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Basic Properties of Unbounded Weighted Conditional Type Operators

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Abstract. In this paper we consider unbounded weighted conditional type (WCT) operators on L^p-space. We provide some conditions under which WCT operators on L^p-spaces are densely defined. Specifically, we obtain a dense subset of their domain. Moreover, we get that a WCT operator is continuous if and only if it is every where defined. A description of polar decomposition, spectrum, spectral radius, normality and hyponormality of WCT operators in this context are provided. Finally, we apply some results of hyperexpansive operators to WCT operators on the Hilbert space $L^2(\Sigma)$. As a consequence hyperexpansive multiplication operators are investigated.

1. Introduction

In the present paper we consider a class of unbounded linear operators on L^p-spaces having the form M_wEM_u, where E is a conditional expectation operator and M_u and M_w are multiplication operators. What follows is a brief review of the operators E and multiplication operators, along with the notational conventions we will be using.

Let (Ω, Σ, μ) be a σ-finite measure space and let $\mathcal{A}$ be a σ-subalgebra of Σ such that $(\Omega, \mathcal{A}, \mu)$ is also σ-finite. We denote the collection of (equivalence classes modulo sets of zero measure of) Σ-measurable complex-valued functions on Ω by $L^0(\Sigma)$ and the support of a function $f \in L^0(\Sigma)$ is defined as $S(f) = \{t \in \Omega; f(t) \neq 0\}$. Moreover, we set $L^p(\Sigma) = L^p(\Omega, \Sigma, \mu)$. We also adopt the convention that all comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. For each σ-finite subalgebra $\mathcal{A}$ of Σ, the conditional expectation, $E^{\mathcal{A}}(f)$, of f with respect to $\mathcal{A}$ is defined whenever $f \geq 0$ or $f \in L^p(\Sigma)$. In any case, $E^{\mathcal{A}}(f)$ is the unique $\mathcal{A}$-measurable function for which

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is an idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. If there is no possibility of confusion we write $E(f)$ in place of $E^{\mathcal{A}}(f)$ [10, 12]. This operator will play a major role in our work and we list here some of its useful properties:

- If g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.

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- $|E(f)|^p \leq E(|f|^p)$.
- If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- $|E(fg)| \leq E(|f|^p)|^{\frac{1}{p}} E(|g|^q)|^{\frac{1}{q}}$, (Hölder inequality) for all $f \in L^p(\Sigma)$ and $g \in L^q(\Sigma)$, in which $\frac{1}{p} + \frac{1}{q} = 1$.
- For each $f \geq 0$, $S(f) \subseteq S(E(f))$.

Let $u \in L^0(\Sigma)$. The corresponding multiplication operator M_u on $L^p(\Sigma)$ is defined by $f \to uf$. Our interest in operators of the form $M_w E M_u$ stems from the fact that such products tend to appear often in the study of those operators related to conditional expectation. This observation was made in [1, 2, 5, 8, 9]. In this paper, first we investigate some properties of unbounded weighted conditional type operators on the space $L^p(\Sigma)$ and then, we apply some results of hyperexpansive operators to WCT operators on the Hilbert space $L^2(\Sigma)$. As a consequence hyperexpansive multiplication operators are investigated.

2. Unbounded weighted conditional type operators

Let X stand for a Banach space and $\mathcal{B}(X)$ for the Banach algebra of all linear operators on X. By an operator in X we understand a linear mapping $T : \mathcal{D}(T) \subseteq X \to X$ defined on a linear subspace $\mathcal{D}(T)$ of X which is called the domain of T. The linear map T is called densely defined if $\mathcal{D}(T)$ is dense in X and it is called closed if its graph $\mathcal{G}(T)$ is closed in $X \times X$, where $\mathcal{G}(T) = \{(f, Tf) : f \in \mathcal{D}(T)\}$. We studied bounded weighted conditional type operators on L^p-spaces in [4]. Also we investigated unbounded weighted conditional type operators of the form EM_u on the Hilbert space $L^2(\Sigma)$ in [3]. Here we consider unbounded weighted conditional type operators of the form of $M_w E M_u$ on $L^p(\Omega, \Sigma, \mu)$, in which (Ω, Σ, μ) is a σ-finite measure space. Let f be a positive Σ-measurable function on Ω. Define the measure $\mu_f : \Sigma \to [0, \infty]$ by

$$\mu_f(E) = \int_E f d\mu, \quad E \in \Sigma.$$

It is clear that the measure μ_f is also σ-finite, since μ is σ-finite. From now on we assume that u and w are conditionable (i.e., $E(u)$ and $E(w)$ are defined). Operators of the form of $M_w E M_u(f) = wE(u.f)$ acting in $L^p(\mu)$ with $\mathcal{D}(M_w E M_u) = \{f \in L^p(\mu) : u.f \in \mathcal{D}(E), \quad wE(u.f) \in L^p(\mu)\}$ are called weighted conditional type operators (or briefly WCT operators). In the first proposition we provide a condition under which the WCT operator $M_w E M_u$ is densely defined on L^p-spaces.

Theorem 2.1. *Let $1 \leq p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $E(|w|^p)^{\frac{1}{p}} E(|u|^q)^{\frac{1}{q}} < \infty$ a.e. Then the linear transformation $M_w E M_u$ is densely defined on $L^p(\Omega, \Sigma, \mu)$.*

Proof. For each $n \in \mathbb{N}$, define

$$A_n = \{t \in \Omega : n - 1 \leq E(|w|^p)(t) E(|u|^q)^{\frac{p}{q}}(t) < n\}.$$

It is clear that each A_n is an $\mathcal{A}$-measurable set and Ω is expressible as the disjoint union of $\{A_n\}_{n=1}^{\infty}$, $\Omega = \cup_{n=1}^{\infty} A_n$.
Let $f \in L^p(\Sigma)$ and $\epsilon > 0$. Then, there exists $N > 0$ such that

$$\int_{\cup_{n=N}^{\infty} A_n} |f|^p d\mu = \sum_{n=N}^{\infty} \int_{A_n} |f|^p d\mu < \epsilon.$$

