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Existence and Ulam-Hyers-Rassias Stability of Stochastic Differential Equations with Random Impulses

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Abstract. In this paper, we investigate the existence and Ulam-Hyers-Rassias stability of solutions for stochastic differential equations with random impulses. Based on the Krasnoselskii's fixed point theorem, we perform investigations on the existence of solutions to the system of stochastic differential equations with random impulses. We apply the integral inequality of Gronwall type to the equations and study their Ulam-Hyers-Rassias stability.

1. Introduction

In 1940, Ulam [32]raised the question concerning the Hyers-Ulam stability of group homomorphisms. This question was first solved by Hyers [11] on Banach spaces. Rassias [22]introduced the concept of unbounded Cauchy difference and extended the Hyers' Theorem to the case of approximate linear mappings. Since then, the theory of Hyers-Ulam stability attracted the attention of many researchers. There are many interesting results on Hyers-Ulam stability have been gained in [1–6, 8, 12–16, 18, 21, 23, 25, 27, 31] and the references therein. Recently, the Hyers-Ulam stability of stochastic differential equations has been investigated by many authors. Zhao [35] studied the Hyers-Ulam stability for a class of first order stochastic differential equations using the Itô formula. Sathiyaraj et al.[26] studied the Ulams stability results for nonlinear Hilfer fractional stochastic differential systems in finite dimensional stochastic setting. Guo et al.[9] studied the Hyers-Ulam stability of the almost periodic solution to the fractional differential equation with impulse and fractional Brownian motion under nonlocal condition based on the basic theory of Hyers-Ulam stability.

Stochastic differential equations have wide applications in describing random or uncertain factors, and as such have advantages over ordinary differential equations in modeling real-world systems with uncertainty. A variety of results on the theory of Stochastic differential equations have been obtained, see [10, 19, 34] and the references therein.

Impulsive differential equations also attracted the attention of many researchers. The theory and application of impulsive differential equations have become an area of active investigation in the literature.

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Great progress has been obtained [29, 30]. Impulses may take place at random moments or fixed moments. Differential equations with these two types of impulses behave differently. The solutions to differential equations with random impulses are stochastic processes, and the solutions to differential equations with impulses at fixed moments are piecewise continuous functions. Wu and Zhou [33] studied the existence and uniqueness of solutions to stochastic differential equations with random impulses and Markovian switching under non-lipschitz conditions using the Bihari inequality. Li et al.[20] studied the existence of upper and lower solutions to a second-order random impulsive differential equation. Shu et al. [28] studied the existence and exponential stability in mean square of mild solutions to second-order neutral stochastic functional differential equations with random impulses in Hilbert space using Mönch fixed point theorem and integral inequality.

Motivated by the above works, we study the existence and Ulam-Hyers-Rassias stability of solutions for stochastic differential equations with random impulses in this article. As far as we know, such type of equations has not been investigated in the literature. We obtain sufficient conditions for the existence of solutions to stochastic differential equations with random impulses using the Krasnoselskii's fixed point. By using the integral inequality of Gronwall type, we study the Ulam-Hyers-Rassias stability for such equations in the mean square.

The rest of this paper is organized as follows. In Section 2, we present some notations and necessary preliminaries. In Section 3, we give the main results of this article. We prove the existence of solutions to the stochastic differential equations with random impulses and the Ulam-Hyers-Rassias stability of this type of equations in mean square.

2. Preliminaries

Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ is a complete probability space with a filtration that satisfies the usual conditions. That is to say, it is increasing and right-continuous. We notice that $\mathcal{F}_{t_0}$ contains all $\mathbb{P}$-zero measure set. We use $\|\cdot\|$ to denote the Euclidean space in $\mathbb{R}^n$ and $\|\cdot\|_M$ to denote some norm in $\mathbb{R}^{n\times m}$ that is compatible with $\|\cdot\|$, i.e., for any $A \in \mathbb{R}^{n\times m}$ and $Y \in \mathbb{R}^n$, $\|AY\| \leq \|A\|_M\|Y\|$. We note that $\|\cdot\|_{L^2(\mathbb{P})} = \sqrt{\mathbb{E}(\|\cdot\|^2)}$, where $\mathbb{E}(Y)$ is the mathematical expectation of random variable Y. Let $T > t_0$ be any fixed time to be determined later and let $L^2_{ad}([t_0, T], \mathbb{R}^n)$ be the space of stochastic processes $f(t,\omega)$ such that each $f(t,\omega)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ and $\mathbb{E}\left(\int_{t_0}^T |f(t)|^2 dt\right) < \infty$.

