



Legendre Wavelet Approximation of Functions Having Derivatives of Lipschitz Class

Shyam Lal^a, Neha Patel^a

^aDepartment of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India

Abstract. In this paper, Legendre Wavelet approximation of functions f having first derivative f' and second derivative f'' of $Lip\alpha$ class, $0 < \alpha \leq 1$, have been determined. These wavelet estimators are sharper, better and best possible in Wavelet Analysis. It is observed that the Legendre Wavelet estimator of f whose $f'' \in Lip\alpha$ is sharper than the estimator of f having $f' \in Lip\alpha$ class.

1. Introduction

The approximation of functions belonging to class $Lip\alpha$, $0 < \alpha \leq 1$ by n^{th} partial sums of its Fourier series has been estimated by several researchers like Raghuvanshi[1], kushwaha[2]. In Modern Analysis, Wavelet analysis is a new branch which has several applications in signal processing, engineering technology and differential equations. In Wavelet Analysis, the wavelet approximations are new tools to estimate the nature of the function and behaviour of convergence of their wavelet series. The wavelet approximation of a function f by its expansion using Haar father wavelet and Haar mother wavelet have been determined by several analyst like Devore[5], Debnath[3], Meyer[8], Lal and Kumar[6]. But till now, no work seems to have been done to obtain wavelet approximations of a function f such that its first derivative f' and second derivative f'' belonging to Lipschitz class of order α , $0 < \alpha \leq 1$ by Legendre Wavelet method. In an attempt to make an advance study in this direction, in this paper, three new Legendre Wavelet approximations of these functions have been estimated. In best of our knowledge, these estimators are new, better and sharper in Wavelet Analysis.

2. Preliminaries and Definition

In this section, we first review the basic definition and properties of Legendre Wavelets and then introduce the $Lip\alpha$ class. Finally we apply the properties of $Lip\alpha$ class to calculate the Legendre Wavelet approximation.

2010 *Mathematics Subject Classification.* 12C40, 65T60, 65L10, 65L60, 65R20.

Keywords. Legendre Wavelet, Legendre Approximation, Function of Lipschitz class, Orthonormal basis.

Received: 17 October 2017; Accepted: 05 April 2021

Communicated by Dragan S. Djordjević

Email addresses: shyam_lal@rediffmail.com (Shyam Lal), patelnehahbu@gmail.com (Neha Patel)

2.1. Legendre Wavelet

A function $\psi \in L^2(\mathbb{R})$ is said to be 'basic wavelet' or 'mother wavelet' if it satisfies the 'admissibility condition'

$$0 \leq C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

In case of mother wavelet ψ , the translation and dilation parameter are denoted by b and a respectively. We have the following family of continuous wavelets as:

$$\psi_{b,a}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ respectively, n , k belonging to positive integer. We have the following family of discrete wavelet:

$$\psi_{k,n}(t) = |a|^{-\frac{k}{2}} \psi(a_0^k t - nb_0)$$

Here $\psi_{k,n}$ forms an orthonormal basis.

Legendre Wavelet $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four argument; $\hat{n} = 2n - 1, n = 1, 2, 3, \dots, 2^{k-1}$, k can be assume any positive integer, m is order of Legendre Polynomial and t is normalized time. They are defined on interval $[0,1]$ as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} L_m(2^k t - \hat{n}), & \frac{\hat{n} - 1}{2^k} \leq t < \frac{\hat{n} + 1}{2^k} \\ 0, & \text{otherwise.} \end{cases}$$

Yousefi[4]

where $n = 1, 2, \dots, 2^{k-1}$ and $m = 0, 1, 2, \dots, M$.

In above definition, the polynomials L_m are Legendre Polynomials of degree m over the interval $[-1,1]$ which are defined as follows:

$$\begin{aligned} L_0(t) &= 1, L_1(t) = t, \\ (m + 1)L_{m+1}(t) &= (2m + 1)tL_m(t) - mL_{m-1}(t), \quad m = 1, 2, 3, \dots \end{aligned}$$

The set $\{L_m(t) : m = 1, 2, 3, \dots\}$ in the Hilbert space $L^2[-1, 1]$ is a complete orthogonal set. Orthogonality of Legendre polynomial on the interval $[-1,1]$ implies that

$$\langle L_m(t), L_n(t) \rangle = \int_{-1}^1 L_m(t) \overline{L_n(t)} dt = \begin{cases} \frac{2}{2m + 1}, & m = n \\ 0, & \text{otherwise.} \end{cases}$$

for $m, n = 0, 1, 2, \dots$

Furthermore, the set of wavelets $\psi_{n,m}$ makes an orthonormal basis in $L^2[0, 1]$, i.e

$$\int_0^1 \psi_{n,m}(t) \overline{\psi_{n',m'}(t)} dt = \delta_{n,n'} \delta_{m,m'}$$

in which δ denotes Kronecker delta function defined by

$$\delta_{n,n'} = \begin{cases} 1, & n=n' \\ 0, & \text{otherwise} \end{cases}$$

2.2. Legendre Expansion and Approximation

The function $f \in L^2[0, 1)$ may be expanded in terms of Legendre wavelet series as follows:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t); \quad \text{where } c_{n,m} = \langle f, \psi_{n,m} \rangle.$$

If this infinite series is truncated then it can be written as:

$$(S_{2^{k-1},M} f)(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \psi_{n,m}(t) = C^T \psi(t),$$

where C and $\psi(t)$ are $2^{k-1}(M + 1)$ vectors given as

$$C = [c_{1,0}, c_{1,1}, \dots, c_{1,M} \dots c_{2,0}, c_{2,1} \dots c_{2^{k-1},0}, \dots c_{2^{k-1},M}]$$

and

$$\psi(t) = [\psi_{1,0}, \psi_{1,1}, \dots, \psi_{1,M} \dots \psi_{2,0}, \psi_{2,1} \dots \psi_{2^{k-1},0}, \dots, \psi_{2^{k-1},M}]^T.$$

We define,

$$\|f\|_2^2 = \frac{1}{2\pi} \int_0^1 |f(t)|^2 dt$$

and the Legendre Wavelet approximation $(E_{2^{k-1},M} f)$ of f by $S_{2^{k-1},M}(f)$ as under norm $\|\cdot\|_2$ is defined as:

$$E_{2^{k-1},M}(f) = \min \|f - S_{2^{k-1},M}(f)\|_2.$$

If $E_{2^{k-1},M}(f) \rightarrow 0$ as $M \rightarrow \infty, k \rightarrow \infty$ then $E_{2^{k-1},M}(f)$ is called the best approximation of f of order $(2^{k-1}, M)$. (Zygmund[9], pp.114)

2.3. Function of Lip α class

A function $f \in \text{Lip}\alpha$, if

$$|f(x) - f(t)| = O(|x - t|^\alpha); \quad 0 < \alpha \leq 1, \forall x, t \in [0, 1)$$

(Titchmarsh[7])

2.4. Examples

1. Let R be the set of all real number, define $f : [0, 1] \rightarrow R$ by

$$f(x) = x^{\frac{1}{2}} \forall x \in [0, 1]$$

$$|f(x) - f(t)| = |x^{\frac{1}{2}} - t^{\frac{1}{2}}| \leq |x - t|^{\frac{1}{2}} \forall x, t \in [0, 1]$$

Then $f \in \text{Lip}_{\frac{1}{2}}[0, 1]$.

