# Characterization of Homological Properties of
# $\theta$-Lau Product of Banach Algebras

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**Abstract.** Let $A$ and $B$ be two Banach algebras and $\theta \in \sigma(B)$. In this paper, we investigate biprojectivity and biflatness of $\theta$-Lau product of Banach algebras $A \times_\theta B$. Indeed, we show that $A \times_\theta B$ is biprojective if and only if $A$ is contractible and $B$ is biprojective. This generalizes some known results. Moreover, we characterize biflatness of $\theta$-Lau product of Banach algebras under some conditions. As an application, we give an example of biflat Banach algebras $A$ and $X$ such that the generalized module extension Banach algebra $X \rtimes A$ is not biflat. Finally, we characterize pseudo-contractibility of $\theta$-Lau product of Banach algebras and give an affirmative answer to an open question.

## 1. Introduction and preliminaries

Let $A$ be a Banach algebra and $E$ be a Banach $A$-bimodule. The dual space $E^*$ can be regarded as a Banach $A$-bimodule with the following module actions:

$$(a \cdot \lambda)(x) = \lambda(x \cdot a), \quad (\lambda \cdot a)(x) = \lambda(a \cdot x) \qquad (a \in A, x \in E, \lambda \in E^*).$$

Also for a Banach algebra $A$, the first Arens product on $A^{**}$ is defined by

$$(F \square G)(\lambda) = F(G \cdot \lambda) \qquad (F, G \in A^{**}, \lambda \in A^*),$$

where $(G \cdot \lambda)(a) = G(\lambda \cdot a)$, for all $a \in A$.

Throughout the paper, $A \hat{\otimes} A$ denotes the *projective tensor product* equipped with the *projective norm* defined as

$$\|u\|_p = \inf\{\sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| : u = \sum_{i=1}^{\infty} x_i \otimes y_i\} \quad (u \in A \hat{\otimes} A).$$

Recall that $A \hat{\otimes} A$ is a Banach $A$-bimodule with the natural module actions, defined as

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

For more details on the projective tensor product of Banach algebras, we refer the reader to [1] and [3].

The concepts of biprojectivity and biflatness of Banach algebras were originally introduced and studied by A. Ya. Helemskii which are the most important notions in Banach homology, see [11]. In fact, a Banach algebra $A$ is called *biprojective* if the product morphism

$$\Delta_A : A\hat{\otimes}A \longrightarrow A, \quad \Delta_A(a \otimes b) = ab \qquad (a, b \in A),$$

has a bounded right inverse which is an $A$-bimodule homomorphism. Moreover, $A$ is called *biflat* if the map $\Delta_A^* : A^* \longrightarrow (A\hat{\otimes}A)^*$ has a bounded left inverse which is an $A$-bimodule homomorphism.

By [19, Lemma 4.3.22], $A$ is biflat if and only if there exists a bounded $A$-bimodule homomorphism $\rho : A \longrightarrow (A\hat{\otimes}A)^{**}$ such that $\Delta_A^{**}\circ\rho = \kappa_A$, where $\kappa_A : A \longrightarrow A^{**}$ is the canonical embedding map. These notions have critical relations with some cohomological properties of Banach algebras defined by B. E. Johnson such as amenability and contractibility [13]. For example, it is shown that $A$ is unital and biprojective if and only if $A$ is contractible, i.e. $\mathcal{H}^1(A, E)$, the first Hochschild cohomology group of $A$ with coefficients in E, is trivial for each Banach $A$-bimodule $E$ [3, Theorem 2.8.48]. Furthermore, $A$ is biflat with a bounded approximate identity if and only if $A$ is amenable, i.e. $\mathcal{H}^1(A, E^*)$ is trivial for each Banach $A$-bimodule $E$, see [3, Theorem 2.9.65]. Biprojectivity and biflatness have been studied for various classes of Banach algebras associated with a locally compact group, Segal algebras and semigroup algebras, see [2], [18] and [20], for more details.