Now, we consider the following three random sequences on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$:

(i) Let $\{\omega_i\}$ be a random sequence defined on $D_i \equiv (0, d_i), 0 < d_i < +\infty$, where ω_i and ω_j are independent of each other for $i \neq j$, $i, j = 1, 2, \ldots$.

(ii) Let $\{\tau_i, i \geq 1\}$ be a $\mathcal{F}_t$-adapted random sequence that satisfies

$$\tau_i = t_0 + \sum_{j=1}^{i} \omega_j,$$

where $t_0 \in \mathbb{R}$ is a given number. Hence, $\{\tau_i, i \geq 1\}$ is a random sequence with independent increments and $t_0 < \tau_1 < \tau_2 < \tau_3 < \cdots$.

(iii) Suppose that $\{\xi_i\}$ is a series of m-dimensional random vectors independent of $\{\tau_i, i \geq 1\}$ that satisfies $\xi_j \in \mathcal{F}^{(2)}_{\tau_j}$, for $j = 1, 2, \ldots$.

Consider the following stochastic differential equation with random impulses (RISDEs)

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, & t \geq t_0, t \neq \tau_i, \\ X_{\tau_i} = I_i(X_{\tau_i^-}, \xi_i) & i = 1, 2, \ldots, \\ X_{t_0} = x_0, & \end{cases} \tag{1}$$

where $b : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times k}, I_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, i = 1, 2, \ldots, \{B_t\}$ is k-dimensional Wiener process, $X_{\tau_i^-} \equiv \lim_{t\to\tau_i-0} X_t$ in $L^2(\mathbb{P})$, and x_0 is a $\mathcal{F}_{t_0}$-measurable, $\mathbb{R}^n$-valued random variable such that $\mathbb{E}(x_0^2) < \infty$.

We use $\mathcal{F}_t^{(1)}$ to denote the minimal σ-filed generated by $\{B_s, s \le t\}$, and use $\mathcal{F}_t^{(2)}$ to denote the minimal σ-filed containing all events $\{\tau_i \le t\}, i = 1, 2, \ldots$, where $\mathcal{F}_t = \mathcal{F}_t^{(1)} \vee \mathcal{F}_t^{(2)}$. It is assumed that $\{\omega_i\}$, $\{\xi_i\}$, $\mathcal{F}_\infty^{(1)}$ and x_0 are mutually independent. Let $L^2_{ad}([t_0, T], \mathbb{R}^n)$ be the Banach space equipped with the norm

$$\|X_t\| = \sup_{t \in [t_0, T]} \left(\mathbb{E}\|X_t\|^2\right)^{1/2}.$$

Remark 2.1. *Obviously, $\omega_i = \tau_i - \tau_{i-1}$ is the i-th waiting time of impulses. Since RISDEs (1) does not require $\sum_{i=1}^{+\infty} \mathbb{E}(\omega_i) = +\infty$, the solutions to RISDEs (1) may have infinite number of impulses with positive probability in any given interval $[t_0, T]$, where $T > 0$ is a given number.*

The function δ is defined as

$$\int_{-\infty}^{t} \delta(s)ds = \begin{cases} 0, & t < 0, \\ 1, & t \ge 0. \end{cases}$$

It thus follows that RISDEs (1) can be expressed as

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t + \sum_{i=1}^{\infty} (I_i(X_{\tau_i^-}, \xi_i) - X_{\tau_i^-}))d\delta(t - \tau_i), & t \ge t_0, \\ X_{t_0} = x_0, \end{cases}$$

which can be considered as a stochastic control system with random impulses. It is easy to see that RISEDs (1) extends stochastic differential equations with jumps.