Define the sets

$$B_N = \cup_{n=N}^{\infty} A_n, \qquad C_N = \cup_{n=1}^{N-1} A_n.$$

Then, $\int_{B_N} |f|^p d\mu < \epsilon$ and $C_N = \{t \in \Omega : E(|w|^p)(t) E(|u|^q)^{\frac{p}{q}}(t) < N - 1\}$. Next, we define $g = f.\chi_{C_N}$. Clearly $g \in L^p(\Sigma)$ and $E(g) = E(f).\chi_{C_N}$. Now, we show that $g \in \mathcal{D} = \mathcal{D}(M_w E M_u)$. By an straightforward calculations we have

$$\begin{aligned}\int_\Omega |wE(ug)|^p d\mu &= \int_\Omega |wE(uf)\chi_{C_N}|^p d\mu \\ &= \int_{C_N} E(|w|^p)|E(uf)|^p d\mu \\ &\leq \int_{C_N} E(|w|^p)E(|u|^q)^{\frac{p}{q}}|f|^p d\mu \\ &\leq (N-1)\int_{C_N} |f|^p d\mu < \infty.\end{aligned}$$

Thus, $wE(uf) \in L^p(\Sigma)$. Now, we show that $\|g - f\|_p < \epsilon$:

$$\begin{aligned}\|g-f\|_p^p &= \int_X |g-f|^p d\mu \\ &= \int_{C_N} |g-f|^p d\mu < \epsilon.\end{aligned}$$

Therefore $\mathcal{D}$ is dense in $L^p(\Sigma)$. □

Here we obtain a dense subset of $L^p(\mu)$ that we need it to proof our next results.

Lemma 2.2. *Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $J = 1 + E(|w|^p)E(|u|^q)^{\frac{p}{q}}$, $E(|w|^p)^{\frac{1}{p}}E(|u|^q)^{\frac{1}{q}} < \infty$ a.e, μ, and $d\nu = Jd\mu$. Then we get that $S(J) = \Omega$ and*
(i) $L^p(\nu) \subseteq \mathcal{D}(M_w EM_u)$,
(ii) $\overline{L^p(\nu)}^{\|.\|_\mu} = \overline{\mathcal{D}(M_w EM_u)}^{\|.\|_\mu} = L^p(\mu)$.

Proof. Let $f \in L^p(\nu)$. Then

$$\|f\|^p d\mu \leq \|f\|_\nu^p < \infty,$$

and so $f \in L^p(\mu)$. Also, by conditional-type Hölder-inequality we have

$$\begin{aligned}\|M_w EM_u(f)\|^p d\mu &\leq \int_\Omega E(|w|^p)E(|u|^q)^{\frac{p}{q}}E(|f|^p)d\mu \\ &= \int_\Omega E(|w|^p)E(|u|^q)^{\frac{p}{q}}|f|^p d\mu \\ &\leq \|f\|_\nu^p < \infty.\end{aligned}$$

This implies that $f \in \mathcal{D}(M_w EM_u)$. Now we prove that $L^p(\nu)$ is dense in $L^p(\mu)$. By Riesz representation theorem we have

$$(L^p(\nu))^\perp = \{g \in L^q(\mu) : \int_\Omega f.g d\mu = 0, \quad \forall f \in L^p(\nu)\}.$$

Suppose that $g \in (L^p(\nu))^\perp$. For $A \in \Sigma$ we set $A_n = \{t \in A : J(t) \leq n\}$. It is clear that $A_n \subseteq A_{n+1}$ and $\Omega = \cup_{n=1}^\infty A_n$. Also, Ω is σ-finite, hence $\Omega = \cup_{n=1}^\infty \Omega_n$ with $\mu(\Omega_n) < \infty$. If we set $B_n = A_n \cap \Omega_n$, then $B_n \nearrow A$ and so $g.\chi_{B_n} \nearrow g.\chi_A$ a.e. μ. Since $\nu(B_n) \leq (n+1)\mu(B_n) < \infty$, we have $\chi_{B_n} \in L^p(\nu)$ and then by our assumptions we have $\int_{B_n} f d\mu = 0$. Therefore by Fatou's lemma we get that $\int_A g d\mu = 0$. Consequently, for all $A \in \Sigma$ we have $\int_A g d\mu = 0$. This means that $g = 0$ a.e. μ and so $L^p(\nu)$ is dense in $L^p(\mu)$. □

By the Lemma 2.2 we get that $L^p(\nu)$ is a core of $M_w EM_u$. Here we give a condition that we will use it in the next theorem.

(★) If $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space and $J - 1 = (E(|u|^q))^{\frac{p}{q}}E(|w|^p) < \infty$ a.e. μ, then there exists a sequence $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$ such that $\mu(A_n) < \infty$ and $J - 1 < n$ a.e. μ on A_n for every $n \in \mathbb{N}$ and $A_n \nearrow \Omega$ as $n \to \infty$.

Theorem 2.3. *If $u, w : \Omega \to \mathbb{C}$ are Σ-measurable and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then the following are equivalent:*

(i) $M_w E M_u$ is densely defined on $L^p(\Sigma)$,

(ii) $J - 1 = E(|w|^p)(E(|u|^q))^{\frac{p}{q}} < \infty$ a.e., μ.

(iii) $\mu_{J-1} |_{\mathcal{A}}$ is σ-finite.

Proof. $(i) \to (ii)$ Let $E = \{E(|w|^p)(E(|u|^q))^{\frac{p}{q}} = \infty\}$. Clearly, we have $f |_E = 0$ a.e., μ for every $f \in L^p(\nu)$. This implies that $f.J |_E = 0$ a.e. So we have $J.\chi_{A \cap E} = 0$ a.e., μ for all $A \in \Sigma$, with $\mu(A) < \infty$. By σ-finiteness of μ, we have $J.\chi_E = 0$ a.e., μ. Since $S(J) = \Omega$, we get that $\mu(E) = 0$.
$(ii) \to (i)$ Evident.