2. Define, $f(x) = x^\alpha \forall x \in [0, 1); \quad 0 < \alpha \leq 1$

$$|f(x) - f(t)| = |x^\alpha - t^\alpha| \leq |x - t|^\alpha \implies f \in \text{Lip}_\alpha[0, 1).$$

3. If $f \in Lip\alpha$; where $\alpha > 1$, then f is constant function for $\alpha > 1$,

$$|f(x+t) - f(x)| = O(|t|^\alpha)$$

$$\left| \frac{f(x+t) - f(x)}{t} \right| = O(|t|^{\alpha-1})$$

Therefore,

$$\lim_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t} = 0$$

$$f'(x) = 0 \forall x \in [0, 1] \implies f \text{ is constant.}$$

3. Theorems

In this paper we prove the following theorems.

Theorem 3.1 If a function f is defined on $[0,1]$ with bounded first derivative i.e $0 \leq |f'(x)| \leq M_1 < \infty \forall x \in [0, 1]$ and its infinite Legendre wavelet series is

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x); \text{ where } c_{n,m} = \langle f, \psi_{n,m} \rangle \tag{1}$$

and $(2^{k-1}, M)^{th}$ partial sum $(S_{2^{k-1}, M} f)(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \psi_{n,m}(x)$

then Legendre Wavelet approximation $E_{2^{k-1}, M}(f)$ of f by $S_{2^{k-1}, M}$ under the norm $\|\cdot\|_2$ satisfies

$$E_{2^{k-1}, M}(f) = \min \|f - S_{2^{k-1}, M}(f)\|_2$$

$$= O\left(\frac{1}{\sqrt{2M+1} 2^k}\right), M = 0, 1, 2, 3, \dots$$

Theorem 3.2 If a function $f \in L^2[0, 1]$, its first derivative $f' \in Lip\alpha; 0 < \alpha \leq 1$, i.e

$$|f'(x+t) - f'(x)| = O(|t|^\alpha); \forall x, t \in [0, 1]$$

then the Legendre wavelet approximations of f satisfy:

$$(i) \text{ For } f(x) = \sum_{n=1}^{\infty} c_{n,0} \psi_{n,0}(x), E_{2^{k-1}, 0}^{(1)}(f) = \min \|f - \sum_{n=1}^{2^{k-1}} c_{n,0} \psi_{n,0}(x)\|_2$$

$$= O\left(\frac{1}{2^{k-1}} \left(1 + \frac{1}{2^{\alpha[k-1]}}\right)\right).$$

$$(ii) \text{ For } f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x), E_{2^{k-1}, M}^{(2)}(f) = \min \|f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \psi_{n,m}(x)\|_2$$

$$= O\left(\frac{1}{2^{k+1} \sqrt{2M+1}} \left(1 + \frac{1}{2^{\alpha k}}\right)\right); M \geq 1.$$

Theorem 3.3 If a function f whose second derivative $f'' \in Lip\alpha$; i.e

$$|f''(x+t) - f''(x)| = O(|t|^\alpha); 0 < \alpha \leq 1, \forall x, t \in [0, 1]$$

then the Legendre wavelet approximations of f satisfy:

$$\begin{aligned} (i) \text{ For } f(x) = \sum_{n=1}^{\infty} c_{n,0} \psi_{n,0}(x), E_{2^{k-1},0}^{(3)}(f) &= \min \|f - \sum_{n=1}^{2^{k-1}} c_{n,0} \psi_{n,0}(x)\|_2 \\ &= O\left(\frac{1}{2^{k-1}} \left(1 + \frac{1}{2^{(\alpha+1)(k-1)}} + \frac{1}{2^{k-1}}\right)\right). \end{aligned}$$

$$\begin{aligned} (ii) \text{ For } f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^1 c_{n,m} \psi_{n,m}(x), E_{2^{k-1},1}^{(4)}(f) &= \min \|f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^1 c_{n,m} \psi_{n,m}(x)\|_2 \\ &= O\left(\frac{1}{2^{2(k-1)}} \left(1 + \frac{1}{2^{\alpha(k-1)}}\right)\right). \end{aligned}$$

$$\begin{aligned} (iii) \text{ For } f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x), E_{2^{k-1},M}^{(5)}(f) &= \min \|f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \psi_{n,m}(x)\|_2 \\ &= O\left(\frac{1}{(2M-1)^{\frac{3}{2}} 2^{2(k+1)}} \left(1 + \frac{1}{2^{\alpha k}}\right)\right); M \geq 2. \end{aligned}$$

4. Proofs

4.1. Proof of the theorem (3.1)

The Legendre Wavelet Series of $f \in L^2[0, 1)$ is

$$f(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x)$$

$$S_{2^{k-1},M}(f) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \psi_{n,m}(x)$$

$$\begin{aligned} f(x) - S_{2^{k-1},M}(f) &= \left(\sum_{n=1}^{2^{k-1}} \left(\sum_{m=0}^M + \sum_{m=M+1}^{\infty} \right) c_{n,m} \psi_{n,m}(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \psi_{n,m}(x) \right) \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m} \psi_{n,m}(x) \end{aligned}$$

$$\begin{aligned}
 E_{2^{k-1},M}^2(f) &= \|f - S_{2^{k-1},M}(f)\|_2^2 \\
 &= \int_0^1 |f(x) - S_{2^{k-1},M}(f)|^2 dx \\
 &= \int_0^1 \left(\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m} \psi_{n,m} \right)^2 dx \\
 &= \int_0^1 \left(\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \psi_{n,m}^2 \right. \\
 &\quad \left. + 2 \sum_{1 \leq n \neq n' \leq 2^{k-1}, M+1 \leq m \neq m' < \infty} c_{n,m} c_{n',m'} \psi_{n,m} \psi_{n',m'} \right) dx \\
 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \int_0^1 \psi_{n,m}^2(x) dx \tag{2}
 \end{aligned}$$

$$\begin{aligned}
 &+ 2 \sum_{1 \leq n \neq n' \leq 2^{k-1}, M+1 \leq m \neq m' < \infty} c_{n,m} c_{n',m'} \int_0^1 \psi_{n,m}(x) \psi_{n',m'}(x) dx \\
 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \|\psi_{n,m}\|_2^2, \text{ other term vanish due to orthonormality of } \psi_{n,m}. \\
 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \tag{3}
 \end{aligned}$$