Definition 2.2. *$\mathbb{R}^n$-valed stochastic process $\{X_t, t \in [t_0, T]\}$ is called a solution to RISDEs (1) with respect to the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \ge t_0}, \mathbb{P})$ if*

(1) X_t is right-continuous and has limit on the left;

(2) $X_t \in L^2_{ad}([t_0, T], \mathbb{R}^n)$ is $\mathcal{F}_t$-adapted;

(3) $X_{t_0} = x_0$ and for every $t_0 \le t \le T$,

$$\begin{cases} X_t = x_0 + \int_{t_0}^t b(s, X_s)ds + \int_{t_0}^t \sigma(s, X_s)dB_s, & t \in [t_0, \tau_1), \\ X_t = I_\ell(X_{\tau_\ell^-}, \xi_\ell) + \int_{\tau_\ell}^t b(s, X_s)ds + \int_{\tau_\ell}^t \sigma(s, X_s)dB_s, & t \in [\tau_\ell, \tau_{\ell+1}),\ \ell \ge 1. \end{cases} \tag{2}$$

Next, we consider the Ulam's type stability concepts for RISDEs (1).

Definition 2.3. *RISDEs (1) is Ulam-Hyers stable in the mean square with respect to $\varepsilon > 0$ if there exists a constant $\kappa > 0$ such that for each solution $Y_t \in L^2_{ad}([t_0, T], \mathbb{R}^n)$ of the inequality*

$$\begin{cases} \|Y_t - x_0 - \int_{t_0}^t b(s, Y_s)ds - \int_{t_0}^t \sigma(s, Y_s)dB_s\|^2 \le \varepsilon, & t \in [t_0, \tau_1), \\ \|Y_t - I_\ell(Y_{\tau_\ell^-}, \xi_\ell) - \int_{\tau_\ell}^t b(s, Y_s)ds - \int_{\tau_\ell}^t \sigma(s, Y_s)dB_s\|^2 \le \varepsilon, & t \in [\tau_\ell, \tau_{\ell+1}),\ \ell \ge 1, \end{cases} \tag{3}$$

there exists a solution $X_t \in L^2_{ad}([t_0, T], \mathbb{R}^n)$ of RISDEs (1) such that

$$\mathbb{E}\|Y_t - X_t\|^2 \le \kappa\varepsilon, \quad t \in [t_0, T].$$

Definition 2.4. *RISDEs (1) is Ulam-Hyers-Rassias stable in the mean square with respect to $\varphi(t)$ if there exists a constant $M_\varphi > 0$ such that for each solution $Y_t \in L^2_{ad}([t_0, T], \mathbb{R}^n)$ of the inequality*

$$\begin{cases} \|Y_t - x_0 - \int_{t_0}^t b(s, Y_s)ds - \int_{t_0}^t \sigma(s, Y_s)dB_s\|^2 \le \varphi(t), & t \in [t_0, \tau_1), \\ \|Y_t - I_\ell(Y_{\tau_\ell^-}, \xi_\ell) - \int_{\tau_\ell}^t b(s, Y_s)ds - \int_{\tau_\ell}^t \sigma(s, Y_s)dB_s\|^2 \le \varphi(t), & t \in [\tau_\ell, \tau_{\ell+1}),\ \ell \ge 1, \end{cases} \tag{4}$$

there exists a solution $X_t \in L^2_{ad}([t_0, T], \mathbb{R}^n)$ of RISDEs (1) such that

$$\mathbb{E}\|Y_t - X_t\|^2 \le M_\varphi \varphi(t), \quad t \in [t_0, T],$$

where M_φ is a constant that does not depend on Y_t.

Remark 2.5. *It is clear that Definition 2.4 $\Longrightarrow$ Definition 2.3.*

Lemma 2.6. *Let $x(t) \in C([t_0, T], \mathbb{R}^+)$ be a solution to the inequality*

$$x(t) \le \varphi(t) + b\int_{t_0}^{t} x(s)ds + \beta_k x(t_k), \quad t_k \in [t_0, T], k \ge 1,$$

where $\varphi(t) \in C([t_0, T], \mathbb{R}^+)$, $\varphi(t)$ is nondecreasing and $b, \beta_k > 0$. Then,

$$x(t) \le \varphi(t)(1 + \beta_k)exp\left(\int_{t_0}^{t} b(s)ds\right).$$

More integral inequalities of Gronwall type can be found in [17, 24].