$(ii) \to (iii)$ Let $\{A_n\}_{n=1}^{\infty}$ be in ($\bigstar$). We have

$$\mu_{J-1} |_{\mathcal{A}} (A_n) = \int_{A_n} E(|w|^p)(E(|u|^q))^{\frac{p}{q}} d\mu \leq n\mu(A_n) < \infty, \quad n \in \mathbb{N}.$$

This yields (iii).

$(iii) \to (i)$ Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $A_n \nearrow \Omega$ as $n \to \infty$ and $\mu_{J-1} |_{\mathcal{A}} (A_n) < \infty$, for every $k \in \mathbb{N}$. It follows from the definition of μ_{J-1} that $J - 1 = E(|w|^p)(E(|u|^q))^{\frac{p}{q}} < \infty$ a.e., μ on Ω. Applying Theorem 2.1, we get (i). □

Let X, Y be Banach spaces and $T : X \to Y$ be a linear operator. If T is densely defined, then there is a unique maximal operator T^* from $\mathcal{D}(T^*) \subset Y^*$ into X^* such that

$$y^*(Tx) = \langle Tx, y^* \rangle = \langle x, T^* y^* \rangle = T^* y^*(x), \quad x \in \mathcal{D}(T), \quad y^* \in \mathcal{D}(T^*).$$

T^* is called the adjoint of T.
Riesz representation theorem for L^p- spaces states that $\langle f, F \rangle = F(f) = \int_{\Omega} f \bar{F} d\mu$, when $f \in L^p(\Sigma)$, $F \in L^q(\Sigma) = (L^p(\Sigma))^*$ and $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 2.3 easily we get that the operator $M_w E M_u$ is densely defined if and only if the operator $M_{\bar{u}} E M_{\bar{w}}$ is densely defined. In the next proposition we obtain the adjoint of the WCT operator $M_w E M_u$ on the Banach space $L^p(\Sigma)$.

Proposition 2.4. *If the linear transformation $T = M_w E M_u$ is densely defined on $L^p(\Sigma)$, then $M_{\bar{u}} E M_{\bar{w}}$ is a densely defined operators on $L^q(\Sigma)$ and $T^* = M_{\bar{u}} E M_{\bar{w}}$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Let $f \in \mathcal{D}(T)$ and $g \in \mathcal{D}(T^*)$. Then we have

$$\begin{aligned} \langle Tf, g \rangle &= \int_{\Omega} wE(uf)\bar{g} d\mu \\ &= \int_{\Omega} fuE(w\bar{g}) d\mu \\ &= \langle f, M_{\bar{u}} E M_{\bar{w}} g \rangle. \end{aligned}$$

Therefore $T^* = M_{\bar{u}} E M_{\bar{w}}$. □

In the next proposition we prove that every densely defined WCT operator is closed.

Proposition 2.5. *If $(E(|u|^q))^{\frac{p}{q}} E(|w|^p) < \infty$ a.e., μ, then the linear transformation $M_w E M_u : \mathcal{D}(M_w E M_u) \to L^p(\Sigma)$ is closed.*

Proof. Assume that $f_n \in \mathcal{D}(M_wEM_u)$, $f_n \to f$, $wE(uf_n) \to g$, and let $h \in \mathcal{D}(M_{\bar{u}}EM_{\bar{w}})$. Then

$$\begin{aligned}\langle f, M_{\bar{u}}EM_{\bar{w}}h\rangle &= \lim_{n\to\infty}\langle f_n, M_{\bar{u}}EM_{\bar{w}}h\rangle \\ &= \lim_{n\to\infty}\langle wE(uf_n), h\rangle = \langle g, h\rangle.\end{aligned}$$

This calculation (which uses the continuity of the inner product and the fact that $f_n \in \mathcal{D}(M_wEM_u)$) shows that $f \in \mathcal{D}(M_wEM_u)$ and $wE(uf) = g$, as required. □

In the next theorem we provide an equivalent condition to continuity of WCT operator M_wEM_u.

Theorem 2.6. *If $(E(|u|^q))^{\frac{p}{q}}E(|w|^p) < \infty$ a.e., μ, then the WCT operator $M_wEM_u : \mathcal{D}(M_wEM_u) \to L^p(\Sigma)$ is continuous if and only if it is every where defined i.e., $\mathcal{D}(M_wEM_u) = L^p(\Sigma)$.*

Proof. Let M_wEM_u be continuous. By Lemma 2.2 it is closed. Hence easily we get that $\mathcal{D}(M_wEM_u)$ is closed and so $\mathcal{D}(M_wEM_u) = L^p(\Sigma)$. The converse is easy by closed graph theorem. □

We denote the range of the operator T as $\mathcal{R}(T)$ i.e., $\mathcal{R}(T) = \{T(x) : x \in \mathcal{D}(T)\}$.

Proposition 2.7. *If $E(|u|^2)E(|w|^2) < \infty$ a.e., μ and $M_wEM_u : \mathcal{D}(M_wEM_u) \subset L^2(\Sigma) \to L^2(\Sigma)$, then $\mathcal{R}(M_wEM_u)$ is closed if and only if $\mathcal{R}(M_{\bar{u}}EM_{\bar{w}})$ is closed.*

Proof. Let $P_1 : L^2(\Sigma) \times L^2(\Sigma) \to \mathcal{G}(M_wEM_u)$ be a projection and $P_2 : L^2(\Sigma) \times L^2(\Sigma) \to \{0\} \times L^2(\Sigma)$ be the canonical projection. It is clear that $\mathcal{R}(M_wEM_u) \cong \mathcal{R}(P_2P_1)$. Also, $\mathcal{R}(M_{\bar{u}}EM_{\bar{w}}) \cong \mathcal{R}((I - P_2)(I - P_1))$. Since P_1 and P_2 are orthogonal projections, then $\mathcal{R}(P_2P_1)$ is closed if and only if $\mathcal{R}((I - P_2)(I - P_1))$. Thus we obtain the desired result. □

It is well-known that for a densely defined closed operator T from $\mathcal{H}_1$ into $\mathcal{H}_2$, there exists a partial isometry U_T with initial space $\mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)} = \overline{\mathcal{R}(|T|)}$ and final space $\mathcal{N}(T^*)^\perp = \overline{\mathcal{R}(T)}$ such that

$$T = U_T|T|.$$

Now we are going to find the polar decomposition of WCT operator M_wEM_u on the Hilbert space $L^2(\Sigma)$.

