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Planar Harmonic Mappings in a Family of Functions Convex in One Direction

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Abstract. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, and h and g be two analytic functions in $\mathbb{D}$. Suppose that $f = h + \overline{g}$ is a harmonic mapping in $\mathbb{D}$ with the usual normalization $h(0) = 0 = g(0)$ and $h'(0) = 1$. In this paper, we consider harmonic mappings f by restricting its analytic part to a family of functions convex in one direction and, in particular, starlike. Some sharp and optimal estimates for coefficient bounds, growth, covering and area bounds are investigated for the class of functions under consideration. Also, we obtain optimal radii of fully convexity, fully starlikeness, uniformly convexity, and uniformly starlikeness of functions belonging to those family.

1. Introduction

A continuous complex valued function $f = u + iv$ is said to be harmonic in a simply connected domain Ω in the complex plane $\mathbb{C}$ if both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are harmonic in Ω. Each such f can uniquely be represented as

$$f = h + \overline{g} \tag{1}$$

where h and g are analytic functions in Ω. Here, h and g are called the analytic and co-analytic part of f, respectively.

Let $\mathcal{H}$ denote the class of all complex valued harmonic mappings f of the form (1) on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{2}$$

Set $\mathcal{H}_0 = \{f \in \mathcal{H} : g'(0) = f_{\overline{z}}(0) = 0\}$. The Jacobian of f is denoted by J_f which is defined by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ and the second complex dilatation of f is defined by $w(z) = g'(z)/h'(z)$. A result of Lewy [17] states that a locally univalent harmonic mapping f is sense-preserving if the Jacobian is positive. The sense-preserving case implies that $|w(z)| < 1$ in Ω. In particular, if $f \in \mathcal{H}_0$, then it is clear that $w(0) = 0$.

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Further, let $\mathcal{S}_H$ denote the class of all univalent and sense-preserving harmonic mappings f in $\mathcal{H}$, and setting $\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : g'(0) = f_{\bar{z}}(0) = 0\}$. The family $\mathcal{S}_H^0$ is known to be compact and normal, whereas $\mathcal{S}_H$ is normal but not compact (see [8, 25]). For many interesting results and expositions on planar harmonic mappings, we refer the monograph of Duren [8], and also the expository articles [7, 25]. The class $\mathcal{S}_H$ for which $g(z) \equiv 0$ reduces to the class $\mathcal{S}$ of the well known normalized univalent analytic functions in $\mathbb{D}$. The geometric subclasses of $\mathcal{S}_H$ consisting of starlike, convex and close-to-convex functions in $\mathbb{D}$ are denoted by $\mathcal{S}_H^*$, $\mathcal{K}_H$ and $\mathcal{C}_H$, respectively, and also we denote by $\mathcal{S}_H^{*0}$, $\mathcal{K}_H^0$ and $\mathcal{C}_H^0$ the classes consisting of those functions f in $\mathcal{S}_H^*$, $\mathcal{K}_H$ and $\mathcal{C}_H$ respectively for which $g'(0) = 0$.

As an analogue of Bieberbach Conjecture proved by de Branges [6] for functions in the class $\mathcal{S}$, Clunie and Sheil-Small [5] proposed a conjecture for the functions in the class $\mathcal{S}_H^0$, as stated below is still open.

Conjecture. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$, where h and g are given by the series representation* (2) *with $b_1 = 0$. Then for each $n \geq 2$,*

$$|a_n| \leq \frac{(2n+1)(n+1)}{6}, \quad |b_n| \leq \frac{(2n-1)(n-1)}{6} \quad \textit{and} \quad ||a_n| - |b_n|| \leq n.$$

Equality holds in each case for the harmonic Koebe function $K(z) = H(z) + \overline{G(z)}$ in $\mathcal{S}_H^0$, where

$$H(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \quad \textit{and} \quad G(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}.$$

Here we see that the function $K(z)$ has the dilatation $w(z) = z$ and maps $\mathbb{D}$ onto the slit plane $\mathbb{C} \setminus \{u + iv : u \leq -1/6,\ v = 0\}$. Although, this conjecture remains an open problem for the full class $\mathcal{S}_H^0$, but the same has been studied for certain subclasses, such as, the family of typically real functions [5], for the family of functions that are convex in one direction [31], for the family of starlike functions [8, sec. 6.6], and for the family of close-to-convex functions [34]. For some recent development on this conjecture for the functions in the class $\mathcal{S}_H^0$, we refer to the article of Ponnusamy et al. [29]. However, the best known bound for $|a_2|$ is 20.5, see [2]. In order to prove the above conjecture, the authors of [27] suggested another conjecture. One of our objectives in this paper is also to deal with such coefficient problems for some specific class of functions that we are considering in this section.

The Class $\mathcal{G}(c)$

Umezawa [33] studied that, if the function h is locally univalent of the form (2) satisfying the relation

$$\alpha > \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{\alpha}{2\alpha - 3} \tag{3}$$

with α not less than 3/2, then h is analytic and univalent in $\mathbb{D}$. Moreover, h maps every subdisk $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r \leq 1\}$ into a domain which is convex in one direction. Several special cases of inequality (3) have been studied for several purposes by allowing different values of $\alpha \geq 3/2$. In particular, when α approaches 3/2 in (3), it generates a subclass of $\mathcal{S}$ denoted by $\mathcal{G}$, in which the function h satisfy the condition

$$\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) < \frac{3}{2}, \ z \in \mathbb{D}. \tag{4}$$

Furthermore, a locally univalent analytic function h is said to belong to $\mathcal{G}(c)$ for some $c > 0$, if it satisfies the condition

$$\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) < 1 + \frac{c}{2}, \ z \in \mathbb{D}. \tag{5}$$

The class $\mathcal{G}(c)$ was introduced and studied by Obradović *et al.* in [21]. Clearly, $\mathcal{G}(1) \equiv \mathcal{G}$ which was introduced by Ozaki [22] and proved that functions in $\mathcal{G}$ are univalent in $\mathbb{D}$. Later on, Singh and Singh [32] proved that functions in $\mathcal{G}$ are starlike in $\mathbb{D}$. By Umezawa [33], each function in $\mathcal{G}$ maps every $\mathbb{D}_r$ onto a domain which is convex in one direction. For more informations about the class $\mathcal{G}$ in different contexts we refer [18, 24, 26, 30].