Lemma 2.7. *(Krasnoselskii's Fixed Point Theorem [7]) Let B be a nonempty closed convex set of a Banach space $(X, \|\cdot\|)$. Suppose that P and Q map B into X such that*
(1) $Px + Qy \in B$ whenever $x, y \in B$;
(2) P is a contraction mapping;
(3) Q is compact and continuous.
Then there exists $z \in B$ such that $z = Pz + Qz$.

3. Main Results

In this section, we present and prove the existence and Ulam-Hyers-Rassias stablility of the solutions to RISDEs (1). Before developing our main results, we introduce the following hypotheses.

(H1) The function $b : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists a constant $L_b > 0$ such that, for any $x, y \in \mathbb{R}^n$, $t \in [t_0, T]$,

$$||b(t, x) - b(t, y)||^2 \le L_b||x - y||^2, \quad ||b(t, x)||^2 \le L_b(1 + ||x||^2).$$

(H2) The function $\sigma : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times k}$ satisfies the following conditions.

(i) For each $x \in \mathbb{R}^n$, $\sigma(\cdot, x) : [t_0, T] \to \mathbb{R}^{n\times k}$ is measurable and for each $t \in [t_0, T]$, $\sigma(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ is continuous.

(ii) There exists a constant $L_\sigma > 0$ such that, for any $x, y \in \mathbb{R}^n$, $t \in [t_0, T]$,

$$||\sigma(t, x) - \sigma(t, y)||^2 \le L_\sigma||x - y||^2, \quad ||\sigma(t, x)||^2 \le L_\sigma(1 + ||x||^2).$$

(H3) The functions $I_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $i = 1, 2, \ldots$, are continuous and there exists a constant $L > 0$ such that, for any $x, y \in \mathbb{R}^n$, $u \in \mathbb{R}^m$,

$$||I_i(x, u) - I_i(y, u)||^2 \le L||x - y||^2.$$

Now, we establish the following existence results via Krasnoselskii's fixed point theorem.

Theorem 3.1. *If assumptions (H1)-(H3) are satisfied, then the system (1) has at least one mild solution defined on $[t_0, T]$ provided that*

$$\rho := max\{3\omega_1(\omega_1 L_b + L_\sigma), 3[L + \omega_{\ell+1}(\omega_{\ell+1}L_b + L_\sigma)]\} < 1, \ \ell \ge 1, \tag{5}$$

and

$$\eta := max\omega_\ell^2 L_b < 1, \ \ell \ge 1. \tag{6}$$

Proof. Letting $B_r = \{X_t \in L^2_{ad}([t_0, T], \mathbb{R}^n) : \|X_t\|^2 \leq r\}$, which stands for a closed ball with radius $r > 0$ and center at X_t, $t \in [t_0, T]$.
Here,

$$r \geq \left\{ \frac{3[\mathbb{E}\|x_0\|^2 + \omega_1(\omega_1 L_b + L_\sigma)]}{1-\rho}, \frac{3\omega_{\ell+1}(\omega_{\ell+1} L_b + L_\sigma)}{1-\rho} \right\}, \ \ell \geq 1.$$

We then Transform problem (1) into a fixed point problem and define an operator $F : L^2_{ad}([t_0, T], \mathbb{R}^n) \to L^2_{ad}([t_0, T], \mathbb{R}^n)$ by

$$FX_t = \begin{cases} x_0 + \int_{t_0}^t b(s, X_s)ds + \int_{t_0}^t \sigma(s, X_s)dB_s, & t \in [t_0, \tau_1), \\ I_\ell(X_{\tau_\ell^-}, \xi_\ell) + \int_{\tau_\ell}^t b(s, X_s)ds + \int_{\tau_\ell}^t \sigma(s, X_s)dB_s, & t \in [\tau_\ell, \tau_{\ell+1}), \ \ell \geq 1. \end{cases}$$