Next,

$$\begin{aligned}
 c_{n,m} &= \int_{-\infty}^{\infty} f(x) \psi_{n,m}(x) dx \\
 &\leq \left(\frac{2m+1}{2} \right)^{\frac{1}{2}} (2)^{\frac{k}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (f(x) L_m(2^k x - \hat{n})) dx \\
 &= \left(\frac{2m+1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{k}{2}} \int_{-1}^1 f\left(\frac{\hat{n}+t}{2^k}\right) L_m(t) dt, \quad 2^k x - \hat{n} = t \\
 &= \left(\frac{1}{2^{k+1}(2m+1)} \right)^{\frac{1}{2}} \int_{-1}^1 f\left(\frac{\hat{n}+t}{2^k}\right) d(L_{m+1} - L_{m-1}) dt \\
 &= \left(\frac{1}{2^{k+1}(2m+1)} \right)^{\frac{1}{2}} \left[\left(f\left(\frac{\hat{n}+1}{2^k}\right) (L_{m+1}(t) - L_{m-1}(t)) \right)_{-1}^1 \right. \\
 &\quad \left. - \int_{-1}^1 f'\left(\frac{\hat{n}+t}{2^k}\right) \frac{1}{2^k} (L_{m+1}(t) - L_{m-1}(t)) dt \right] \tag{4}
 \end{aligned}$$

$$\begin{aligned}
 |c_{n,m}| &\leq \left| -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^1 \left|f'\left(\frac{\hat{n}+t}{2^k}\right)(L_{m+1}-L_{m-1})\right| dt, m \geq 1 \right. \\
 &\leq \left| -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \left(\int_{-1}^1 \left|f'\left(\frac{\hat{n}+t}{2^k}\right)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_{-1}^1 |L_{m+1}-L_{m-1}|^2 dt\right)^{\frac{1}{2}} \right. \\
 &\qquad\qquad\qquad \text{by cauchy schwarz inequality} \\
 &\leq \left| -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \left| \sqrt{2}M_1 \left[\frac{2}{2m+1} + \frac{2}{2m-1}\right]^{\frac{1}{2}} \right| \right. \\
 &\leq \left| -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \left| \frac{2\sqrt{2}M_1}{\sqrt{2m-1}} \right| \right. \\
 &\leq \frac{2M_1}{2^{\frac{3k}{2}}(2m-1)}
 \end{aligned} \tag{5}$$

Using equation 5 and equation 4.1,

$$\begin{aligned}
 E_{2^{k-1},M}^2(f) &\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \left(\frac{4M_1^2}{2^{3k}}\right) \left(\frac{1}{(2m-1)^2}\right) \\
 &= 4 \left(\sum_{n=1}^{2^{k-1}} \frac{M_1^2}{2^{3k}}\right) \sum_{m=M+1}^{\infty} \left(\frac{1}{(2m-1)^2}\right) \\
 &= 4 \left(\frac{(2^{k-1})M_1^2}{2^{3k}}\right) \left(\frac{1}{(2M+1)^2} + \int_{M+1}^{\infty} \frac{dm}{(2m+1)^2}\right) \\
 &= \left(\frac{2M_1^2}{2^{2k}}\right) \left(\frac{1}{2M+1} + \frac{1}{2M+1}\right) \\
 &= \frac{4M_1^2}{2^{2k}} \left(\frac{1}{2M+1}\right)
 \end{aligned}$$

Hence,

$$\begin{aligned}
 E_{2^{k-1},M}(f) &= \left(\frac{2M_1}{2^k(2M+1)^{\frac{1}{2}}}\right) \\
 &= O\left(\frac{1}{\sqrt{2M+1} 2^k}\right), M = 0, 1, 2, 3, \dots
 \end{aligned}$$

Thus Theorem (3.1) is proved.

4.2. Proof of Theorem (3.2)

$$\begin{aligned}
 (i) \quad f(x) &= \sum_{n=1}^{\infty} c_{n,0} \psi_{n,0}(x) \\
 e_n(x) &= c_{n,0} \psi_{n,0}(x) - f(x) \quad \forall x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)
 \end{aligned}$$

$$\begin{aligned}
 \|e_n(x)\|_2^2 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} e_n^2(x) dx \\
 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (c_{n,0}\psi_{n,0} - f(x))^2 dx \\
 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (c_{n,0}^2\psi_{n,0}^2 - 2c_{n,0}\psi_{n,0}(x)f(x) + f^2(x)) dx \\
 &= c_{n,0}^2\|\psi_{n,0}\|_2^2 - 2c_{n,0} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x)\psi_{n,0}(x) dx + \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f^2(x) dx \\
 &= c_{n,0}^2 - 2c_{n,0}^2 + \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f^2(x) dx \\
 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f^2(x) dx - c_{n,0}^2 \\
 &= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k} + h\right) \right)^2 dh - c_{n,0}^2; \quad x = \frac{\hat{n}-1}{2^k} + h, \quad 0 \leq h \leq \frac{1}{2^{k-1}} \\
 &= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k} + hf'\left(\frac{\hat{n}-1}{2^k} + \theta h\right)\right) \right)^2 dh - c_{n,0}^2; \quad 0 < \theta < 1 \\
 &\quad \text{By mean value theorem} \\
 &= \frac{1}{2^{k-1}} \left(f\left(\frac{\hat{n}-1}{2^k}\right) \right)^2 + 2f\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} hf'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\
 &\quad + \int_0^{\frac{1}{2^{k-1}}} h^2 \left(f'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \right)^2 dh - c_{n,0}^2 \\
 &= \frac{1}{2^{k-1}} \left(f\left(\frac{\hat{n}-1}{2^k}\right) \right)^2 + 2f\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} hf'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\
 &\quad + \int_0^{\frac{1}{2^{k-1}}} h^2 \left(f'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \right)^2 dh - \left(2^{\frac{k-1}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x) dx \right)^2 \\
 &= \frac{1}{2^{k-1}} \left(f\left(\frac{\hat{n}-1}{2^k}\right) \right)^2 + 2f\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} hf'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\
 &\quad + \int_0^{\frac{1}{2^{k-1}}} h^2 \left(f'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \right)^2 dh - \left(2^{\frac{k-1}{2}} \int_0^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k} + h\right) dh \right)^2 \\
 &= \frac{1}{2^{k-1}} \left(f\left(\frac{\hat{n}-1}{2^k}\right) \right)^2 + 2f\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} hf'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\
 &\quad + \int_0^{\frac{1}{2^{k-1}}} h^2 \left(f'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \right)^2 dh \\
 &\quad - \left(2^{\frac{k-1}{2}} \int_0^{\frac{1}{2^{k-1}}} f\left(\frac{\hat{n}-1}{2^k}\right) dh + 2^{\frac{k-1}{2}} \int_0^{\frac{1}{2^{k-1}}} hf'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right)^2 \\
 &= \int_0^{\frac{1}{2^{k-1}}} h^2 \left(f'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \right)^2 dh - 2^{k-1} \left(\int_0^{\frac{1}{2^{k-1}}} hf'\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right)^2 \\
 &= I_1 - I_2 \quad (\text{say})
 \end{aligned}$$