The analytic part of harmonic mappings plays a vital role in shaping their geometric properties. For instance, if $f = h + \overline{g}$ is sense preserving harmonic mapping and h is convex univalent, then $f \in \mathcal{S}_H$ and maps $\mathbb{D}$ onto a close-to-convex domain (see [5, Theorem 5.17, p.20]). It is pertinent that for a fixed analytic function h, an interesting problem arises to describe about all functions g such that $f = h + \overline{g} \in \mathcal{H}$. Not much known on the geometric properties of such planner harmonic mappings. Very recently, by restricting the analytic part h as a member of univalent starlike functions, convex functions, functions of bounded boundary rotation and functions convex in one direction, few authors studied some geometric properties of certain subfamily of $\mathcal{S}_H$ (see [9, 13–16, 28] and also [1] for a variety of such results).

Motivated by all these studies, we introduce here the class of function $\mathcal{F}^{\alpha}_{G(c)}$ as stated below and obtain the coefficient estimates of the co-analytic part of $f \in \mathcal{F}^{\alpha}_{G(c)}$ in Section 2. Structure of rest of the paper are as follows. Section 3 is devoted to the results about the radii of fully convexity, fully starlikeness, uniformly convexity and uniformly starlikeness for the class of functions under consideration. Furthermore, the growth theorem, covering theorem and area theorem are obtained in the last section.

The Class $\mathcal{F}^{\alpha}_{G(c)}$

Let $0 \le \alpha < 1$. For an analytic function w in $\mathbb{D}$ with $|w(z)| < 1$, we denote by $\mathcal{F}^{\alpha}_{G(c)}$ the set of all functions $f = h + \overline{g} \in \mathcal{S}_H$, for which $h \in \mathcal{G}(c)$ and $g'(0) = b_1$, $|b_1| = \alpha$, satisfying

$$g'(z) = w(z)h'(z) \quad \text{for all } z \in \mathbb{D}. \tag{6}$$

Set $\mathcal{F}^{0}_{G(c)} = \{f = h + \overline{g} \in \mathcal{F}^{\alpha}_{G(c)} : g'(0) = 0\}$. In particular, for $c = 1$ we denote the class of functions $\mathcal{F}^{\alpha}_{G(c)}$ and $\mathcal{F}^{0}_{G(c)}$ by $\mathcal{F}^{\alpha}_{G}$ and $\mathcal{F}^{0}_{G}$, respectively.

2. Coefficient Inequality

In this section, we find coefficient bounds for the co-analytic part of the function $f = h + \overline{g}$ belonging to the class $\mathcal{F}^{\alpha}_{G(c)}$. The following coefficient bound obtained by Obradović *et al.* for the functions $h \in \mathcal{G}(c)$ is useful to obtain our main result in this section.

Lemma 2.1. *[21] Let $h \in \mathcal{G}(c)$ for some $0 < c \le 1$ and $h(z)$ be of the form* (2). *Then*

$$|a_n| \le \frac{c}{n(n-1)} \quad \text{for all } \ n \ge 2.$$

Equality is attained for the function $f_n(z)$ such that $f_n'(z) = \left(1 - z^{n-1}\right)^{c/(n-1)}$, $n \ge 2$.

Now we state our main result of this section.

Theorem 2.2. *Let $f = h + \overline{g} \in \mathcal{F}^{\alpha}_{G(c)}$ for some $0 < c \le 1$, where h and g are given by* (2). *Then*

$$|b_n| \le \frac{c\alpha}{n(n-1)} + \frac{\left(1-\alpha^2\right)(1 + c\gamma + c\Psi(n-1))}{n}, \tag{7}$$

where γ is the well known Euler's constant, *and Ψ is the well known* digamma function *which is the logarithmic derivative of the classical gamma function given by $\Psi(z) = \Gamma'(z)/\Gamma(z)$.*

Moreover, we have the sharp estimate for the second coefficient b_2 as

$$|b_2| \le \frac{1 + c\alpha - \alpha^2}{2}, \tag{8}$$

and the extremal function is

$$f(z) = z + c(z + (1-z)\log(1-z)) + \overline{\int_0^z \frac{(\zeta+\alpha)(1-c\log(1-\zeta))}{(1+\alpha\zeta)}d\zeta}. \tag{9}$$

In particular when $\alpha = 0$ and $c = 1$, we have respectively the following results for the classes $\mathcal{F}^0_{G(c)}$, $\mathcal{F}^\alpha_G$ and $\mathcal{F}^0_G$ as follows.

Corollary 2.3. *Let $f = h + \overline{g} \in \mathcal{F}^0_{G(c)}$ for some $0 < c \leq 1$, where h and g are given by* (2). *Then, we have*

$$|b_n| \leq \frac{1 + c\gamma + c\Psi(n-1)}{n}.$$

Corollary 2.4. *Let $f = h + \overline{g} \in \mathcal{F}^\alpha_G$, where h and g are given by* (2). *Then*

$$|b_n| \leq \frac{\alpha}{n(n-1)} + \frac{(1-\alpha^2)(1+\gamma+\Psi(n-1))}{n}. \tag{10}$$

Moreover, we have the sharp estimate for the second coefficient b_2 as

$$|b_2| \leq \frac{1+\alpha-\alpha^2}{2}, \tag{11}$$

and the extremal function is

$$f(z) = z - \frac{z^2}{2} + \overline{\int_0^z \frac{(\zeta-\alpha)(1-\zeta)}{(1-\alpha\zeta)}d\zeta},$$

which in particular gives

$$f(z) = \begin{cases} z - \frac{z^2}{2} + \overline{(-1-\frac{1}{\alpha}+\frac{1}{\alpha^2})z + \frac{1}{2\alpha}z^2 + (1-\frac{1}{\alpha}-\frac{1}{\alpha^2}+\frac{1}{\alpha^3})\ln(1-\alpha z)}, & \alpha \neq 0, \\ z - \frac{z^2}{2} + \overline{\frac{z^2}{2} - \frac{z^3}{3}}, & \alpha = 0. \end{cases} \tag{12}$$

Corollary 2.5. *Let $f = h + \overline{g} \in \mathcal{F}^0_G$, where h and g are given by* (2). *Then, we have*

$$|b_n| \leq \frac{1+\gamma+\Psi(n-1)}{n}.$$

Proof of Theorem 2.2.