Defining the operators P and Q on B_r yields

$$PX_t = \begin{cases} x_0 + \int_{t_0}^t b(s, X_s)ds, & t \in [t_0, \tau_1), \\ \int_{\tau_\ell}^t b(s, X_s)ds, & t \in [\tau_\ell, \tau_{\ell+1}), \ \ell \geq 1, \end{cases}$$

and

$$QX_t = \begin{cases} \int_{t_0}^t \sigma(s, X_s)dB_s, & t \in [t_0, \tau_1), \\ I_\ell(X_{\tau_\ell^-}, \xi_\ell) + \int_{\tau_\ell}^t \sigma(s, X_s)dB_s, & t \in [\tau_\ell, \tau_{\ell+1}), \ \ell \geq 1. \end{cases}$$

For better readability, we divide the proof into 4 steps.
Step 1. For any $X_t \in B_r$, $t \in [t_0, T]$, we prove that $FX_t = PX_t + QX_t \in B_r$.
Case 1. For each $t \in [t_0, \tau_1)$, we have

$$\begin{aligned} \mathbb{E}\|PX_t + QX_t\|^2 &\leq 3\mathbb{E}\|x_0\|^2 + 3\mathbb{E}\|\int_{t_0}^t b(s, X_s)ds\|^2 + 3\mathbb{E}\|\int_{t_0}^t \sigma(s, X_s)dB_s\|^2 \\ &\leq 3\mathbb{E}\|x_0\|^2 + 3(t - t_0)\int_{t_0}^t \mathbb{E}\|b(s, X_s)\|^2 ds + 3\int_{t_0}^t \mathbb{E}\|\sigma(s, X_s)\|^2 ds \\ &\leq 3\mathbb{E}\|x_0\|^2 + 3\omega_1^2 L_b(1 + r) + 3\omega_1 L_\sigma(1 + r) \leq r. \end{aligned}$$

Case 2. For each $t \in [\tau_\ell, \tau_{\ell+1}), \ell \geq 1$, we have

$$\begin{aligned} \mathbb{E}\|PX_t + QX_t\|^2 &\leq 3\mathbb{E}\|I_\ell(X_{\tau_\ell^-}, \xi_\ell)\|^2 + 3\mathbb{E}\|\int_{\tau_\ell}^t b(s, X_s)ds\|^2 + 3\mathbb{E}\|\int_{\tau_\ell}^t \sigma(s, X_s)dB_s\|^2 \\ &\leq 3L\mathbb{E}\|X_{\tau_\ell^-}\|^2 + 3(t - \tau_\ell)\int_{\tau_\ell}^t \mathbb{E}\|b(s, X_s)\|^2 ds + 3\int_{\tau_\ell}^t \mathbb{E}\|\sigma(s, X_s)\|^2 ds \\ &\leq 3Lr + 3\omega_{\ell+1}^2 L_b(1 + r) + 3\omega_{\ell+1} L_\sigma(1 + r) \leq r. \end{aligned}$$

The above argument shows that $PX_t + QX_t \in B_r$.
Step 2. We prove that P is a contraction mapping.
Case 1. For $X_t, Y_t \in B_r$ and $t \in [t_0, \tau_1)$, we have

$$\mathbb{E}\|PX_t - PY_t\|^2 \leq (t - t_0)\int_{t_0}^t \mathbb{E}\|b(s, X_s) - b(s, Y_s)\|^2 ds \leq \omega_1^2 L_b\|X_t - Y_t\|^2.$$

Case 2. For $X_t, Y_t \in B_r$ and $t \in [\tau_\ell, \tau_{\ell+1}), \ell \geq 1$, we have

$$\mathbb{E}\|PX_t - PY_t\|^2 \leq (t - \tau_\ell)\int_{\tau_\ell}^t \mathbb{E}\|b(s, X_s) - b(s, Y_s)\|^2 ds \leq \omega_{\ell+1}^2 L_b\|X_t - Y_t\|^2.$$

It follows from (6) that

$$\mathbb{E}\|PX_t - PY_t\|^2 \leq \eta\|X_t - Y_t\|^2 < \|X_t - Y_t\|^2.$$

The above inequalities imply that P is contraction on B_r.