(6)

$$\begin{aligned}
 |I_1| &\leq \int_0^{\frac{1}{2^{k-1}}} \left| h^2 \left(f' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right)^2 \right| dh \\
 &= \int_0^{\frac{1}{2^{k-1}}} \left| h^2 \left[\left(f' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) - f' \left(\frac{\hat{n}-1}{2^k} \right) \right) + f' \left(\frac{\hat{n}-1}{2^k} \right) \right]^2 \right| dh \\
 &\leq \int_0^{\frac{1}{2^{k-1}}} h^2 \left[\left| \left(f' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) - f' \left(\frac{\hat{n}-1}{2^k} \right) \right)^2 \right| + \left| f' \left(\frac{\hat{n}-1}{2^k} \right) \right|^2 \right. \\
 &\quad \left. + 2 \left| \left(f' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) - f' \left(\frac{\hat{n}-1}{2^k} \right) \right) f' \left(\frac{\hat{n}-1}{2^k} \right) \right| \right] dh \\
 &\leq \int_0^{\frac{1}{2^{k-1}}} h^2 \left[(O(\theta h)^\alpha)^2 + M_2^2 + 2(\theta h)^\alpha M_2 \right] dh \\
 &\leq \int_0^{\frac{1}{2^{k-1}}} h^2 \left(M_1^2 h^{2\alpha} + M_2^2 + 2M_1 h^\alpha M_2 \right) dh \\
 &= \left(M_1^2 \frac{h^{2\alpha+3}}{2\alpha+3} + \frac{M_2^2 + h^3}{3} + 2M_1 M_2 \frac{h^{\alpha+3}}{\alpha+3} \right)_0^{\frac{1}{2^{k-1}}} \\
 &= \frac{M_1^2}{2\alpha+3} \frac{1}{2^{2(\alpha+3)(k-1)}} + \frac{1}{3} \frac{M_2^2}{2^{3(k-1)}} + \frac{2M_1 M_2}{\alpha+3} \frac{1}{2^{(\alpha+3)(k-1)}}
 \end{aligned}$$

$$\begin{aligned}
 |I_2| &= 2^{k-1} \left| \int_0^{\frac{1}{2^{k-1}}} h f' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) dh \right|^2 \\
 &\geq 2^{k-1} \left[\int_0^{\frac{1}{2^{k-1}}} h M_2 dh \right]^2 \\
 &= \frac{M_2^2}{4} \frac{1}{2^3(k-1)}
 \end{aligned}$$

Then,

$$\begin{aligned}
 \|e_n\|_2^2 &= \frac{M_1^2}{2\alpha+3} \left(\frac{1}{2^{2(\alpha+3)(k-1)}} \right) + \frac{M_2^2}{12} \left(\frac{1}{2^{3(k-1)}} \right) + \frac{2M_1 M_2}{\alpha+3} \left(\frac{1}{2^{(\alpha+3)(k-1)}} \right) \\
 &\leq \left(\frac{M_1^2}{2^{2(\alpha+3)(k-1)}} + \frac{M_2^2}{2^{3(k-1)}} + \frac{2M_1 M_2}{2^{(\alpha+3)(k-1)}} \right)
 \end{aligned}$$

$$\begin{aligned}
 E_{2^{k-1}, M}^{(1)2}(f) &= \sum_{n=1}^{2^{k-1}} \|e_n\|^2 \\
 &= \left(\frac{M_1^2 2^{k-1}}{2^{2(\alpha+3)(k-1)}} + \frac{M_2^2 2^{k-1}}{2^{3(k-1)}} + \frac{2M_1 M_2 2^{k-1}}{2^{(\alpha+3)(k-1)}} \right) \\
 &= \left(\frac{M_1^2}{2^{2(\alpha+2)(k-1)}} + \frac{M_2^2}{2^{2(k-1)}} + \frac{2M_1 M_2}{2^{(\alpha+2)(k-1)}} \right) \\
 &= \left(\frac{M_1}{2^{(\alpha+1)(k-1)}} + \frac{M_2}{2^{(k-1)}} \right)^2 \\
 &= \frac{(M_1 + M_2)^2}{2^{2(k-1)}} \left(\frac{1}{2^{\alpha(k-1)}} + 1 \right)^2
 \end{aligned}$$

Hence, $E_{2^{k-1},M}^{(1)}(f) = O\left(\frac{1}{2^{k-1}}\left(1 + \frac{1}{2^\alpha(k-1)}\right)\right)$

(ii) $f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x)$

Following the proof on Theorem (3.1) and equation 4, we have;

$$\begin{aligned} c_{n,m} &= -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^1 \left(f'\left(\frac{\hat{n}+t}{2^k}\right) - f'\left(\frac{\hat{n}}{2^k}\right) + f'\left(\frac{\hat{n}}{2^k}\right)\right) (L_{m+1} - L_{m-1}) dt \\ &= -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^1 \left(f'\left(\frac{\hat{n}+t}{2^k}\right) - f'\left(\frac{\hat{n}}{2^k}\right)\right) (L_{m+1} - L_{m-1}) dt \\ &\quad - \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} f'\left(\frac{\hat{n}}{2^k}\right) \int_{-1}^1 (L_{m+1} - L_{m-1}) dt \\ |C_{n,m}| &= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^1 \left|f'\left(\frac{\hat{n}+t}{2^k}\right) - f'\left(\frac{\hat{n}}{2^k}\right)\right| |L_{m+1} - L_{m-1}| dt \\ &\quad + \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \left|f'\left(\frac{\hat{n}}{2^k}\right)\right| \int_{-1}^1 |L_{m+1} - L_{m-1}| dt \\ &= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{2^{k\alpha}} \int_{-1}^1 |L_{m+1} - L_{m-1}| dt \\ &\quad + \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} M_1 \int_{-1}^1 |L_{m+1} - L_{m-1}| dt \\ &= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \left(\frac{1}{2^{k\alpha}} + M_1\right) \int_{-1}^1 |L_{m+1} - L_{m-1}| dt \\ &= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \left(\frac{1}{2^{k\alpha}} + M_1\right) \sqrt{2} \left(\int_{-1}^1 |L_{m+1}^2(t) + L_{m-1}^2(t) 2L_{m+1}(t)L_{m-1}(t)| dt\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \left(\frac{1}{2^{k\alpha}} + M_1\right) \sqrt{2} \left(\frac{2}{2m+3} + \frac{2}{2m-1}\right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \left(\frac{1}{2^{k\alpha}} + M_1\right) \sqrt{2} \frac{2}{\sqrt{2m-1}} \\ &\leq \left(\frac{8}{2^{3k+1}}\right)^{\frac{1}{2}} \left(\frac{1}{2^{k\alpha}} + 1\right) \frac{1+M_1}{2m-1} \end{aligned}$$