Let $f \in \mathcal{H}$. Then f is sense preserving locally univalent harmonic mapping and therefore its Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ is positive. Hence, its dilatation $w(z) = g'(z)/h'(z)$ is analytic in $\mathbb{D}$ and $|w(z)| < 1$. The function w has the series expansion of the form

$$w(z) = c_0 + c_1 z + c_2 z^2 + \cdots \tag{13}$$

for $z \in \mathbb{D}$, $c_i \in \mathbb{C}$, $i = 1, 2, \ldots$. Since $w(z)h'(z) = g'(z)$, by comparing the constant coefficients in their series expansions, obtained from (2) and (13), we have $c_0 = b_1$ with $|b_1| = \alpha$. Therefore, for $z \in \mathbb{D}$, we have

$$\frac{|r-\alpha|}{1-\alpha r} \leq |w(z)| \leq \frac{r+\alpha}{1+\alpha r}, \tag{14}$$

and

$$|c_n| \leq 1 - |c_0|^2, \quad |w'(z)| \leq \frac{1-|w(z)|^2}{1-|z|^2} \tag{15}$$

for $n = 1, 2, \ldots$, see e.g. Nehari [19, p. 172] and Avkhadiev and Wirths [3, p. 30 & p. 53].

Again in the series expansions of the relation $g'(z) = w(z)h'(z)$, equating the coefficient of z^{n-1} on both sides, we obtain

$$nb_n = a_1c_{n-1} + 2a_2c_{n-2} + 3a_3c_{n-3} + \cdots + (n-1)a_{n-1}c_1 + na_nc_0, \tag{16}$$

with $a_1 = 1$, where $c_n \in \mathbb{C},\ n = 0, 1, 2, \ldots$. In particular, for $n = 2$, we have

$$2b_2 = a_1c_1 + 2a_2c_0.$$

By the help of (15) and Lemma 2.1, we estimate

$$2|b_2| \leq |a_1|\,|c_1| + 2|a_2|\,|c_0| \leq (1 - |c_0|^2) + c|c_0| \leq 1 - \alpha^2 + c\alpha,$$

which yields the inequality (8). The inequality (11) clearly follows when $c = 1$.

Now, we show that the inequality given by (8) is sharp. For $0 \leq \alpha < 1$, consider the function $f(z) = h(z) + \overline{g(z)}$ with

$$h(z) = z + \sum_{n=2}^{\infty} \frac{c}{n(n-1)} z^n \in \mathcal{G}(c),$$

and the dilatation $w(z)$ as

$$w(z) = \frac{z+\alpha}{1+\alpha z}, \quad z \in \mathbb{D}.$$

Using the relation $g'(z) = w(z)h'(z)$, we obtain

$$\begin{aligned} g'(z) &= \frac{(z+\alpha)(1 + cz + \frac{c}{2}z^2 + \frac{c}{3}z^3 + \cdots)}{1+\alpha z} \\ &= \frac{\alpha + (1 + c\alpha)z + (c + \frac{c\alpha}{2})z^2 + (\frac{c}{2} + \frac{c\alpha}{3})z^3 + \cdots}{1+\alpha z} \\ &= \alpha + (1 + c\alpha - \alpha^2)z + (c - \alpha + \frac{c\alpha}{2} - c\alpha^2 + \alpha^3)z^3 \cdots, \quad z \in \mathbb{D}. \end{aligned}$$

This implies that the estimate (8) is sharp. Since $g(0) = 0$, by integration, we uniquely deduce for $z \in \mathbb{D}$ that

$$g(z) = \int_0^z \frac{(\zeta + \alpha)(1 - c\log(1-\zeta))}{(1+\alpha\zeta)} d\zeta,$$

which yields the extremal function (9).

Since the inequality (15) holds for all $n \geq 1$, from (16), it immediately follows that

$$n|b_n| \leq n|a_n|\,|c_0| + (1 - |c_0|^2) \sum_{k=1}^{n-1} k|a_k|,$$

with $a_1 = 1$. Since $|c_0| = |b_1| = \alpha$, Lemma 2.1 gives us in concluding the desired inequality

$$\begin{aligned} |b_n| &\leq \frac{c\alpha}{n(n-1)} + \frac{1-\alpha^2}{n} + \frac{1-\alpha^2}{n} \sum_{k=2}^{n-1} k|a_k| \\ &\leq \frac{c\alpha}{n(n-1)} + \frac{1-\alpha^2}{n} + \frac{1-\alpha^2}{n} \sum_{k=2}^{n-1} \frac{c}{k-1} \\ &= \frac{c\alpha}{n(n-1)} + \frac{1-\alpha^2}{n} + \frac{1-\alpha^2}{n} c(\gamma + \Psi(n-1)), \end{aligned}$$

where γ is the *Euler constant* and Ψ is the *digamma function* defined in the statement of our theorem. $\square$

3. Radius Properties

For the radii problems in harmonic mappings, one of the basic papers which we refer is [12]. In this section we study the radii of fully convexity, fully starlikeness, uniformly convexity and uniformly starlikeness for functions $f = h + \overline{g}$ belonging to the class $\mathcal{F}^{\alpha}_{G(c)}$. In order to obtain such type of results under consideration, we collect here some basic definitions.

Convexity and starlikeness are not the hereditary properties for the univalent harmonic mappings, i.e., if a harmonic mapping maps $\mathbb{D}$ onto a convex (or a starlike) domain, then it does not always map each concentric subdisk onto a convex (or a starlike) domain (see [4, 8]). Chuaqui, Duren and Osgood [4] introduced the notion of fully starlike and fully convex harmonic mappings that do inherits the properties of starlikeness and convexity respectively. Later on this has been generalized in terms of fully convex of order β and fully starlike of order β, $\beta \in [0, 1)$, by Jahangiri in [10] and [11], respectively. The definitions of fully convexity and fully starlikeness are given below.

