.P. Duggal, R.E. Harte, Extensions of Jacobson's lemma, Comm. Algebra 41(2) (2013) 520–531.
[7] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506–514.
[8] O. Gürgün, Properties of generalized strongly Drazin invertible elements in general rings, J. Algebra Appl. 16(8) (2017) 1750207 (13 pages).
[9] Y. Jiang, Y.X. Wen, Q.P. Zeng, Generalizations of Cline's formula for three generalized inverses, Rev. Un. Mat. Argentina, 58(1) (2017) 127-134.
[10] J.J. Koliha, P. Patrício, Elements of rings with equal spectral idempotents, J. Aust. Math. Soc. 72(1) (2002) 137–152.
[11] T.Y. Lam, P.P. Nielsen, Jacobson's lemma for Drazin inverses, in: Contemp. Math.: Ring theory and its applications 609 (2014) 185–196.
[12] H.F. Lian, Q.P. Zeng, An extension of Cline's formula for a generalized Drazin inverse, Turk. J. Math. 40(1) (2016) 161–165.
[13] Y.H. Liao, J.L. Chen, J. Cui, Cline's formula for the generalized Drazin inverse, Bull. Malays. Math. Sci. Soc. (2) 37(1) (2014) 37–42.
[14] D. Mosić, Reverse order laws for the generalized strong Drazin inverses, Appl. Math. Comput. 284 (2016) 37–46.
[15] D. Mosić, Extensions of Jacobson's lemma for Drazin inverses, Aequat. Math. 91(3) (2017) 419–428.
[16] P. Patrício, R.E. Hartwig, Some additive results on Drazin inverses, Appl. Math. Comput. 215(2) (2009) 530–538.
[17] P. Patrício, R.E. Hartwig, The link between regularity and strong-pi-regularity, J. Aust. Math. Soc. 89(1) (2010) 17–22.
[18] P. Patrício, A. Veloso da Costa, On the Drazin index of regular elements, Cent. Eur. J. Math. 7(2) (2009) 200–205.
[19] Z. Wang, A class of Drazin inverses in rings, Filomat 31(6) (2017) 1781–1789.
[20] K. Yan, X.C. Fang, Common properties of the operator products in local spectral theory, Acta Math. Sin. (Engl. Ser.) 31(11) (2015) 1715–1724.
[21] K. Yan, Q.P. Zeng, Y.C. Zhu, Generalized Jacobson's lemma for Drazin inverses and its applications, Linear Multilinear Algebra, DOI: 10.1080/03081087.2018.1498828.
[22] Q.P. Zeng, Z.Y. Wu, Y.X. Wen, New extensions of Cline's formula for generalized inverses, Filomat 31(7) (2017) 1973–1980.
[23] Q.P. Zeng, H.J. Zhong, New results on common properties of the products AC and BA, J. Math. Anal. Appl. 427(2) (2015) 830–840.
[24] G.F. Zhuang, J.L. Chen, J. Cui, Jacobson's lemma for the generalized Drazin inverse, Linear Algebra Appl. 436(3) (2012) 742–746.