$$\begin{aligned} \|f(x) - S_{2^{k-1},M}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} |c_{n,m}|^2 \\ &\leq \left(\frac{8(1+M_1)^2}{2^{3k+1}(2m-1)^2}\right) \left(\frac{1}{2^{k\alpha}} + 1\right) \\ &\leq 8(M_1+1)^2 \sum_{n=1}^{2^{k-1}} \frac{1}{2^{3k+1}} \left(\frac{1}{2^{k\alpha}} + 1\right)^2 \sum_{m=M+1}^{\infty} \frac{1}{(2m-1)^2} \\ &= 8(M_1+1)^2 \frac{1}{2^{3k+1}} \left(\frac{1}{2^{k\alpha}} + 1\right)^2 2^{k-1} \left(\frac{1}{(2M+1)^2} + \int_{M+1}^{\infty} \frac{dm}{(2m-1)^2}\right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{8(M_1 + 1)^2}{2^{2k+2}} \left(\frac{1}{2^{k\alpha}} + 1\right)^2 \left(\frac{1}{(2M + 1)^2} + \frac{1}{2M + 1}\right) \\ &\leq \frac{8(M_1 + 1)^2}{2^{2k+2}} \left(\frac{1}{2^{k\alpha}} + 1\right)^2 \left(\frac{2}{2M + 1}\right) \end{aligned}$$

$$\begin{aligned} E_{2^{k-1},M} f &= \|f(x) - S_{2^{k-1},M}(f)\|_2 \\ &= \frac{4(M_1 + 1)}{2^{k+1}} \left(\frac{1}{2^{k\alpha}} + 1\right) \left(\frac{1}{(2M + 1)^{\frac{1}{2}}}\right) \\ &= O\left(\frac{1}{2^{k+1} \sqrt{2M + 1}} \left(1 + \frac{1}{2^{\alpha k}}\right)\right); M \geq 1. \end{aligned}$$

Thus theorem (3.2) is completely established.

4.3. Proof of Theorem (3.3)

$$(i) f(x) = \sum_{n=1}^{\infty} c_{n,0} \psi_{n,0}(x)$$

$$e_n(x) = c_{n,0} \psi_{n,0}(x) - f(x) \quad \forall x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$$

$$e_n(x) = c_{n,0} \psi_{n,0} - f(x) \quad \forall x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}\right)$$

$$\begin{aligned} \|e_n(x)\|_2^2 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f^2(x) - c_{n,0}^2 \\ &= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k} + h\right)\right)^2 dh - c_{n,0}^2; \quad x = \frac{\hat{n}-1}{2^k} + h, 0 \leq h \leq \frac{1}{2^{k-1}} \\ &= \frac{1}{2^{k-1}} \left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{1}{3} \frac{1}{2^{3(k-1)}} \left(f'\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4} \left(f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right)\right)^2 dh \\ &+ \frac{1}{2^{2(k-1)}} f\left(\frac{\hat{n}-1}{2^k}\right) f'\left(\frac{\hat{n}-1}{2^k}\right) + f'\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} h^3 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\ &+ f\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh - c_{n,0}^2 \end{aligned} \tag{7}$$

Consider,

$$c_{n,0} = \frac{1}{2^{\frac{k-1}{2}}} f\left(\frac{\hat{n}-1}{2^k}\right) + \frac{1}{2} \frac{1}{2^{\frac{3(k-1)}{2}}} f'\left(\frac{\hat{n}-1}{2^k}\right) dh + \frac{1}{2} 2^{\frac{k-1}{2}} \int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh$$

$$\begin{aligned} c_{n,0}^2 &= \frac{1}{2^{k-1}} \left(f\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{1}{4} \frac{1}{2^{3(k-1)}} \left(f'\left(\frac{\hat{n}-1}{2^k}\right)\right)^2 + \frac{1}{4} 2^{k-1} \left(\int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh\right)^2 \\ &+ \frac{1}{2^{2(k-1)}} f\left(\frac{\hat{n}-1}{2^k}\right) f'\left(\frac{\hat{n}-1}{2^k}\right) + f\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\ &+ \frac{1}{2^k} f'\left(\frac{\hat{n}-1}{2^k}\right) \left(\int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh\right) \end{aligned}$$

Next,

$$\begin{aligned}
 \|e_n(x)\|_2^2 &= \frac{1}{12} \frac{1}{2^{3(k-1)}} \left(f' \left(\frac{\hat{n}-1}{2^k} \right) \right)^2 + \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4} \left(f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right)^2 dh \\
 &+ f' \left(\frac{\hat{n}-1}{2^k} \right) \int_0^{\frac{1}{2^{k-1}}} h^3 f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) dh - \frac{1}{4} 2^{k-1} \left(\int_0^{\frac{1}{2^{k-1}}} h^2 f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) dh \right)^2 \\
 &- \frac{1}{2^k} f' \left(\frac{\hat{n}-1}{2^k} \right) \left(\int_0^{\frac{1}{2^{k-1}}} h^2 f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) dh \right) \\
 &= I_1 + I_2 + I_3 - I_4 - I_5 \quad (\text{say})
 \end{aligned} \tag{8}$$

$$\begin{aligned}
 |I_1| &\leq \frac{1}{12} \frac{1}{2^{3(k-1)}} \left| f' \left(\frac{\hat{n}-1}{2^k} \right) \right|^2 \\
 &\leq \frac{M_2^2}{3} \frac{1}{2^{3k-1}}
 \end{aligned}$$

$$\begin{aligned}
 |I_2| &\leq \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4} \left[(O(\theta h)^\alpha)^2 + M_4^2 + 2(\theta h)^\alpha M_4 \right] dh \\
 &\leq \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4} \left[M_3^2 h^{2\alpha} + M_4^2 + 2M_3 h^\alpha M_4 \right] dh \\
 &= \frac{M_3^2}{4(2\alpha+5)} \frac{1}{2^{(k-1)(2\alpha+5)}} + \frac{M_4^2}{20} \frac{1}{2^{5(k-1)}} + \frac{2M_3 M_4}{4(\alpha+5)2^{(k-1)(\alpha+5)}}
 \end{aligned}$$

$$\begin{aligned}
 |I_3| &\leq \left| f' \left(\frac{\hat{n}-1}{2^k} \right) \right| \int_0^{\frac{1}{2^{k-1}}} h^3 \left| f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right| dh \\
 &\leq M_2 \int_0^{\frac{1}{2^{k-1}}} h^3 (M_3 h^\alpha + M_4) \\
 &= \frac{M_2 M_3}{(\alpha+3)2^{(k-1)(\alpha+4)}} + \frac{1}{4} \frac{M_2 M_4}{2^{4(k-1)}}
 \end{aligned}$$

$$\begin{aligned}
 |I_4| &= \frac{1}{4} 2^{k-1} \left| \int_0^{\frac{1}{2^{k-1}}} h^2 f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) dh \right|^2 \\
 &\geq \frac{1}{4} 2^{k-1} M_4^2 \left(\int_0^{\frac{1}{2^{k-1}}} h dh \right)^2 \\
 &= \frac{M_4^2}{36} \frac{1}{2^{5(k-1)}}
 \end{aligned}$$

$$\begin{aligned}
 |I_5| &= \frac{1}{2^k} \left| f' \left(\frac{\hat{n}-1}{2^k} \right) \left(\int_0^{\frac{1}{2^{k-1}}} h^2 f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) dh \right) \right| \\
 &\geq \frac{1}{2^k} M_2 \int_0^{\frac{1}{2^{k-1}}} h^2 M_4 dh \\
 &= \frac{M_2 M_4}{3} \frac{1}{2^{4k-3}}
 \end{aligned}$$