Definition 3.1. *A harmonic mapping $f \in \mathcal{H}$ is said to be fully convex of order β $(0 \le \beta < 1)$ if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve satisfying*

$$\frac{\partial}{\partial\theta}\left\{\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right\} > \beta, \quad 0 \le \theta \le 2\pi. \tag{17}$$

If $\beta = 0$, then the harmonic mapping f satisfying (17) *is said to be fully convex (univalent) in $\mathbb{D}$.*

Definition 3.2. *A harmonic mapping $f \in \mathcal{H}$ is said to be fully starlike of order β $(0 \le \beta < 1)$ if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve satisfying*

$$\frac{\partial}{\partial\theta}\left\{\arg\left(f(re^{i\theta})\right)\right\} > \beta, \quad 0 \le \theta \le 2\pi. \tag{18}$$

If $\beta = 0$, then the harmonic mapping f satisfying (18) *is said to be fully starlike (univalent) in $\mathbb{D}$.*

It is very clear that fully convex mappings are fully starlike but not the converse as the function $f(z) = z + (1/n)\overline{z}^n$ $(n \ge 2)$ shows. Radó-kneser Choquet Theorem [8, Sec.3.1] ensures that fully convex harmonic mappings are necessarily univalent in $\mathbb{D}$. However, a fully starlike mappings need not be univalent in $\mathbb{D}$ (see [4, p.14]).

Other family of functions we consider in this paper are the family of uniformly convex and uniformly starlike functions. To understand their definitions, we need the concept of convex and starlike arcs. An arc is said to be starlike with respect to a point if the line segment joining the point with any other point on the arc does not intersect the arc at any other point.

Definition 3.3. *A locally univalent function $f = h + \overline{g}$ is said to be uniformly convex in $\mathbb{D}$, if f is fully convex in $\mathbb{D}$, and maps every circular arc γ_ζ contained in $\mathbb{D}$ with center ζ also in $\mathbb{D}$, to a convex arc $f(\gamma_\zeta)$.*

Definition 3.4. *A locally univalent function $f = h + \overline{g}$ is said to be uniformly starlike in $\mathbb{D}$, if f is fully starlike in $\mathbb{D}$, and maps every circular arc γ_ζ contained in $\mathbb{D}$ with center ζ also in $\mathbb{D}$, to the arc $f(\gamma_\zeta)$ which is starlike with respect to $f(\zeta)$.*

Clearly the class of uniformly starlike functions are contained in the class of fully starlike functions.

Definition 3.5. *Let $F = \mathcal{S}_H$ or any one of the geometric subclasses of $\mathcal{H}$ under consideration. For $r \in (0, 1)$, we say that $f = h + \overline{g} \in F$ in $|z| < r$ if $f_r(z) = \frac{1}{r} f(rz) \in F$ in $|z| < 1$ in the usual sense.*

3.1. Radii of fully convexity and fully starlikeness

Recall the sufficient conditions obtained by Jahangiri for functions $f \in \mathcal{H}$ to be fully harmonic convex of order β and fully harmonic starlike of order β, respectively, as stated below.

Lemma 3.6. *[10] Let $f = h + \overline{g} \in \mathcal{H}$, where h and g are given by* (2). *Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta}|b_n| \le 1,$$

and $\beta \in [0,1)$. Then, f is a fully harmonic convex function of order β.

Lemma 3.7. *[11] Let $f = h + \overline{g} \in \mathcal{H}$, where h and g are given by* (2). *Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n+\beta}{1-\beta}|b_n| \le 1,$$

and $\beta \in [0,1)$. Then, f is a fully harmonic starlike function of order β.

For the sake of convenience, we collect some elementary identities in the form of a proposition that are useful in the proofs of our main results in this section.

Proposition 3.8. *We have*

(a) $$\sum_{n=2}^{\infty} r^{n-1} = \frac{r}{1-r}, \quad \sum_{n=2}^{\infty} nr^{n-1} = \frac{2r-r^2}{(1-r)^2},$$

(b) $$\sum_{n=2}^{\infty} \frac{r^{n-1}}{n} = \frac{-r-\log(1-r)}{r}, \quad \sum_{n=2}^{\infty} \frac{nr^{n-1}}{n-1} = \frac{r-(1-r)\log(1-r)}{1-r},$$

(c) $$\sum_{n=2}^{\infty} \frac{r^{n-1}}{n-1} = -\log(1-r), \quad \sum_{n=2}^{\infty} \frac{r^{n}}{n-1} = -r\log(1-r), \quad \sum_{n=2}^{\infty} \frac{r^{n}}{n} = -r-\log(1-r),$$

(d) $$\sum_{n=2}^{\infty} n\Psi(n-1)r^{n-1} = \frac{r(\gamma(-2+r)+r+(-2+r)\log(1-r))}{(1-r)^2},$$

(e) $$\sum_{n=2}^{\infty} \Psi(n-1)r^{n-1} = -\frac{r(\gamma+\log(1-r))}{1-r},$$

(f) $$\sum_{n=2}^{\infty} \frac{r^{n-1}}{n(n-1)} = \frac{r+(1-r)\log(1-r)}{r}, \quad \sum_{n=2}^{\infty} \frac{r^{n}}{n(n-1)} = r+(1-r)\log(1-r),$$

(g) $$\sum_{n=2}^{\infty} \frac{\Psi(n-1)}{n} r^{n-1} = \frac{(-1+\gamma)r+(\gamma+r)\log(1-r)}{r},$$

where γ is the Euler constant and Ψ is the digamma function defined in Theorem 2.2.

Theorem 3.9. *Let $f = h + \overline{g}$, where h and g are given by* (2). *Then for all $f \in \mathcal{F}^0_{G(c)}$, we have*

(a) *the radius of fully convexity of order β is $r_c(\beta)$, where $r_c(\beta)$ is the unique real root of the equation*

$$1-\beta-(4+c-\beta)r+2r^2+c\left\{1-\beta(1-3r+2r^2)\right\}\log(1-r) = 0 \tag{19}$$

lying in the interval $(0,1)$.

(b) *the radius of fully convexity is r_c, where r_c is the unique real root of the equation*

$$1-(4+c)r+2r^2+c\log(1-r) = 0 \tag{20}$$

lying in the interval $(0,1)$.