Step 3. We prove that Q is continuous on B_r.

Let $\{X_t^n\} \subset B_r$ with $X_t^n \to X_t$ as $n \to \infty$, $t \in [t_0, T]$. By (H2) and (H3), we get

(i) $\sigma(t, X_t^n) \to \sigma(t, X_t)$, $n \to \infty$,

(ii) $\mathbb{E}\|I_\ell(X^n_{\tau_\ell^-}, \xi_\ell) - I_\ell(X_{\tau_\ell^-}, \xi_\ell)\|^2 \to 0$, $n \to \infty$, $\ell \geq 1$.

It thus follows from the dominated convergence theorem that for each $t \in [t_0, \tau_1)$,

$$\mathbb{E}\|QX_t^n - QX_t\|^2 \leq \int_{t_0}^t \mathbb{E}\|\sigma(s, X_s^n - \sigma(s, X_s))\|^2 ds \to 0, \qquad \textit{as } n \to \infty.$$

For each $t \in [\tau_\ell, \tau_{\ell+1})$, $\ell \geq 1$, we have

$$\mathbb{E}\|QX_t^n - QX_t\|^2 \leq 2\mathbb{E}\|I_\ell(X^n_{\tau_\ell^-}, \xi_\ell) - I_\ell(X_{\tau_\ell^-}, \xi_\ell)\|^2 + 2\int_{\tau_\ell}^t \mathbb{E}\|\sigma(s, X_s^n - \sigma(s, X_s))\|^2 ds \to 0, \textit{ as } n \to \infty.$$

Combining the above arguments, we obtain that Q is continuous on B_r.

Step 4. We prove that Q is compact.

First, we verify that $Q(B_r)$ is uniformly bounded.

From Step 1, we derive that $\mathbb{E}\|QX_t\|^2 \leq r$ for $t \in [t_0, T]$.

Second, we prove that Q maps bounded sets into equicontinuous sets of B_r.

For $t \in [t_0, T]$, $t_0 < t_1 < t_2 < T$, $X_t \in B_r$, we get

$$\mathbb{E}\|QX_{t_2} - QX_{t_1}\|^2 = \mathbb{E}\|\int_{t_0}^{t_2} \sigma(s, X_s)dB_s - \int_{t_0}^{t_1} \sigma(s, X_s)dB_s\|^2 \leq \int_{t_1}^{t_2} \mathbb{E}\|\sigma(s, X_s)\|^2 ds \to 0, \textit{ as } t_2 \to t_1.$$

This implies that Q maps bounded sets into equicontinuous sets of B_r.

Finally, we show that Q maps B_r into a precompact set for every $t \in [t_0, T]$.

Let $0 < \epsilon < t < T$, we define

$$U_t^\epsilon = \{Q^\epsilon X_t : X_t \in B_r\}. \tag{7}$$

The set $U_t^\epsilon = \{Q^\epsilon X_t : X_t \in B_r\}$ is relatively compact in $L^2_{ad}([t_0, T], \mathbb{R}^n)$ for each ϵ with $\epsilon \in (0, t)$.

Case 1. For interval $t \in [t_0, \tau_1)$, (7) reduces to

$$U_t^\epsilon = \left\{\int_{t_0}^{t-\epsilon} \sigma(s, X_s)dB_s : X_t \in B_r\right\}.$$

Then

$$\begin{aligned}\mathbb{E}\|QX_t - Q^\epsilon X_t\|^2 &= \mathbb{E}\|\int_{t_0}^{t} \sigma(s, X_s)dB_s - \int_{t_0}^{t-\epsilon} \sigma(s, X_s)dB_s\|^2 \leq \int_{t-\epsilon}^{t} \mathbb{E}\|\sigma(s, X_s)\|^2 ds \\ &\leq \epsilon L_\sigma(1 + r).\end{aligned} \tag{8}$$

Case 2. For interval $t \in [\tau_\ell, \tau_{\ell+1})$, $\ell \geq 1$, (7) reduces to

$$U_t^\epsilon = \left\{I_\ell(X^-_{\tau_\ell}, \xi_\ell) + \int_{\tau_\ell}^{t-\epsilon} \sigma(s, X_s)dB_s : X_t \in B_r\right\}.$$