Then,

$$\begin{aligned} \|e_n(x)\|_2^2 &= \frac{1}{3} \frac{M_2^2}{2^{3k-1}} + \frac{1}{4(2\alpha + 5)} \left(\frac{M_3^2}{2^{(k-1)(2\alpha+5)}} \right) + \frac{M_4^2}{(4)2^{5(k-1)}} \left(\frac{1}{5} - \frac{1}{9} \right) + \frac{2M_3M_4}{(4)(\alpha + 5)2^{(k-1)(\alpha+5)}} \\ &+ \frac{M_2M_3}{(\alpha + 3)2^{(k-1)(\alpha+4)}} + \frac{M_2M_4}{2^{(4k-3)}} \left(\frac{1}{2} - \frac{1}{3} \right) \\ &\leq \left(\frac{M_2^2}{2^{3k-1}} + \frac{M_3^2}{2^{2k\alpha+5k-2\alpha-5}} + \frac{M_4^2}{2^{5k-3}} + \frac{M_3M_4}{2^{k\alpha+5k-\alpha-5}} + \frac{M_2M_3}{2^{k\alpha+4k-\alpha-4}} + \frac{M_2M_4}{2^{4k-3}} \right) \end{aligned}$$

$$\begin{aligned} E_{2^{k-1},M}^{(3)2}(f) &= \sum_{n=1}^{2^{k-1}} \|e_n\|^2 \\ &= \left(\frac{M_2^2}{2^{2k}} + \frac{M_3^2}{2^{2k\alpha+4k-2\alpha-4}} + \frac{M_4^2}{2^{4k-2}} + \frac{M_3M_4}{(2)2^{k\alpha+4k-\alpha-4}} + \frac{M_2M_3}{2^{k\alpha+3k-\alpha-3}} + \frac{M_2M_4}{2^{3k-2}} \right) \\ &\leq (M_2 + M_3 + M_4)^2 \left(\frac{1}{2^{\frac{2k}{2}}} + \frac{1}{2^{\frac{2k\alpha+4k-2\alpha-4}{2}}} + \frac{1}{2^{\frac{4k-2}{2}}} \right)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} E_{2^{k-1},M}^{(3)}(f) &= (M_2 + M_3 + M_4) \left(\frac{1}{2^k} + \frac{1}{2^{k\alpha+2k-\alpha-2}} + \frac{1}{2^{2k-1}} \right) \\ &= (M_2 + M_3 + M_4) \left(\frac{1}{2^{k-1}} \left(1 + \frac{1}{2^{(k-1)(\alpha+1)}} + \frac{1}{2^k} \right) \right) \\ &= O \left(\frac{1}{2^{k-1}} \left(1 + \frac{1}{2^{(\alpha+1)(k-1)}} + \frac{1}{2^{k-1}} \right) \right). \end{aligned}$$

(ii) $f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^1 c_{n,m} \psi_{n,m}(x)$

$e_n(x) = c_{n,0} \psi_{n,0} - f(x) \quad \forall x \in \left[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k} \right]$

$$\begin{aligned} \|e_n(x)\|_2^2 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} e_n^2(x) dx \\ &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f^2(x) - c_{n,0}^2 - c_{n,1}^2 \\ &= \int_0^{\frac{1}{2^{k-1}}} \left(f \left(\frac{\hat{n}-1}{2^k} + h \right) \right)^2 dh - c_{n,0}^2 - c_{n,1}^2 \end{aligned} \tag{9}$$

consider,

$$\begin{aligned} c_{n,1} &= \langle f(x), \psi_{n,1} \rangle \\ &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x) \psi_{n,1}(x) dx \\ &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x) \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} (2^k x - \hat{n}) dx \\ &= \int_0^{\frac{1}{2^{k-1}}} f \left(\frac{\hat{n}-1}{2^k} + h \right) \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} \left(2^k \left(\frac{\hat{n}-1}{2^k} + h \right) - \hat{n} \right) dh \\ &= \int_0^{\frac{1}{2^{k-1}}} f \left(\frac{\hat{n}-1}{2^k} + h \right) \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} (2^k h - 1) dh \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\frac{1}{2^{k-1}}} \left(f\left(\frac{\hat{n}-1}{2^k}\right) + hf'\left(\frac{\hat{n}-1}{2^k}\right) + \frac{h^2}{2} f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \right) \sqrt{\frac{3}{2}} 2^{\frac{k}{2}} (2^k h - 1) dh \\
 &= \frac{1}{\sqrt{32}^{\frac{3k-1}{2}}} f'\left(\frac{\hat{n}-1}{2^k}\right) + \sqrt{32}^{\frac{3(k-1)}{2}} \int_0^{\frac{1}{2^{k-1}}} h^3 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\
 &- \sqrt{32}^{\frac{(k-1)}{2}} \int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\
 c_{n,1}^2 &= \frac{2^{1-3k}}{3} \left(f'\left(\frac{\hat{n}-1}{2^k}\right) \right)^2 + (3)2^{3(k-1)} \left(\int_0^{\frac{1}{2^{k-1}}} h^3 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right)^2 \\
 &+ (3)2^{k-1} \left(\int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right)^2 + f'\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} h^3 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\
 &- 2^{-k} f'\left(\frac{\hat{n}-1}{2^k}\right) \int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \\
 &- (3)2^{2(k-1)} \int_0^{\frac{1}{2^{k-1}}} h^3 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh
 \end{aligned}$$

Then,

$$\begin{aligned}
 \|e_n(f)\|_2^2 &= \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4} \left(f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \right)^2 dh - 2^{k-1} \left(\int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right)^2 \\
 &- (3)2^{3(k-1)} \left(\int_0^{\frac{1}{2^{k-1}}} h^3 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right)^2 \\
 &+ (3)2^{2(k-1)} \left(\int_0^{\frac{1}{2^{k-1}}} h^3 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right) \left(\int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right) \\
 &= I_1 - I_2 - I_3 + I_4 \text{ (say)}
 \end{aligned}$$

$$\begin{aligned}
 |I_1| &\leq \int_0^{\frac{1}{2^{k-1}}} \frac{h^4}{4} \left| f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) \right|^2 dh \\
 &\leq \frac{M_3^2}{4(2\alpha + 5)} \frac{1}{2^{(k-1)(2\alpha+5)}} + \frac{M_4^2}{20} \frac{1}{2^{5(k-1)}} + \frac{2M_3M_4}{4(\alpha + 5)2^{(k-1)(\alpha+5)}}
 \end{aligned}$$

$$\begin{aligned}
 |I_2| &= 2^{k-1} \left| \int_0^{\frac{1}{2^{k-1}}} h^2 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right|^2 \\
 &\geq \frac{M_4^2}{9} \frac{1}{2^{5(k-1)}}
 \end{aligned}$$