Proof. Let $f = h + \overline{g} \in \mathcal{F}^0_{G(c)}$, where h and g are given by (2). First we observe that $b_1 = 0$, and the coefficients of the series satisfy the conditions of Lemma 2.1 and Corollary 2.3. These conditions imply that the series (2) are convergent in $\mathbb{D}$, and hence, h and g are analytic in $\mathbb{D}$. For $0 < r < 1$, consider the function

$$f_r(z) = \frac{1}{r}f(rz) = \frac{1}{r}h(rz) + \frac{1}{r}\overline{g(rz)},$$

so that

$$f_r(z) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=1}^{\infty} b_n r^{n-1} z^n}, \quad z \in \mathbb{D}. \tag{21}$$

Thus, by Definition 3.5, it is now needed to show that the function $f_r(z)$ defined by (21) is fully convex of order β. Using the coefficient estimates from Lemma 2.1 and Corollary 2.3, now estimate

$$\begin{aligned} S_1 &= \sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta}|a_n|r^{n-1} + \sum_{n=2}^{\infty} \frac{n(n+\beta)}{1-\beta}|b_n|r^{n-1} \\ &\leq \sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta}\left(\frac{c}{n(n-1)}\right)r^{n-1} + \sum_{n=2}^{\infty} \frac{n(n+\beta)}{1-\beta}\left(\frac{1+c\gamma+c\Psi(n-1)}{n}\right)r^{n-1} \\ &= \frac{1}{1-\beta}\left[c\sum_{n=2}^{\infty}\frac{nr^{n-1}}{n-1} - c\beta\sum_{n=2}^{\infty}\frac{r^{n-1}}{n-1} + (1+c\gamma)\sum_{n=2}^{\infty} nr^{n-1} + c\sum_{n=2}^{\infty} n\Psi(n-1)r^{n-1}\right. \\ &\qquad \left. + \beta(1+c\gamma)\sum_{n=2}^{\infty} r^{n-1} + c\beta\sum_{n=2}^{\infty}\Psi(n-1)r^{n-1}\right]. \end{aligned}$$

In view of Lemma 3.6, it suffices to show that $S_1 \leq 1$, or equivalently r satisfies the inequality

$$\begin{aligned} &\frac{c(r-(1-r)\log(1-r))}{1-r} + c\beta\log(1-r) + \frac{(1+c\gamma)(2r-r^2)}{(1-r)^2} \\ &+ \frac{cr(\gamma(-2+r)+r+(-2+r)\log(1-r))}{(1-r)^2} + \frac{\beta(1+c\gamma)r}{1-r} - \frac{c\beta r(\gamma+\log(1-r))}{1-r} \leq 1-\beta, \end{aligned}$$

where we have used Proposition 3.8 (a)-(e) to get the left side expression of the last inequality. A simple calculation brings this inequality to the form

$$1-\beta-(4+c-\beta)r+2r^2+c\left\{1-\beta(1-3r+2r^2)\right\}\log(1-r) \geq 0.$$

Thus, by Lemma 3.6, f is fully convex of order β for $|z| = r \leq r_c(\beta)$, where $r_c(\beta)$ is the unique real root of (19) lying in the interval $(0, 1)$.

Note that for a fixed $c \in (0, 1]$ the roots of the equation given by (19) lying in $(0, 1)$ are clearly decreasing as a function of $\beta \in [0, 1)$. Consequently, $r_c(\beta) \leq r_c(0)$. Therefore, taking $\beta = 0$ in (19), we obtain (20). Then Lemma 3.6 gives that the harmonic function f is convex and univalent in $|z| = r \leq r_c$, where r_c is the unique real root of (20) lying in the interval $(0, 1)$. This completes the proof. □

Theorem 3.10. *Let $f = h + \overline{g}$, where h and g are given by* (2). *Then for all $f \in \mathcal{F}^0_{G(c)}$, we have*

(a) *the radius of fully starlikeness of order β is $r_s(\beta)$, where $r_s(\beta)$ is the unique real root of the equation*

$$1+2c\beta-2(1+c\beta)r+\{c-\beta(1-1/r)(1+c-2cr)\}\log(1-r) = 0 \tag{22}$$

lying in the interval $(0, 1)$.

(b) *the radius of fully starlikeness is* r_u*, where* r_u *is the unique real root of the equation*

$$1 - 2r + c\log(1-r) = 0 \tag{23}$$

lying in the interval $(0, 1)$.

Proof. Under the same hypothesis of Theorem 3.9, and by Definition 3.5, it is now needed to show that the function $f_r(z)$ defined by (21) is fully starlike of order β. Using the coefficient estimates stated in Lemma 2.1 and Corollary 2.3, we estimate

$$\begin{aligned}
S_2 &= \sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta}|a_n|r^{n-1} + \sum_{n=2}^{\infty} \frac{n+\beta}{1-\beta}|b_n|r^{n-1} \\
&\le \sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta}\left(\frac{c}{n(n-1)}\right)r^{n-1} + \sum_{n=2}^{\infty} \frac{n+\beta}{1-\beta}\left(\frac{1+c\gamma+c\Psi(n-1)}{n}\right)r^{n-1} \\
&= \frac{1}{1-\beta}\left[c\sum_{n=2}^{\infty}\frac{r^{n-1}}{n-1} - c\beta\sum_{n=2}^{\infty}\frac{r^{n-1}}{n(n-1)} + (1+c\gamma)\sum_{n=2}^{\infty} r^{n-1} + c\sum_{n=2}^{\infty}\Psi(n-1)r^{n-1}\right. \\
&\qquad \left. +\beta(1+c\gamma)\sum_{n=2}^{\infty}\frac{r^{n-1}}{n} + c\beta\sum_{n=2}^{\infty}\frac{\Psi(n-1)}{n}r^{n-1}\right].
\end{aligned}$$

In view of Lemma 3.7, it suffices to show that $S_2 \le 1$, or equivalently, r satisfies the inequality

$$\begin{aligned}
&-c\log(1-r) - \frac{c\beta(r+(1-r)\log(1-r))}{r} + \frac{(1+c\gamma)r}{1-r} - \frac{cr(\gamma+\log(1-r))}{1-r} \\
&+ \frac{\beta(1+c\gamma)(-r-\log(1-r))}{r} + \frac{c\beta((-1+\gamma)r+(\gamma+r)\log(1-r))}{r} \le 1-\beta,
\end{aligned}$$

where we have used Proposition 3.8 (a)-(c) and (e)-(g) to obtain the left side expression of the last inequality. On simplification, the last inequality reduces to

$$(1+2c\beta)r - 2(1+c\beta)r^2 + \{cr + \beta(1-r)(1+c-2cr)\}\log(1-r) \ge 0.$$

Thus, by Lemma 3.7, f is fully starlike of order β for $|z| = r \le r_s(\beta)$, where $r_s(\beta)$ is the unique real root of (22) lying in the interval $(0, 1)$.