Throughout the paper, for a Banach algebra $B$ we assume that $\sigma(B)$, the set of all non-zero homomorphisms from $B$ onto $\mathbb{C}$, is non-empty. Let $A$ and $B$ be Banach algebras and $\theta \in \sigma(B)$. Consider the Banach space $A \times B$ equipped with the $\ell^1$-norm $\|(a, b)\| = \|a\| + \|b\|$ and the product defined by

$$(a, b) \odot (a', b') = (aa' + \theta(b)a' + \theta(b')a, bb') \qquad (a, a' \in A, b, b' \in B).$$

Then $A \times B$ is a Banach algebra called $\theta$-*Lau product of Banach algebras* $A$ and $B$ and denoted by $A \times_\theta B$. This class of Banach algebras was originally constructed and studied by A. T. Lau in [15]. Afterward, Sangani Monfared in [21] investigated some cohomological properties of $\theta$-Lau product of Banach algebras. For example, $A \times_\theta B$ is amenable (contractible) if and only if $A$ and $B$ are amenable (contractible). Also, weak amenability of $A$ and $B$ implies that $A \times_\theta B$ is weakly amenable but the converse is not true, in the general case. Indeed, if $A\times_\theta B$ is weakly amenable, then $B$ is weakly amenable and $A$ is cyclic amenable [21, Theorem 2.11]. Moreover, Khoddami and Vishki in [14] studied and characterized biprojectivity and biflatness of $A \times_\theta B$, in the special case where $A$ is unital. It is worthwhile to mention that all proofs in [14] are heavily based on the hypothesis that $A$ is unital. Our aim in this paper is to study biprojectivity and biflatness of $A \times_\theta B$, in the general case and without any extra condition. This paper is organized as follows:

In Section 2, we study biprojectivity of $\theta$-Lau product of Banach algebras. Indeed, we show that if $A\times_\theta B$ is biprojective, then $A$ is unital. Using this, we characterize biprojectivity of $\theta$-Lau product of Banach algebras $A \times_\theta B$, where $A$ and $B$ are arbitrary Banach algebras and $\theta \in \sigma(B)$. Precisely, we show that $A \times_\theta B$ is biprojective if and only if $A$ is contractible and $B$ is biprojective. This is a generalization of a well-known result in [14].

In Section 3, we study biflatness of $\theta$-Lau product of Banach algebras and give a characterization under some conditions. As a main result, we prove that if $A \times_\theta B$ is biflat, then $A$ has a bounded approximate identity. Furthermore, we show that if $A \times_\theta B$ is biflat, then $B$ is biflat and $A$ is amenable. As a consequence, in the case where $B$ has a bounded approximate identity, we conclude that the biflatness of $A \times_\theta B$ is equivalent to the amenability of $A$ and $B$. This result gives a negative answer to an open question whether biflatness of $A$ and $X$ implies biflatness of the generalized module extension Banach algebra $X \rtimes A$, which is arised in [6].

In Section 4, we characterize pseudo-contractibility of $\theta$-Lau product of Banach algebras. In [8], the authors addressed an open question whether or not $A \times_\theta B$ is pseudo-contractible if $A$ is contractible and $B$ is pseudo-contractible. Here, we give an affirmative answer to this question.

## 2. Biprojectivity of $\theta$-Lau product of Banach algebras

Our aim in this section is to study of biprojectivity of $\theta$-Lau product of Banach algebras $A \times_\theta B$, where $A$ and $B$ are arbitrary Banach algebras and $\theta \in \sigma(B)$. First, we introduce the generalized module extension

Banach algebras which can be regarded as a generalization of $\theta$-Lau product of Banach algebras, see [6], [7], [12] and [17], for more details. Recall that we shall use the notation, introduced in [6].

Suppose that $A$ and $X$ are Banach algebras such that $X$ is a Banach $A$-bimodule. Following [6], we say that $X$ is an *algebraic Banach A-bimodule* if the left and right actions of $A$ on $X$ are compatible, i.e. for all $a \in A$ and $x, x' \in X$,

$$a \cdot (xx') = (a \cdot x)x', \qquad (x \cdot a)x' = x(a \cdot x'), \qquad (xx') \cdot a = x(x' \cdot a).$$

Then the *generalized module extension Banach algebra*, denoted by $X \bowtie A$, is defined as the set $X \times A$ equipped with the norm $\|(x, a)\| = \|x\| + \|a\|$ and the multiplication

$$(x, a) \bullet (x', a') = (a \cdot x' + x \cdot a' + xx', aa') \qquad (a, a' \in A, x, x' \in X).$$

**Example 2.1.** *With the above hypothesis,*

(i) *the usual direct sum $X \oplus_1 A$ with the coordinatewise product defined by*

$$(x, a)(x', a') = (xx', aa') \quad (a, a' \in A, x, x' \in X),$$

*can be considered as the generalized module extension Banach algebra, in the case where $X$ is a Banach $A$-bimodule with trivial module actions.*