$$\begin{aligned}
 |I_3| &= (3)2^{3(k-1)} \left| \int_0^{\frac{1}{2^{k-1}}} h^3 f''\left(\frac{\hat{n}-1}{2^k} + \theta h\right) dh \right|^2 \\
 &\geq \frac{3M_4^2}{16} \frac{1}{2^{5(k-1)}}
 \end{aligned}$$

$$|I_4| \leq (3)2^{2(k-1)} \left(\int_0^{\frac{1}{2^{k-1}}} \left| h^3 f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right| dh \right) \left(\int_0^{\frac{1}{2^{k-1}}} \left| h^2 f'' \left(\frac{\hat{n}-1}{2^k} + \theta h \right) \right| dh \right) \\ \leq (3)2^{2(k-1)} \left(\frac{M_3}{(\alpha+4)2^{(k-1)(\alpha+4)}} + \frac{M_4}{(4)(2^{4(k-1)})} \right) \left(\frac{M_3}{(\alpha+3)2^{(k-1)(\alpha+3)}} + \frac{M_4}{(3)(2^{3(k-1)})} \right)$$

Then,

$$\|e_n(f)\|_2^2 = \frac{1}{2^{(k-1)(2\alpha+5)}} \left(\frac{M_3^2}{4(2\alpha+5)} + \frac{3M_3^2}{(\alpha+4)(\alpha+3)} \right) + \frac{1}{2^{5(k-1)}} \left(\frac{M_4^2}{20} - \frac{M_4^2}{9} - \frac{3M_4^2}{16} + \frac{M_4^2}{4} \right) \\ + \frac{1}{2^{(k-1)(\alpha+5)}} \left(\frac{2M_3M_4}{4(\alpha+5)} + \frac{M_3M_4}{(\alpha+4)} + \frac{3M_3M_4}{4(\alpha+3)} \right) \\ \leq \left(\frac{M_3^2}{2^{(k-1)(2\alpha+5)}} + \frac{M_4^2}{2^{5(k-1)}} + \frac{2M_3M_4}{2^{(k-1)(\alpha+5)}} \right)$$

$$E_{2^{k-1},M}^{(4)2}(f) = \sum_{n=1}^{2^{k-1}} \|e_n(f)\|_2^2 \\ = \left(\frac{M_3^2 2^{k-1}}{2^{(k-1)(2\alpha+5)}} + \frac{M_4^2 2^{k-1}}{2^{5(k-1)}} + \frac{2M_3M_4 2^{k-1}}{2^{(k-1)(\alpha+5)}} \right) \\ = \frac{1}{2^{4(k-1)}} \left(\frac{M_3^2}{2^{2\alpha(k-1)}} + M_4^2 + \frac{2M_3M_4}{2^{\alpha(k-1)}} \right) \\ \leq \frac{(M_3 + M_4)^2}{2^{4(k-1)}} \left(\frac{1}{2^{\alpha(k-1)}} + 1 \right)^2$$

$$E_{2^{k-1},M}^{(4)}(f) = \frac{M_3 + M_4}{2^{2(k-1)}} \left(\frac{1}{2^{\alpha(k-1)}} + 1 \right) \\ = O \left(\frac{1}{2^{2(k-1)}} \left(1 + \frac{1}{2^{\alpha(k-1)}} \right) \right).$$

(iii) $f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x)$

Following the proof on Theorem (3.2)(ii),

$$c_{n,m} \leq \left(\frac{1}{2^{5k+1}(2m+1)} \right)^{\frac{1}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f''(x) \left[\frac{L_{m+2}(2^k x - \hat{n}) - L_m(2^k x - \hat{n})}{2m+3} \right] 2^k dx \\ - \left(\frac{1}{2^{5k+1}(2m+1)} \right)^{\frac{1}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f''(x) \left[\frac{L_m(2^k x - \hat{n}) - L_{m-2}(2^k x - \hat{n})}{2m-1} \right] 2^k dx \\ = \left(\frac{1}{2^{5k+1}(2m+1)} \right)^{\frac{1}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left| f''(x) - f'' \left(\frac{\hat{n}-1}{2^k} \right) \right| \left[\frac{L_{m+2}(2^k x - \hat{n}) - L_m(2^k x - \hat{n})}{2m+3} \right. \\ \left. - \frac{L_m(2^k x - \hat{n}) - L_{m-2}(2^k x - \hat{n})}{2m-1} \right] 2^k dx \\ + \left(\frac{1}{2^{5k+1}(2m+1)} \right)^{\frac{1}{2}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f'' \left(\frac{\hat{n}-1}{2^k} \right) \left[\frac{L_{m+2}(2^k x - \hat{n}) - L_m(2^k x - \hat{n})}{2m+3} \right. \\ \left. - \frac{L_m(2^k x - \hat{n}) - L_{m-1}(2^k x - \hat{n})}{2m-1} \right] 2^k dx$$

$$\begin{aligned}
 &= \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{2}{2^{k\alpha}} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left[\frac{L_{m+2}(2^kx - \hat{n}) - L_m(2^kx - \hat{n})}{2m+3} \right. \\
 &\quad \left. - \frac{L_m(2^kx - \hat{n}) - L_{m-2}(2^kx - \hat{n})}{2m-1} \right] 2^k dx + \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} M_2 \\
 &\quad \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left[\frac{L_{m+2}(2^kx - \hat{n}) - L_m(2^kx - \hat{n})}{2m+3} - \frac{L_m(2^kx - \hat{n}) - L_{m-2}(2^kx - \hat{n})}{2m-1} \right] 2^k dx \\
 &= \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{2}{2^{k\alpha}} \int_{-1}^1 \left[\frac{L_{m+2}(t) - L_m(t)}{2m+3} - \frac{L_m(t) - L_{m-2}(t)}{2m-1} \right] dt + \\
 &\quad + \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} M_2 \int_{-1}^1 \left[\frac{L_{m+2}(t) - L_m(t)}{2m+3} - \frac{L_m(t) - L_{m-2}(t)}{2m-1} \right] dt \\
 &\leq \frac{1}{2^{\frac{5k+1}{2}}(2m+1)^{\frac{1}{2}}} \frac{2}{2^{k\alpha}} \left[\frac{\sqrt{12}}{(2m-1)(2m-3)^{\frac{1}{2}}} \right] + \frac{M_2}{2^{\frac{5k+1}{2}}(2m+1)^{\frac{1}{2}}} \frac{2}{2^{k\alpha}} \left[\frac{\sqrt{12}}{(2m-1)(2m-3)^{\frac{1}{2}}} \right] \\
 &\leq \frac{\sqrt{12}(2+M_2)}{2^{\frac{5k+1}{2}}(2m-3)^2} \left(\frac{1}{2^{k\alpha}} + 1 \right)
 \end{aligned}$$