Note that for a fixed $c \in (0, 1]$ the roots of the equation given by (22) lying in $(0, 1)$ are clearly decreasing as a function of $\beta \in [0, 1)$. Consequently, $r_s(\beta) \le r_s(0)$. Therefore, taking $\beta = 0$ in (22), we obtain (23). Then Lemma 3.7 yields that the harmonic function f is starlike and univalent in $|z| = r \le r_u$, where r_u is the unique real root of (23) lying in the interval $(0, 1)$. This completes the proof. □

In a similar manner, we can show the following two results which find radii of fully convexity and starlikeness of functions belonging to the family $\mathcal{F}^{\alpha}_{G(c)}$.

Theorem 3.11. *Let* $f = h + \overline{g}$*, where* h *and* g *are given by* (2). *Then for all* $f \in \mathcal{F}^{\alpha}_{G(c)}$*, we have*

(a) *the radius of fully convexity of order* β *is* $r_c(\beta, \alpha)$*, where* $r_c(\beta, \alpha)$ *is the unique real root of the equation*

$$\begin{aligned}
&(1-\beta)(1-\alpha) - \Big(4 + (1+\alpha)c - (1+\alpha)2\alpha - (1+2\alpha+\alpha^2)\beta\Big)r \\
&+ \Big(2 - (1-c)\alpha - (1-c)\alpha^2 - (1+\alpha)\alpha\beta\Big)r^2 \\
&+ c\Big\{(1-r)^2(1+\alpha-\beta+\alpha\beta) - (1-\alpha^2)(-2+r-\beta+\beta r)r\Big\}\log(1-r) = 0
\end{aligned}$$

lying in the interval $(0, 1)$.

(b) *the radius of fully convexity is $r_c(\alpha)$, where $r_c(\alpha)$ is the unique real root of the equation*

$$\begin{aligned}&1-\alpha-(4+(1+\alpha)c-(1+\alpha)2\alpha)\,r+\left(2-(1-c)\alpha-(1-c)\alpha^2\right)r^2\\&\quad+c\left\{(1-r)^2(1+\alpha)-(1-\alpha^2)(-2+r)r\right\}\log(1-r)=0\end{aligned}$$

lying in the interval $(0,1)$.

Theorem 3.12. *Let $f=h+\overline{g}$, where h and g are given by* (2). *Then for all $f\in\mathcal{F}^{\alpha}_{G(c)}$, we have*

(a) *the radius of fully starlikeness of order β is $r_s(\beta,\alpha)$, where $r_s(\beta,\alpha)$ is the unique real root of the equation*

$$\begin{aligned}&\left(1+2c\beta-(1+\beta+c\beta)\alpha-\beta(1+c)\alpha^2\right)r\\&\quad-\left(2+2c\beta-(1+\beta+c\beta+\alpha)\alpha-\beta(1+c)\alpha^2\right)r^2+\{(1-r)[cr(1+\alpha)\\&\quad+\beta(1-\alpha)(1+c+\alpha-2cr-\alpha cr)]+(1-\alpha^2)cr^2\Big\}\log(1-r)=0\end{aligned}$$

lying in the interval $(0,1)$.

(b) *the radius of fully starlikeness is $r_u(\alpha)$, where $r_u(\alpha)$ is the unique real root of the equation*

$$1-\alpha-(2-\alpha-\alpha^2)r+c(1+\alpha)(1-\alpha r)\log(1-r)=0$$

lying in the interval $(0,1)$.

3.2. Radii of uniformly starlikeness and uniformly convexity

Ponnusamy *et al.* obtained the following useful sufficient conditions for functions $f\in\mathcal{H}$ to be uniformly starlike and uniformly convex, respectively.

Lemma 3.13. *[23] Let $f=h+\overline{g}\in\mathcal{H}$, where h and g are given by* (2), *and satisfy the condition*

$$\sum_{n=2}^{\infty}n|a_n|+\sum_{n=1}^{\infty}n|b_n|\le\frac{1}{2}.$$

Then f is a uniformly starlike function.

Lemma 3.14. *[23] Let $f=h+\overline{g}\in\mathcal{H}$, where h and g are given by* (2), *and satisfy the condition*

$$\sum_{n=2}^{\infty}n(2n-1)|a_n|+\sum_{n=1}^{\infty}n(2n-1)|b_n|\le 1.$$

Then f is a uniformly convex function.

Theorem 3.15. *Let $f=h+\overline{g}$, where h and g are given by* (2). *Then for all $f\in\mathcal{F}^{0}_{G(c)}$, the radius of uniformly starlikeness is r_{us}, where r_{us} is the unique positive root of the equation*

$$1-3r+2c\log(1-r)=0 \tag{24}$$

lying in the interval $(0,1)$.

Proof. Under the same hypothesis of Theorem 3.9, and by Definition 3.5, it is now needed to show that the function $f_r(z)$ defined by (21) is uniformly starlike. Using the coefficient estimates from Lemma 2.1 and Corollary 2.3, we estimate

$$\begin{aligned} S_3 &= \sum_{n=2}^{\infty} n|a_n|r^{n-1} + \sum_{n=2}^{\infty} n|b_n|r^{n-1} \\ &\le \sum_{n=2}^{\infty} \frac{c}{n-1} r^{n-1} + \sum_{n=2}^{\infty}(1 + c\gamma + c\Psi(n-1))r^{n-1} \\ &= c\sum_{n=2}^{\infty} \frac{r^{n-1}}{n-1} + (1 + c\gamma)\sum_{n=2}^{\infty} r^{n-1} + c\sum_{n=2}^{\infty} \Psi(n-1)r^{n-1}. \end{aligned}$$