Theorem 2.8. *Let M_wEM_u be densely defined on $L^2(\Sigma)$ and $M_wEM_u = U|M_wEM_u|$ be its polar decomposition. Then*

(i) $|M_wEM_u| = M_{u'}EM_u$, where $u' = (\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}.\chi_S.\bar{u}$ and $S = S(E(|u|^2))$,

(ii) $U = M_{w'}EM_u$, where $w' : \Omega \to \mathbb{C}$ is an a.e. μ well-defined Σ-measurable function such that

$$w' = \frac{w}{(E(|w|^2)E(|u|^2))^{\frac{1}{2}}}.\chi_{S\cap G},$$

in which $G = S(E(|w|^2))$.

Proof. (i). For every $f \in \mathcal{D}(M_{u'}EM_u)$, we have

$$\|M_{u'}EM_u(f)(f)\|^2 = \||M_wEM_u|(f)\|^2.$$

Also, by Lemma 2.2 we conclude that $\mathcal{D}(M_{u'}EM_u) = \mathcal{D}(|M_wEM_u|)$ and it is easily seen that $M_{u'}EM_u$ is a positive operator. These observations imply that $|M_wEM_u| = M_{u'}EM_u$.
(ii). For $f \in L^2(\Sigma)$ we have

$$\int_\Omega |w'E(uf)|^2 d\mu = \int_\Omega \frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}|wE(uf)|^2 d\mu,$$

which implies that the operator $M_{w'}EM_u$ is well-defined and $\mathcal{N}(M_wEM_u) = \mathcal{N}(M_{w'}EM_u)$. Also, for $f \in \mathcal{D}(M_wEM_u) \ominus \mathcal{N}(M_wEM_u)$ we have

$$U(|M_wEM_u|(f)) = wE(uf).\chi_{S\cap G} = wE(uf).$$

Thus $\|U(f)\| = \|f\|$ for all $f \in \mathcal{R}(|M_wEM_u|)$ and since U is a contraction, then it holds for all $f \in \mathcal{N}(M_wEM_u)^{\perp} = \overline{\mathcal{R}(|M_wEM_u|)}$. □

Here we remind that: if $T : \mathcal{D}(T) \subset X \to X$ is a closed linear operator on the Banach space X, then a complex number λ belongs to the resolvent set $\rho(T)$ of T, if the operator $\lambda I - T$ has a bounded everywhere on X defined inverse $(\lambda I - T)^{-1}$, called the resolvent of T at λ and denoted by $R_{\lambda}(T)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the spectrum of the operator T.
It is known that, if a, b are elements of a unital algebra A, then $1 - ab$ is invertible if and only if $1 - ba$ is invertible. A consequence of this equivalence is that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$. Now, in the next theorem we compute the spectrum of WCT operator M_wEM_u as a densely defined operator on $L^2(\Sigma)$.

Proposition 2.9. *Let M_wEM_u be densely defined and $\mathcal{A} \subsetneq \Sigma$. Then*

(i) essrange(E(uw)) $\setminus\{0\} \subseteq \sigma(M_wEM_u)$.

(ii) If $L^2(\mathcal{A}) \subseteq \mathcal{D}(EM_{uw})$, then $\sigma(M_wEM_u) \setminus \{0\} \subseteq$ essrange(E(uw)) $\setminus\{0\}$.

Proof. Since $\sigma(M_wEM_u) \setminus \{0\} = \sigma(EM_uM_w) \setminus \{0\} = \sigma(EM_{uw}) \setminus \{0\}$, then by Theorem 2.8 of [3] we get the proof. □

By a similar method that we used in the proof of Theorem 2.8 of [3] we have the same assertion for the spectrum of the densely defined operator EM_u on the space $L^p(\Sigma)$, i.e.,
(i) essrange(E(u)) $\cup\{0\} \subseteq \sigma(EM_u)$.

(ii) If $L^p(\mathcal{A}) \subseteq \mathcal{D}(EM_u)$, then $\sigma(EM_u) \subseteq$ essrange(E(u)) $\cup\{0\}$.
By these observations we have the next remark.

Remark 2.10. *Let M_wEM_u be densely defined operator on $L^p(\Sigma)$ and $\mathcal{A} \subsetneq \Sigma$. Then*

(i) essrange(E(uw)) $\setminus\{0\} \subseteq \sigma(M_wEM_u)$.

(ii) If $L^p(\mathcal{A}) \subseteq \mathcal{D}(EM_{uw})$, then $\sigma(M_wEM_u) \setminus \{0\} \subseteq$ essrange(E(uw)) $\setminus\{0\}$.

As we know the spectral radius of a densely defined operator T is denoted by $r(T)$ and is defined as: $r(T) = \sup_{\lambda\in\sigma(T)} |\lambda|$. Hence we have the next corollary.