Then

$$\mathbb{E}\|QX_t - Q^{\epsilon}X_t\|^2 = \mathbb{E}\|\int_{\tau_\ell}^{t}\sigma(s,X_s)dB_s - \int_{\tau_\ell}^{t-\epsilon}\sigma(s,X_s)dB_s\|^2 \leq \int_{t-\epsilon}^{t}\mathbb{E}\|\sigma(s,X_s)\|^2 ds$$
$$\leq \epsilon L_\sigma(1+r). \tag{9}$$

As $\epsilon \to 0$, the right hand side of inequalities (8) and (9) tend to zero. Hence, there are relatively compact sets arbitrary close to set $U_t = \{QX_t : X_t \in B_r\}$ and U_t is relatively compact in $L^2_{ad}([t_0,T],\mathbb{R}^n)$. Thus, by the Arzela-Ascoli theorem, we know that Q is compact.

By Lemma 2.7, we know that $FX_t = PX_t + QX_t$ has a fixed point in B_r. The proof is thus complete. □

Next, we verify the following Ulam-Hyers-Rassias stable via Gronwall lemma approach.

Theorem 3.2. *Assume that assumptions (H1)-(H3) are satisfied and $\varphi : [t_0,T] \to \mathbb{R}^+$ is a continuous nondecreasing function. Then RISDEs (1) is Ulam-Hyers-Rassias stable in the mean square.*

Proof. Let Y_t be a solution of inequality (4) and let X_t be the solution to RISDEs (1).

Case 1. For interval $t \in [t_0, \tau_1)$, we have

$$\begin{aligned}\mathbb{E}\|Y_t - X_t\|^2 &= \mathbb{E}\|Y_t - x_0 - \int_{t_0}^{t} b(s,X_s)ds - \int_{t_0}^{t}\sigma(s,X_s)dB_s\|^2\\ &\leq 3\mathbb{E}\|Y_t - x_0 - \int_{t_0}^{t} b(s,Y_s)ds - \int_{t_0}^{t}\sigma(s,Y_s)dB_s\|^2\\ &\quad + 3\mathbb{E}\|\int_{t_0}^{t} b(s,Y_s)ds - \int_{t_0}^{t} b(s,X_s)ds\|^2\\ &\quad + 3\mathbb{E}\|\int_{t_0}^{t}\sigma(s,Y_s)dB_s - \int_{t_0}^{t}\sigma(s,X_s)dB_s\|^2.\end{aligned}$$

From Definition 2.4 and the fact that $\varphi : [t_0,T] \to \mathbb{R}^+$ is a continuous function, we get

$$\mathbb{E}\|Y_t - x_0 - \int_{t_0}^{t} b(s,Y_s)ds - \int_{t_0}^{t}\sigma(s,Y_s)dB_s\|^2 \leq \varphi(t).$$

It follows from (H1) and (H2) that

$$\begin{aligned}\mathbb{E}\|Y_t - X_t\|^2 &\leq 3\varphi(t) + 3(t-t_0)\int_{t_0}^{t}\mathbb{E}\|b(s,Y_s) - b(s,X_s)\|^2 ds\\ &\quad + 3\int_{t_0}^{t}\mathbb{E}\|b(s,Y_s) - b(s,X_s)\|^2 ds\\ &\leq 3\varphi(t) + 3(\omega_1 L_b + L_\sigma)\int_{t_0}^{t}\mathbb{E}\|Y_s - X_s\|^2 ds.\end{aligned}$$

According to Lemma 2.6, we obtain

$$\mathbb{E}\|Y_t - X_t\|^2 \leq 3\varphi(t) exp\left(\int_{t_0}^{t} 3(\omega_1 L_b + L_\sigma)ds\right) \leq 3\varphi(t) exp\left(3\omega_1(\omega_1 L_b + L_\sigma)\right).$$

Thus,

$$\mathbb{E}\|Y_t - X_t\|^2 \leq M_{\varphi_1}\varphi(t),$$

where $M_{\varphi_1} = 3exp\left(3\omega_1(\omega_1 L_b + L_\sigma)\right)$.