In view of Lemma 3.13, it suffices to show that $S_3 \le 1/2$, or equivalently, r satisfies the inequality

$$-c\log(1-r) + \frac{(1+c\gamma)r}{1-r} - \frac{cr(\gamma + \log(1-r))}{1-r} \le 1/2,$$

where we have used Proposition 3.8 to obtain the left side expression of the last inequality. A simple calculation brings the last inequality to

$$1 - 3r + 2c\log(1-r) \ge 0.$$

Thus, by Lemma 3.13, f is uniformly starlike for $|z| = r \le r_{us}$, where r_{us} is the unique real root of (24) lying in the interval $(0, 1)$. Clearly, we can check that the roots of (24) are monotonically decreasing in $(0, 1)$, and hence it has exactly one root in $(0, 1)$. Therefore, we conclude that f is uniformly starlike in $|z| < r_{us}$. □

Theorem 3.16. *Let $f = h + \overline{g}$, where h and g are given by* (2)*. Then for all $f \in \mathcal{F}^0_{G(c)}$, the radius of uniformly convexity is r_{uc}, where r_{uc} is the unique positive root of the equation*

$$1 - (5 + 2c)r + 2r^2 + c(1 + r)\log(1-r) = 0 \tag{25}$$

lying in the interval $(0, 1)$.

Proof. Under the same hypothesis of Theorem 3.9, and by Definition 3.5, it is now needed to show that the function $f_r(z)$ defined by (21) is uniformly convex. Using the coefficient estimates from Lemma 2.1 and Corollary 2.3, we estimate

$$\begin{aligned} S_4 &= \sum_{n=2}^{\infty} n(2n-1)|a_n|r^{n-1} + \sum_{n=2}^{\infty} n(2n-1)|b_n|r^{n-1} \\ &\le \sum_{n=2}^{\infty} \frac{c(2n-1)}{n-1} r^{n-1} + \sum_{n=2}^{\infty}(2n-1)(1 + c\gamma + c\Psi(n-1))r^{n-1} \\ &= 2c\sum_{n=2}^{\infty} \frac{nr^{n-1}}{n-1} - c\sum_{n=2}^{\infty} \frac{r^{n-1}}{n-1} + 2(1+c\gamma)\sum_{n=2}^{\infty} nr^{n-1} + 2c\sum_{n=2}^{\infty} n\Psi(n-1)r^{n-1} \\ &\qquad - (1+c\gamma)\sum_{n=2}^{\infty} r^{n-1} - c\sum_{n=2}^{\infty} \Psi(n-1)r^{n-1}. \end{aligned}$$

In view of Lemma 3.14, it suffices to show that $S_4 \le 1$, or equivalently, r satisfies the inequality

$$\begin{aligned} &\frac{2c(r - (1-r)\log(1-r))}{1-r} + c\log(1-r) + \frac{2(1+c\gamma)(2r - r^2)}{(1-r)^2} \\ &+ \frac{2cr(\gamma(-2+r) + r + (-2+r)\log(1-r))}{(1-r)^2} - \frac{(1+c\gamma)r}{1-r} + \frac{cr(\gamma + \log(1-r)}{1-r} \le 1, \end{aligned}$$

where we have used Proposition 3.8 to obtain the left side expression of the last inequality. After a simple calculation, the last inequality reduces to

$$1-(5+2c)r+2r^2+c(1+r)\log(1-r)\geq 0.$$

Thus, by Lemma 3.14, f is uniformly convex for $|z|=r\leq r_{uc}$, where r_{uc} is the unique real root of (25) lying in the interval $(0,1)$. Clearly, one can check that the roots of (25) are monotonically decreasing in $(0,1)$, and hence it has exactly one root in $(0,1)$. Hence, f is uniformly convex in $|z|<r_{uc}$. □

In a similar manner, we now state (without proof) two results related to radii of uniformly starlikeness and uniformly convexity for the class $\mathcal{F}^{\alpha}_{G(c)}$.

Theorem 3.17. *Let $f=h+\overline{g}$, where h and g are given by* (2). *Then for all $f\in\mathcal{F}^{\alpha}_{G(c)}$, the radius of uniformly starlikeness is $r_{us}(\alpha)$, where $r_{us}(\alpha)$ is the unique positive root of the equation*

$$1-2\alpha-(3-2\alpha-2\alpha^2)r+2c(1+\alpha)(1-r\alpha)\log(1-r)=0$$

lying in the interval $(0,1)$.

Theorem 3.18. *Let $f=h+\overline{g}$, where h and g are given by* (2). *Then for all $f\in\mathcal{F}^{\alpha}_{G(c)}$, the radius of uniformly convexity is $r_{uc}(\alpha)$, where $r_{uc}(\alpha)$ is the unique positive root of the equation*

$$\begin{aligned}&1-(5+2c+(2c-2-3\alpha)\alpha)r+(2+(2c-1)(1+\alpha)\alpha)r^2\\&\qquad+c\left\{(1-r)^2(1+\alpha)+(1-\alpha^2)(3-r)r\right\}\log(1-r)=0\end{aligned}$$

lying in the interval $(0,1)$.

4. Growth, Covering and Area Theorems for $\mathcal{F}^{\alpha}_{G}$

The idea of growth of a function f refers to the size of its image domain, that is, estimation of $|f(z)|$. It is well known that the Jacobian of a smooth mapping may be viewed as the local magnification factor of area. Thus, for $0<r<1$, the area of the image of the disk $\mathbb{D}_r$ under the mapping f is denoted by $A(f(\mathbb{D}_r))$ and is given by the formula

$$A(f(\mathbb{D}_r))=\int\int_{\mathbb{D}_r}J_f(z)\,dxdy.$$

Computing this area became widely known as the area problem in the function theory. Recently, Ponnusamy and Sairam [28] have obtained the growth, covering and area theorems for certain harmonic close-to-convex functions. As a motivation from their work, we have studied here the growth, covering and area problems for the class of functions f in $\mathcal{F}^{\alpha}_{G}$. The following lemma from [18] is useful in this section (see also [20]).

Lemma 4.1. *[18] Let $h\in\mathcal{G}$ be of the form* (2). *Then for $|z|=r<1$, we have*

$$1-r\leq|h'(z)|\leq 1+r.$$

The inequality is sharp and equality is attained for the function given by $h(z)=z-z^2/2$.