Corollary 2.11. *If the WCT operator M_wEM_u is densely defined on $L^p(\Sigma)$ and $L^p(\mathcal{A}) \subseteq \mathcal{D}(EM_{uw})$, then $\sigma(M_wEM_u) \setminus \{0\} =$ essrange(E(uw)) $\setminus\{0\}$ and $r(M_wEM_u) = \|E(uw)\|_{\infty}$.*

A densely defined operator T on the Hilbert space $\mathcal{H}$ is said to be *hyponormal* if $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ and $\|T^*(f)\| \leq \|T(f)\|$, for all $f \in \mathcal{D}(T)$. Also, it is to be *normal* if T is closed and $T^*T = TT^*$. For the WCT operator $T = M_wEM_u$ on $L^2(\Sigma)$ we have $T^* = M_{\bar{u}}EM_{\bar{w}}$ and we recall that T is densely defined if and only if T^* is densely defined. If T is densely defined, then by the Lemma 2.2 we get that $L^2(\nu) \subseteq \mathcal{D}(T)$, $L^2(\nu) \subseteq \mathcal{D}(T^*)$ and

$$\overline{L^2(\nu)}^{\|.\|_\mu} = \overline{\mathcal{D}(T)}^{\|.\|_\mu} = \overline{\mathcal{D}(T^*)}^{\|.\|_\mu} = L^2(\mu),$$

in which $d\nu = Jd\mu$ and $J = 1 + E(|w|^2)E(|u|^2)$. Also, we have $T^*T = M_{E(|w|^2)\bar{u}}EM_u$ and $TT^* = M_{E(|u|^2)w}EM_{\bar{w}}$. Similarly, we have $L^2(\nu') \subseteq \mathcal{D}(T^*T)$, $L^2(\nu') \subseteq \mathcal{D}(TT^*)$ and

$$\overline{L^2(\nu')}^{\|.\|_\mu} = \overline{\mathcal{D}(T^*T)}^{\|.\|_\mu} = \overline{\mathcal{D}(TT^*)}^{\|.\|_\mu} = L^2(\mu),$$

in which $d\nu' = J'd\mu$ and $J' = 1 + (E(|w|^2))^2(E(|u|^2))^2$. By these observations we have next assertions.

Proposition 2.12. *Let WCT operator M_wEM_u be densely defined on $L^2(\Sigma)$. Then we have the followings:*

(i) If $u(E(|w|^2))^{\frac{1}{2}} = \bar{w}(E(|u|^2))^{\frac{1}{2}}$ with respect to the measure μ, then $T = M_wEM_u$ is normal.

(ii) If $T = M_wEM_u$ is normal, then $E(|w|^2)|E(u)|^2 = E(|u|^2)|E(w)|^2$ with respect to the measure μ.

Proof. (i) Direct computations show that

$$T^*T - TT^* = M_{\bar{u}E(|w|^2)}EM_u - M_{wE(|u|^2)}EM_{\bar{w}},$$

on $L^2(\nu')$. Hence for every $f \in L^2(\nu')$,

$$\begin{aligned}\langle T^*T - TT^*(f), f\rangle &= \int_X E(|w|^2)E(uf)\bar{u}\bar{f} - E(|u|^2)E(\bar{w}f)w\bar{f}d\mu \\ &= \int_X |E(u(E(|w|^2))^{\frac{1}{2}}f)|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w}f)|^2 d\mu.\end{aligned}$$

This implies that if

$$(E(|u|^2))^{\frac{1}{2}}\bar{w} = u(E(|w|^2))^{\frac{1}{2}},$$

then $\langle T^*T - TT^*(f), f\rangle = 0$, for all $f \in L^2(\nu')$. Thus $T^*T = TT^*$.

(ii) Let T be normal. By (i), we have

$$\int_X |E(u(E(|w|^2))^{\frac{1}{2}}f)|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w}f)|^2 d\mu = 0,$$

for all $f \in L^2(\nu')$. Now, let $A \in \mathcal{A}$, with $0 < \nu'(A) < \infty$. By replacing f with χ_A, we have

$$\int_A |E(u(E(|w|^2))^{\frac{1}{2}})|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w})|^2 d\mu = 0$$

and so

$$\int_A |E(u)|^2E(|w|^2) - |E(w)|^2E(|u|^2)d\mu = 0.$$

Since $A \in \mathcal{A}$ is arbitrary and $\mu \ll \nu'$ (absolutely continuous), then $|E(u)|^2E(|w|^2) = |E(w)|^2E(|u|^2)$ with respect to μ. □

In the next proposition we obtain some necessary and sufficient conditions for hyponormality of WCT operators.

Proposition 2.13. *Let the WCT operator M_wEM_u be densely defined on $L^2(\Sigma)$. Then we have the followings:*

(i) If $u(E(|w|^2))^{\frac{1}{2}} \geq \bar{w}(E(|u|^2))^{\frac{1}{2}}$ with respect to μ, then $T = M_wEM_u$ is hyponormal.

(ii) If $T = M_wEM_u$ is hyponormal, then $E(|w|^2)|E(u)|^2 \geq E(|u|^2)|E(w)|^2$ with respect to the measure μ.

Proof. By a similar method of 2.12 we can get the proof. □

If we set $w \equiv 1$, then we have the next remark.

Remark 2.14. *Let EM_u be a densely defined operator on $L^2(\Sigma)$. Then EMu is norma if and only if $u \in L^0(\mathcal{A})$ if and only if EM_u is hyponormal.*

3. Hyperexpansive WCT operators

In this section we provide some conditions under which WCT operator M_wEM_u on $L^2(\Sigma)$ is k-isometry, k-expansive, k-hyperexpansive and completely hyperexpansive. For an operator T on the Hilbert space $\mathcal{H}$ we set

$$\Theta_{T,n}(f) = \sum_{0\leq i\leq n}(-1)^i \left(\begin{array}{c} n \\ i \end{array}\right)\|T^i(f)\|^2, \quad f \in \mathcal{D}(T^n), \ n \geq 1.$$

By means of this definition an operator T on $\mathcal{H}$ is said to be:

(i) k-isometry ($k \geq 1$) if $\Theta_{T,k}(f) = 0$, for $f \in \mathcal{D}(T^k)$,

(ii) k-expansive ($k \geq 1$) if $\Theta_{T,k}(f) \leq 0$, for $f \in \mathcal{D}(T^k)$,

(iii) k-hyperexpansive ($k \geq 1$) if $\Theta_{T,n}(f) \leq 0$, for $f \in \mathcal{D}(T^n)$ and $n = 1, 2, ..., k$.

(iv) completely hyperexpansive if $\Theta_{T,n}(f) \leq 0$, for $f \in \mathcal{D}(T^n)$ and $n \geq 1$.