Case 2. For interval $t \in [\tau_\ell, \tau_{\ell+1}), \ell \geq 1$, we have

$$\begin{aligned}\mathbb{E}\|Y_t - X_t\|^2 &= \mathbb{E}\|Y_t - I_\ell(X_{\tau_\ell^-}, \xi_\ell) - \int_{\tau_\ell}^t b(s, X_s)ds - \int_{\tau_\ell}^t \sigma(s, X_s)dB_s\|^2 \\ &\leq 4\mathbb{E}\|Y_t - I_\ell(Y_{\tau_\ell^-}, \xi_\ell) - \int_{\tau_\ell}^t b(s, Y_s)ds - \int_{\tau_\ell}^t \sigma(s, Y_s)dB_s\|^2 \\ &\quad + 4\mathbb{E}\|I_\ell(Y_{\tau_\ell^-}, \xi_\ell) - I_\ell(X_{\tau_\ell^-}, \xi_\ell)\|^2 \\ &\quad + 4\mathbb{E}\|\int_{\tau_\ell}^t b(s, Y_s)ds - \int_{\tau_\ell}^t b(s, X_s)ds\|^2 \\ &\quad + 4\mathbb{E}\|\int_{\tau_\ell}^t \sigma(s, Y_s)dB_s - \int_{\tau_\ell}^t \sigma(s, X_s)dB_s\|^2.\end{aligned}$$

From Definition 2.4 and the fact that $\varphi : [t_0, T] \to \mathbb{R}^+$ is a continuous function, we have

$$\mathbb{E}\|Y_t - I_\ell(Y_{\tau_\ell^-}, \xi_\ell) - \int_{\tau_\ell}^t b(s, Y_s)ds - \int_{\tau_\ell}^t \sigma(s, Y_s)dB_s\|^2 \leq \varphi(t).$$

By (H1)-(H3), we get

$$\begin{aligned}\mathbb{E}\|Y_t - X_t\|^2 &\leq 4\varphi(t) + 4L\mathbb{E}\|Y_{\tau_\ell^-} - X_{\tau_\ell^-}\|^2 + 4(t - \tau_\ell)\int_{\tau_\ell}^t \mathbb{E}\|b(s, Y_s)ds - b(s, X_s)\|^2 ds \\ &\quad + 4\int_{\tau_\ell}^t \mathbb{E}\|\sigma(s, Y_s) - \sigma(s, X_s)\|^2 ds \\ &\leq 4\varphi(t) + 4L\mathbb{E}\|Y_{\tau_\ell^-} - X_{\tau_\ell^-}\|^2 + 4(\omega_{\ell+1}L_b + L_\sigma)\int_{\tau_\ell}^t \mathbb{E}\|Y_s - X_s\|^2 ds.\end{aligned}$$

It follows from Lemma 2.6 that

$$\mathbb{E}\|Y_t - X_t\|^2 \leq 4\varphi(t)(1 + 4L)exp\left(\int_{\tau_\ell}^t 4(\omega_{\ell+1}L_b + L_\sigma)ds\right) \leq 4\varphi(t)(1 + 4L)exp\left(4\omega_{\ell+1}(\omega_{\ell+1}L_b + L_\sigma)\right).$$

Hence,

$$\mathbb{E}\|Y_t - X_t\|^2 \leq M_{\varphi_2}\varphi(t),$$

where $M_{\varphi_2} = 4(1 + 4L)exp\left(4\omega_{\ell+1}(\omega_{\ell+1}L_b + L_\sigma)\right)$.

Let $M_\varphi = \max\{M_{\varphi_1}, M_{\varphi_2}\}$. Then for all $t \in [t_0, T]$, we obtain

$$\mathbb{E}\|Y_t - X_t\|^2 \leq M_\varphi\varphi(t).$$

Summarizing the above, RISDEs (1) is Ulam-Hyers-Rassias stable in the mean square. This completes the proof. □

Remark 3.3. *Under the assumption of Theorem 3.2, we consider RISDEs (1) and inequality (3). One can repeat the same process to verify that RISDEs (1) is Ulam-Hyers stable in the mean square.*

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