Theorem 4.2. (Growth Theorem) *If $f=h+\overline{g}\in\mathcal{F}^{\alpha}_{G}$, then*

$$|f(z)|\geq\begin{cases}\frac{(1-\alpha)}{2\alpha^3}\left\{r\alpha(-2-(4-r)\alpha)+2(1+\alpha)^2\log(1+r\alpha)\right\}, & \alpha\neq 0,\\ r-r^2+r^3/3, & \alpha=0,\end{cases}\tag{26}$$

and

$$|f(z)|\leq\begin{cases}\frac{(1+\alpha)}{2\alpha^3}\left\{r\alpha(-2+(4+r)\alpha)+2(1-\alpha)^2\log(1+r\alpha)\right\}, & \alpha\neq 0,\\ r+r^2+r^3/3, & \alpha=0,\end{cases}\tag{27}$$

for all $z\in\mathbb{D}$.

Proof. For any point $z = re^{i\theta} \in \mathbb{D}$, let $m := \min\{|f(\mathbb{D}_r)| : z \in \mathbb{D}_r\}$. Then $\mathbb{D}_m \subseteq f(\mathbb{D}_r) \subseteq f(\mathbb{D})$. It is due to the minimum modulus principle that there exists $z_r \in \partial\mathbb{D}_r := \{z \in \mathbb{C} : |z| = r,\ r < 1\}$ such that $m = |f(z_r)|$.

Let $\Gamma(t) = tf(z_r)$, $t \in [0, 1]$. Then $\gamma(t) := f^{-1}(\Gamma(t)), t \in [0, 1]$, is a well defined Jordan arc. Since $f(z) = h(z) + \overline{g(z)}$, we have

$$m = |f(z_r)| = \int_\Gamma |dw| = \int_\gamma |df| = \int_\gamma \left|h'(\eta)d\eta + \overline{g'(\eta)}d\bar{\eta}\right|$$
$$\geq \int_\gamma (|h'(\eta)| - |g'(\eta)|)\,|d\eta|.$$

By using the relation $g'(z) = w(z)h'(z)$, together with Lemma 4.1 and (14), the integrand of the right side expression of the above inequality simplifies to

$$|h'(\eta)| - |g'(\eta)| = |h'(\eta)|(1 - |w(\eta)|) \geq (1 - |\eta|)\left(1 - \frac{\alpha + |\eta|}{1 + \alpha|\eta|}\right) = \frac{(1-\alpha)(1-|\eta|)^2}{1 + \alpha|\eta|}.$$

Therefore, we obtain

$$m \geq \int_\gamma \frac{(1-\alpha)(1-|\eta|)^2}{1+\alpha|\eta|}|d\eta| = \int_0^1 \frac{(1-\alpha)(1-|\gamma(t)|)^2}{1+\alpha|\gamma(t)|}dt = \int_0^r \frac{(1-\alpha)(1-\rho)^2}{1+\alpha\rho}d\rho.$$

On integrating, we obtain the estimation (26).

Next, note that

$$|f(z)| = |h(z) + \overline{g(z)}| \leq |h(z)| + |g(z)|.$$

Thus, the inequality (27) follows from the application of the relation $g'(z) = w(z)h'(z)$, together with Lemma 4.1 and (14). □

Theorem 4.3. (Covering Theorem) *Let $f = h + \overline{g} \in \mathcal{F}_G^\alpha$. Then the range $f(\mathbb{D})$ contains the disk $|w| < r_\alpha$, where*

$$r_\alpha = \begin{cases} \frac{1}{2\alpha^3}\left[-\alpha(2+3\alpha) + 2(1+\alpha)^2\log(1+\alpha)\right], & \alpha \neq 0 \\ 1/3 \approx 0.33333, & \alpha = 0. \end{cases}$$

Proof. The radius r_α follows by allowing the limit $r \to 1^-$ in the lower bound of $|f(z)|$ obtained in Theorem 4.2. □

Theorem 4.4. (Area Theorem) *Let $f(z) = h(z) + \overline{g(z)} \in \mathcal{F}_G^\alpha$, where $h(z)$ and $g(z)$ be of the form* (2). *Then the estimation of area $A(f(\mathbb{D}_r))$ for $0 < r < 1$ satisfies the following inequalities*

$$2\pi(1-\alpha)^2 \int_0^1 \frac{r(1-r)^3(1+r)}{(1+r\alpha)^2}dr \leq A(f(\mathbb{D}_r)) \leq 2\pi(1-\alpha)^2 \int_0^1 \frac{r(1+r)^3(1-r)}{(1-r\alpha)^2}dr. \tag{28}$$

Proof. Let $f(z) = h(z) + \overline{g(z)} \in \mathcal{F}_G^\alpha$. Then $h'(z)$ is nonvanishing in $\mathbb{D}$ and the Jacobian becomes $J_f(z) = |h'(z)|^2(1 - |w(z)|^2)$, where $w(z) = g'(z)/h'(z)$ is the dilatation of f.

By using the inequality (14), together with Lemma 4.1, we obtain

$$\begin{aligned}
A(f(\mathbb{D}_r)) := \int\int_{\mathbb{D}_r} J_f(z)dx\,dy &= \int_0^{2\pi}\int_0^1 J_f(re^{i\theta})r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r|h'(re^{i\theta})|^2\left(1-|w(re^{i\theta})|^2\right)dr\,d\theta \\
&\geq \int_0^{2\pi} d\theta \int_0^1 r(1-r)^2\left(1-\left(\frac{r+\alpha}{1+\alpha r}\right)^2\right)dr \\
&= 2\pi\int_0^1 r(1-r)^2\frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2}\,dr \\
&= 2\pi(1-\alpha^2)\int_0^1 \frac{r(1-r)^3(1+r)}{(1+\alpha r)^2}\,dr.
\end{aligned}$$

This yields the left hand side inequality of (28). Next, to find its right hand side estimation, we compute

$$\begin{aligned}
A(f(\mathbb{D}_r)) &:= 2\pi\int_0^1 r|h'(re^{i\theta})|^2\left(1-|w(re^{i\theta})|^2\right)\,dr \\
&\leq 2\pi\int_0^1 r(1+r)^2\left(1-\left(\frac{\alpha-r}{1-\alpha r}\right)^2\right)\,dr \\
&= 2\pi\int_0^1 r(1+r)^2\frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2}\,dr \\
&= 2\pi(1-\alpha^2)\int_0^1 \frac{r(1+r)^3(1-r)}{(1-\alpha r)^2}dr.
\end{aligned}$$

This completes the proof of our theorem. □

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