For more details one can see [6, 7, 11]. It is easily seen that for each $f \in L^2(\Sigma)$,

$$\|M_wEM_u(f)\|_2 = \|EM_v(f)\|_2,$$

where $v = u(E(|w|^2))^{\frac{1}{2}}$.

Let $T_1 = M_wEM_u$ and $T_2 = EM_v$. By the above information we have, T_1 is k-isomerty if and only if T_2 is k-isometry, T_1 is k-expansive if and only if T_2 is k-expansive, T_1 is k-hyperexpansive if and only if T_2 is k-hyperexpansive and T_1 is completely hyperexpansive if and only if T_2 is completely hyperexpansive. Thus without loss of generality we can consider the operator EM_v instead of M_wEM_u in our discussion. Now we present our main results. The next lemma is a direct consequence of Theorem 2.3.

Lemma 3.1. *For every $n \in \mathbb{N}$ the operator $(EM_v)^n$ on $L^2(\Sigma)$ is densely-defined if and only if the operator EM_v is densely defined on $L^2(\Sigma)$.*

In the Theorem 3.2 we give some necessary and sufficient conditions for k-isometry and k-expansive WCT operators EM_v.

Theorem 3.2. *Let $\mathcal{D}(EM_v)$ be dense in $L^2(\mu)$. Then we have the followings.*

(i) If the operator EM_v is k-isometry ($k \geq 1$), then $A^0_k(|E(v)|^2) = 0$, a.e.

(ii) If $(1 + E(|v|^2)A^1_k(|E(v)|^2)) = 0$, a.e., and $|E(vf)|^2 = E(|v|^2)E(|f|^2)$, a.e., for all $f \in \mathcal{D}(EM_v)$, then the operator EM_v is k-isometry.

(iii) If the operator EM_v is k-expansive, then $A^0_k(|E(v)|^2) \leq 0$, a.e.

(iv) If $(1 + E(|v|^2)A^1_k(|E(v)|^2)) \leq 0$, a.e., and $|E(vf)|^2 = E(|v|^2)E(|f|^2)$, a.e., for each $f \in \mathcal{D}(EM_v)$, then the operator EM_v is k-expansive, in which

$$A^0_k(|E(v)|^2) = \sum_{0\leq i\leq k}(-1)^i \left(\begin{array}{c} k \\ i \end{array}\right)|E(v)|^{2i}, \quad A^1_k(|E(v)|^2) = \sum_{1\leq i\leq k}(-1)^i \left(\begin{array}{c} k \\ i \end{array}\right)|E(v)|^{2(i-1)}.$$

Proof. Suppose that the operator EM_v is k-isometry. So for all $f \in \mathcal{D}((EM_v)^k)$ we have

$$\begin{aligned} 0 &= \Theta_{T,k}(f) \\ &= \sum_{0\leq i\leq k} (-1)^i \binom{n}{i} \|(EM_v)^i(f)\|^2 \\ &= \int_\Omega |f|^2 d\mu + \sum_{1\leq i\leq k} (-1)^i \binom{n}{i} \int_\Omega |E(v)|^{2(i-1)} |E(vf)|^2 d\mu. \end{aligned}$$

Hence for all $\mathcal{A}$-measurable functions $f \in \mathcal{D}((EM_v)^k)$

$$\begin{aligned} 0 &= \int_\Omega |f|^2 d\mu + \sum_{1\leq i\leq k} (-1)^i \binom{n}{i} \int_\Omega |E(v)|^{2(i-1)} |E(v)|^2 |f|^2 d\mu \\ &= \int_\Omega \left(\sum_{0\leq i\leq k} (-1)^i \binom{n}{i} |E(v)|^{2i} \right) |f|^2 d\mu. \end{aligned}$$

Since $(EM_v)^k$ is densely defined, then we get that $A_k(|E(v)|^2) = 0$, a.e.
(ii) Let $1 + E(|v|^2)A^1_k(|E(v)|^2) = 0$ and $|E(vf)|^2 = E(|v|^2)E(|f|^2)$, a.e., for all $f \in \mathcal{D}((EM_v)^k)$. Then for each $f \in \mathcal{D}((EM_v)^k)$,

$$\begin{aligned} \Theta_{T,k}(f) &= \sum_{0\leq i\leq k} (-1)^i \binom{n}{i} \|(EM_v)^i(f)\|^2 \\ &= \int_\Omega |f|^2 d\mu + \sum_{1\leq i\leq k} (-1)^i \binom{n}{i} \int_\Omega |E(v)|^{2(i-1)} |E(vf)|^2 d\mu \\ &= \int_\Omega |f|^2 d\mu + \int_\Omega \left(\sum_{1\leq i\leq k} (-1)^i \binom{n}{i} (E(|v|^2))^{2(i-1)} \right) E(|v|^2)E(|f|^2) d\mu \\ &= \int_\Omega (1 + E(|v|^2)A_k(|E(v)|^2))|f|^2 d\mu \\ &= 0. \end{aligned}$$

This implies that the operator EM_v is k-isometry.
(iii), (iv). By the same method that is used in (i) and (ii), easily we get (iii) and (iv).
□

Here we recall that if the linear transformation $T = EM_v$ is densely defined on $L^2(\Sigma)$, then $T = EM_v$ is closed and $T^* = M_{\bar{v}}E$. Also, if $\mathcal{D}(EM_v)$ is dense in $L^2(\Sigma)$ and v is almost every where finite valued, then the operator EM_v is normal if and only if $v \in L^0(\mathcal{A})$ [3]. Hence we have the Remark 3.3 for normal WCT operators.

Remark 3.3. *Suppose that the operator EM_v is normal and $\mathcal{D}(EM_v)$ is dense in $L^2(\mu)$, for a fixed $k \geq 1$. If $|E(f)|^2 = E(|f|^2)$, a.e., on $S(v)$ for all $f \in \mathcal{D}((EM_v)^k)$, then:*

(i) The operator EM_v is k-isometry ($k \geq 1$) if and only if $A_k(|v|^2) = 0$, a.e,;

(ii) The operator EM_v is k-expansive if and only if $A_k(|v|^2) \leq 0$, a.e.