$$\begin{aligned}
 \|S_{2^{k-1},M}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \frac{122 + M_2^2}{2^{5k+1}(2m-3)^4} \left(\frac{1}{2^{k\alpha}} + 1 \right)^2 \\
 &= 12(2 + M_2)^2 \sum_{n=1}^{2^{k-1}} \frac{1}{2^{5k+1}} \left(\frac{1}{2^{k\alpha}} + 1 \right)^2 \sum_{m=M+1}^{\infty} \frac{1}{(2m-3)^4} \\
 &= 12(2 + M_2)^2 \frac{1}{2^{4k+2}} \left(\frac{1}{2^{k\alpha}} + 1 \right)^2 \left[\frac{1}{(2M-1)^4} + \int_{m=M+1}^{\infty} \frac{dm}{(2m-3)^4} \right] \\
 &\leq 12(2 + M_2)^2 \frac{1}{2^{4k+2}} \left(\frac{1}{2^{k\alpha}} + 1 \right)^2 \left[\frac{1}{(2M-1)^4} + \frac{1}{(2M-1)^3} \right] \\
 &\leq 12(2 + M_2)^2 \frac{1}{2^{4k+2}} \left(\frac{1}{2^{k\alpha}} + 1 \right)^2 \left[\frac{2}{(2M-1)^3} \right]
 \end{aligned}$$

$$\begin{aligned}
 E_{2^{k-1},M}^{(5)}(f) &= \|f - S_{2^{k-1},M}(f)\|_2 \\
 &= \sqrt{12}(2 + M_2) \frac{1}{2^{2k+1}} \left(\frac{1}{2^{k\alpha}} + 1 \right) \left[\frac{\sqrt{2}}{(2M-1)^{\frac{3}{2}}} \right] \\
 &= O\left(\frac{1}{(2M-1)^{\frac{3}{2}} 2^{2(k+1)}} \left(1 + \frac{1}{2^{k\alpha}} \right) \right); \quad M \geq 2.
 \end{aligned}$$

5. Conclusions

- 1.(i) $E_{2^{k-1},M}(f) = O\left(\frac{1}{\sqrt{2M+1} 2^k}\right) \rightarrow 0$ as $M \rightarrow \infty, k \rightarrow \infty$.
- (ii) $E_{2^{k-1},0}^{(1)}(f) = O\left(\frac{1}{2^{k-1}} \left(1 + \frac{1}{2^{\alpha(k-1)}}\right)\right) \rightarrow 0$ as $k \rightarrow \infty$.
- (iii) $E_{2^{k-1},M}^{(2)}(f) = O\left(\frac{1}{2^{k+1} \sqrt{2M+1}} \left(1 + \frac{1}{2^{k\alpha}}\right)\right) \rightarrow 0$ as $M \rightarrow \infty, k \rightarrow \infty$.
- (iv) $E_{2^{k-1},0}^{(3)}(f) = O\left(\frac{1}{2^{k-1}} \left(1 + \frac{1}{2^{(\alpha+1)(k-1)}} + \frac{1}{2^{k-1}}\right)\right) \rightarrow 0$ as $k \rightarrow \infty$.
- (v) $E_{2^{k-1},1}^{(4)}(f) = O\left(\frac{1}{2^{2(k-1)}} \left(1 + \frac{1}{2^{\alpha(k-1)}}\right)\right) \rightarrow 0$ as $k \rightarrow \infty$.

$$(vi) E_{2^{k-1},1}^{(5)}(f) = O\left(\frac{1}{(2M+1)^{\frac{3}{2}} 2^{2(k+1)}} \left(1 + \frac{1}{2^{ak}}\right)\right) \rightarrow 0 \quad M \rightarrow \infty, k \rightarrow \infty.$$

Hence, these are best possible estimators in wavelet analysis.

2. Estimator of a function f with $f' \in Lip\alpha$ are

$$(i) E_{2^{k-1},0}^{(1)}(f) = O\left(\frac{1}{2^{k-1}} \left(1 + \frac{1}{2^{a(k-1)}}\right)\right).$$

$$(ii) E_{2^{k-1},M}^{(2)}(f) = O\left(\frac{1}{2^{k+1} \sqrt{2M+1}} \left(1 + \frac{1}{2^{ak}}\right)\right).$$

and estimator of a function f with $f'' \in Lip\alpha$ are

$$(i) E_{2^{k-1},0}^{(3)}(f) = O\left(\frac{1}{2^{k-1}} \left(1 + \frac{1}{2^{(a+1)(k-1)}} + \frac{1}{2^{k-1}}\right)\right).$$

$$(ii) E_{2^{k-1},1}^{(4)}(f) = O\left(\frac{1}{2^{2(k-1)}} \left(1 + \frac{1}{2^{a(k-1)}}\right)\right).$$

$$(iii) E_{2^{k-1},1}^{(5)}(f) = O\left(\frac{1}{(2M+1)^{\frac{3}{2}} 2^{2(k+1)}} \left(1 + \frac{1}{2^{ak}}\right)\right).$$

The estimators of a function f with $f'' \in Lip\alpha$ are better and sharper than the estimator of a function f with $f' \in Lip\alpha$.

3. The estimators of a function having more derivatives are sharper and better than the estimators of function having less derivatives.

6. Acknowledgments

Shyam Lal, one of the authors, is thankful to DST - CIMS for encouragement to this work. Authors are grateful to the editor and reviewers for their valuable comments and suggestions which have improved the quality and presentation of this paper.

References

- [1] A. K. Raghuvanshi, B. K. Singh, R. Kumar, "On the degree of Approximation of function Belonging to the Lip(α , r) Class by $(C,2)$ $(E,1)$ ", Vol.2, No. 11, 2013.
- [2] J. K. Kushwaha, "On the Approximation of Generalized Lipschitz Function by Euler Means of Conjugate Series of Fourier Series", The Scientific World Journal, Vol. 2013, Article ID 508026, pp. 1-4, 2013."
- [3] L. Debnath, Wavelet Transform and their applications, Birkhauser Boston, Massachusetts-2002.
- [4] M. Razzaghi and S. Yousefi, "Legendre wavelets direct method for variational problems," Mathematics and Computers in Simulation, Vol. 53, no. 3, pp. 185-192, 2000.
- [5] R.A.Devore, Nonlinear approximation, Acta Numerica, Vol.7, Cambridge University Press, Cambridge, 1998, pp. 51-150.
- [6] Shyam Lal and Susheel Kumar "Best Wavelet Approximation of functions belonging to Generalized Lipschitz Class using Haar Scaling function", Thai Journal of Mathematics, Vol. 13, no.2 (2015). A., Trigonometric Series Volume I, Cambridge University Press.
- [7] E.C.Titchmarsh, The Theory of Functions, Second Edition, Oxford University Press, (1939).
- [8] Y.Meyer, Wavelets; their past and their future, Progress in Wavelet Analysis and (Applications) (Toulouse, 1992) (Y.Meyer and S.Roques, eds) Frontieres, Gif-sur-Yvette, 1993, pp. 9-18.
- [9] Zygmund A. , Trigonometric Series Volume I, Cambridge University Press, 1959.