Proof. Since EM_v is normal, then $|E(v)|^2 = E(|v|^2) = |v|^2$, a.e. Thus by Theorem 3.2 we have (i) and (ii). □

Here we give some properties of 2-expansive WCT operators and as a corollary for 2-expansive multiplication operators.

Proposition 3.4. *If $\mathcal{D}(EM_v)$ is dense in $L^2(\mu)$ and EM_v is 2-expansive, then:*

(i) EM_v leaves its domain invariant:

(ii) $|E(v)|^{2k} \geq |E(v)|^{2(k-1)}$ a.e., μ, for all $k \geq 1$.

Proof. (i). Since EM_v is 2-expansive, we get that for each $f \in \mathcal{D}(EM_v)$,

$$\begin{aligned}\|(EM_v)^2(f)\|^2 &= \int_\Omega |E(v)|^2|E(vf)|^2 d\mu \\ &\leq 2\int_\Omega |E(vf)|^2 d\mu - \int_\Omega |f|^2 d\mu \\ &< \infty,\end{aligned}$$

so $EM_v(f) \in \mathcal{D}(EM_v)$.

(ii) Since EM_v leaves its domain invariant, then $\mathcal{D}(EM_v) \subseteq \mathcal{D}^\infty(EM_v)$. So by lemma 3.2 (iii) of [7] we get that $\|(EM_v)^k(f)\|^2 \geq \|(EM_v)^{k-1}(f)\|^2$, for all $f \in \mathcal{D}(EM_v)$ and $k \geq 1$. Also, we have

$$\int_\Omega |E(v)|^{2(k-1)}|E(vf)|^2 d\mu \geq \int_\Omega |E(v)|^{2(k-2)}|E(vf)|^2 d\mu.$$

Hence

$$\int_\Omega (|E(v)|^{2(k-1)} - |E(v)|^{2(k-2)})|E(vf)|^2 d\mu \geq 0,$$

for all $f \in \mathcal{D}(EM_v)$. This leads to $|E(v)|^{2k} \geq |E(v)|^{2(k-1)}$ a.e., μ. □

Corollary 3.5. *If $\mathcal{D}(M_v)$ is dense in $L^2(\mu)$ and M_v is 2-expansive, then we have:*

(i) M_v leaves its domain invariant:

(ii) $v^{2k} \geq v^{2(k-1)}$ a.e. μ for all $k \geq 1$.

Recall that a real-valued map ϕ on $\mathbb{N}$ is said to be completely alternating if $\sum_{0\leq i\leq n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} \phi(m+i) \leq 0$ for all $m \geq 0$ and $n \geq 1$. The next remark is a direct consequence of Lemma 3.1 and Theorem 3.2.

Remark 3.6. *If $\mathcal{D}(EM_v)$ is dense in $L^2(\mu)$ and $k \geq 1$ is fixed, then:*

(i) If the operator EM_v is k-hyperexpansive ($k \geq 1$), then $A^0_n(|E(v)|^2) \leq 0$ for $n = 1, 2, ..., k$;

(ii) If $(1 + E(|v|^2)A^1_n(|E(v)|^2)) \leq 0$ and $|E(vf)|^2 = E(|v|^2)E(|f|^2)$ for all $f \in \mathcal{D}(EM_v)^n$ and $n = 1, 2, ..., k$, then the operator EM_v is k-hyperexpansive ($k \geq 1$);

(iii) If the operator EM_v is completely hyperexpansive, then

(a) the sequence $\{|E(v)(t)|^2\}^\infty_{n=0}$ is a completely alternating sequence for almost every $t \in \Omega$,

(b) $A_n^0(|E(v)|^2) \leq 0$ *for* $n \geq 1$.

(iv) If $(1 + E(|v|^2)A_n^1(|E(v)|^2)) \leq 0$ *and* $|E(vf)|^2 = E(|v|^2)E(|f|^2)$ *for all* $f \in \mathcal{D}((EM_v)^n)$ *and* $n \geq 1$, *then the operator* EM_v *is completely hyperexpansive.*

By Remark 3.6 and some properties of normal WCT operators we get the next remark for k-hyperexpansive and completely hyperexpansive normal WCT operators.

Remark 3.7. *Let the operator* EM_v *be normal,* $\mathcal{D}(EM_v)$ *be dense in* $L^2(\mu)$ *and* $k \geq 1$ *be fixed. If* $|E(f)|^2 = E(|f|^2)$ *on* $S(v)$ *for all* $f \in \mathcal{D}((EM_v)^k)$, *then*

(i) EM_v *is k-hyperexpansive* $(k \geq 1)$ *if and only if* $A_n(|v|^2) \leq 0$ *for* $f \in \mathcal{D}(T^n)$ *and* $n = 1, 2, ..., k$.

(ii) EM_v *is completely hyperexpansive if and only if the sequence* $\{|u(t)|^2\}_{n=0}^{\infty}$ *is a completely alternating sequence for almost every* $t \in \Omega$,

If all functions v^{2i} for $i = 1, ..., n$ are finite valued, then we set

$$\triangle_{v,n}(x) = \sum_{0 \leq i \leq n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} |v|^{2i}(t).$$

Also, if $\mathcal{A} = \Sigma$, then $E = I$. So we have next two corollaries.

Corollary 3.8. *If* $\mathcal{D}(M_v)$ *is dense in* $L^2(\mu)$ *for a fixed* $n \geq 1$, *then:*

(i) M_v *is k-expansive if and only if* $\triangle_{v,n}(x) \leq 0$ *a.e.* μ.

(ii) M_v *is k-isometry if and only* $\triangle_{v,n}(x) = 0$ *a.e.* μ.

Corollary 3.9. *Let* $\mathcal{D}(M_v)$ *be dense in* $L^2(\mu)$ *and* $k \geq 1$ *be fixed. Then*

(i) M_v *is k-hyperexpansive* $(k \geq 1)$ *if and only if* $\triangle_{v,n}(t) \leq 0$ *a.e.,* μ *for* $n = 1, 2, ..., k$.

(ii) M_v *is completely hyperexpansive if and only if the sequence* $\{|u(t)|^2\}_{n=0}^{\infty}$ *is a completely alternating sequence for almost every* $t \in \Omega$.

Finally we give some examples.

Example 3.10. *Let* $\Omega = [-1, 1]$, $d\mu = \frac{1}{2}dx$ *and* $\mathcal{A} =< \{(-a, a) : 0 \leq a \leq 1\} >$ *(Sigma algebra generated by symmetric intervals). Then*

$$E^{\mathcal{A}}(f)(t) = \frac{f(t) + f(-t)}{2}, \quad t \in \Omega,$$

where $E^{\mathcal{A}}(f)$ *is defined. If* $v(t) = e^t$, *then* $E^{\mathcal{A}}(v)(t) = \cosh(t)$ *and we have the followings:*

1) $E^{\mathcal{A}}M_v$ *is densely defined and closed on* $L^p(\Omega)$.

2) $\sigma(E^{\mathcal{A}}M_v) = \mathcal{R}(\cosh(t))$.

3) $E^{\mathcal{A}}M_v$ *is not 2-expansive, since*

$$\begin{aligned} 1 - 2|E(v)|^2(t) + |E(v)|^4(t) &= 1 - 2\cosh^2(t) + \cosh^4(t) \\ &= (\cosh^2(t) - 1)^2 \geq 0. \end{aligned}$$

Example 3.11. *Let $\Omega = \mathbb{N}$, $\mathcal{G} = 2^{\mathbb{N}}$ and let $\mu(\{t\}) = pq^{t-1}$, for each $t \in \Omega =$, $0 \leq p \leq 1$ and $q = 1 - p$. Elementary calculations show that μ is a probability measure on $\mathcal{G}$. Let $\mathcal{A}$ be the σ-algebra generated by the partition $B = \{\Omega_1 = \{3n : n \geq 1\}, \Omega_1^c\}$ of Ω. So, for every $f \in \mathcal{D}(E^{\mathcal{A}})$ we have*

$$E(f) = \alpha_1 \chi_{\Omega_1} + \alpha_2 \chi_{\Omega_1^c}$$

and direct computations show that

$$\alpha_1(f) = \frac{\sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{3n-1}}$$

and

$$\alpha_2(f) = \frac{\sum_{n\geq 1} f(n)pq^{n-1} - \sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{n-1} - \sum_{n\geq 1} pq^{3n-1}}.$$

So, if u and w are real functions on Ω. Then we have the followings:

1) If $\alpha_1((|u|^q)^{\frac{p}{q}})\alpha_1(|w|^p) < \infty$ and $\alpha_2((|u|^q)^{\frac{p}{q}})\alpha_2(|w|^p) < \infty$, then the operator M_wEM_u is a densely defined and closed operator on $L^p(\Omega)$.

2) $\sigma(M_wEM_u) = \{\alpha_1(E(uw)), \alpha_2(E(uw))\}$.

Example 3.12. *Let $\Omega = [0, 1] \times [0, 1]$, $d\mu = dtdt'$, Σ the Lebesgue subsets of Ω and let $\mathcal{A} = \{A \times [0, 1] : A$ is a Lebesgue set in $[0, 1]\}$. Then, for each f in $L^2(\Sigma)$, $(Ef)(t, t') = \int_0^1 f(t, s)ds$, which is independent of the second coordinate. Hence for $v(t, t') = t^m$ we get that v is $\mathcal{A}$-measurable and EM_v is k-expansive and k-isometry if*

$$\sum_{0\leq i\leq k} (-1)^i \binom{k}{i} x^{2mi} \leq 0, \quad \sum_{0\leq i\leq k} (-1)^i \binom{k}{i} t^{2mi} = 0,$$

respectively. This example is valid in the general case as follows:
Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and $\Omega = \Omega_1 \times \Omega_2$, $\Sigma = \Sigma_1 \times \Sigma_2$ and $\mu = \mu_1 \times \mu_2$. Put $\mathcal{A} = \{A \times \Omega_2 : A \in \Sigma_1\}$. Then $\mathcal{A}$ is a sub-σ-algebra of Σ. Then for all f in domain $E^{\mathcal{A}}$ we have

$$E^{\mathcal{A}}(f)(t_1) = E^{\mathcal{A}}(f)(t_1, t_2) = \int_{\Omega_2} f(t_1, s)d\mu_2(s) \quad \mu - a.e.$$

on Ω.

Also, if (Ω, Σ, μ) is a finite measure space and $k : \Omega \times \Omega \to \mathbb{C}$ is a $\Sigma \otimes \Sigma$-measurable function such that

$$\int_\Omega |k(., s)f(s)|d\mu(s) \in L^2(\Sigma)$$

for all $f \in L^2(\Sigma)$. Then the operator $T : L^2(\Sigma) \to L^2(\Sigma)$ defined by

$$Tf(t) = \int_\Omega k(t, s)f(s)d\mu, \quad f \in L^2(\Sigma),$$

is called kernel operator on $L^2(\Sigma)$). We show that T is a weighted conditional type operator.[5] Since $L^2(\Sigma)\times\{1\} \cong L^2(\Sigma)$ and vf is a $\Sigma \otimes \Sigma$-measurable function, when $f \in L^2(\Sigma)$. Then by taking $v := k$ and $f'(t, s) = f(s)$, we get that

$$\begin{aligned} E^{\mathcal{A}}(vf)(t) &= E^{\mathcal{A}}(vf')(t, s) \\ &= \int_\Omega v(t, t')f'(, t')d\mu(t') \\ &= \int_\Omega v(t, t')f(t')d\mu(t') \\ &= Tf(t). \end{aligned}$$

Hence $T = EM_v$, i.e, T is a weighted conditional type operator. This means all assertions of this paper are valid for a class of integral type operators.

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