(ii) *for each $\theta \in \sigma(B)$, we can regard the $\theta$-Lau product of Banach algebra $A \times_\theta B$ as the generalized module extension Banach algebra, where $A$ is a Banach $B$-bimodule with the following module actions:*

$$a \cdot b = b \cdot a = \theta(b)a, \qquad (a \in A, b \in B).$$

(iii) *Let $I$ be a closed two-sided ideal of $B$ and $\phi : A \longrightarrow B$ be a bounded homomorphism. Following [16], we consider the Banach algebra*

$$A \bowtie^\phi I = \{(a, i) : a \in A, i \in I\},$$

*equipped with the $\ell^1$-norm and the following product formula:*

$$(a, i)(a', i') = (aa', \phi(a)i' + i\phi(a') + ii') \qquad (a, a' \in A, i, i' \in I).$$

*The Banach algebra $A \bowtie^\phi I$ is called the amalgamation of $A$ with $B$ along $I$ with respect to $\phi$. It is easy to see that $A \bowtie^\phi I$ can be regarded as the generalized module extension Banach algebra $I \bowtie A$, where $I$ is a Banach $A$-bimodule with the following module actions:*

$$a \cdot i = \phi(a)i, \quad i \cdot a = i\phi(a) \quad (a \in A, i \in I).$$

In the sequel, we prove that if $X$ is a unital Banach algebra, then $X \bowtie A \cong X \oplus_1 A$. Therefore, some results in [7], [12] and [17] immediately follow from this fact.

**Theorem 2.2.** *Let $A$ and $X$ be two Banach algebras and let $X$ be an algebraic Banach $A$-bimodule. If $X$ is unital, then $X \bowtie A$ is isomorphic as a Banach algebra to the usual direct sum $X \oplus_1 A$.*

*Proof.* Suppose that $e_X$ is the identity of $X$ and define the map $\phi : X \bowtie A \longrightarrow X \oplus_1 A$ by

$$\phi((x, a)) = (x + a \cdot e_X, a) \qquad (a \in A, x \in X).$$

It is easy to check that $\phi$ is linear, continuous and bijective. It suffices to show that the map $\phi$ is a homomorphism. For each $(x, a), (x', a') \in X \bowtie A$ we have,

$$\begin{aligned}
\phi((x, a) \bullet (x', a')) &= \phi((a \cdot x' + x \cdot a' + xx', aa')) \\
&= (a \cdot x' + x \cdot a' + xx' + (aa') \cdot e_X, aa') \\
&= (a \cdot (e_X x') + (x \cdot a')e_X + xx' + a \cdot (e_X(a' \cdot e_X)), aa') \\
&= ((a \cdot e_X)x' + x(a' \cdot e_X) + xx' + (a \cdot e_X)(a' \cdot e_X), aa') \\
&= (x + a \cdot e_X, a)(x' + a' \cdot e_X, a') \\
&= \phi((x, a))\phi((x', a')).
\end{aligned}$$

This follows that the map $\phi$ is a isomorphism. So $X \rtimes A \cong X \oplus_1 A$ as Banach algebras and the proof is complete. $\square$

**Remark 2.3.** *Recently, the authors in [6] considered biprojectivity of $X \rtimes A$, for an essential Banach algebra $X$ equipped with non-zero idempotent element $p \in Z(X)$, where $Z(X)$ denotes algebraic center of $X$. It seems that part of the proofs in [6, Theorem 3.1, Theorem 3.3] has a gap. Indeed, it is claimed that the map $\phi : (Xp \cup \langle p \rangle) \rtimes A \longrightarrow X \rtimes A$ defined by*

$$\phi((xp, a)) = (x, a) \qquad (x \in X, a \in A), \tag{1}$$

*is well-defined, where $Xp = \{xp : x \in X\}$. But, it is easy to see that the map $\phi$ is not necessarily well-defined, in the general case. For example, suppose that $X$ is a non-unital and essential Banach algebra with non-zero idempotent element $p \in Z(X)$. Assume towards a contradiction that the map $\phi$ defined in (1) is well-defined. Then for each $x \in X$ we have*

$$(x, a) = \phi((xp, a)) = \phi((xpp, a)) = (xp, a).$$

*It follows that $X$ is unital which is a contradiction.*

Furthermore, the author in [7] studied some homological and cohomological properties of the generalized module extension Banach algebra under some extra conditions. Indeed, in the case where $X$ has an identity $e_X$, Ettefagh in [7, Theorem 3.1(ii), Theorem 3.2] studied biprojectivity of $X \rtimes A$ with the assumption

$$a \cdot e_X = e_X \cdot a \qquad (a \in A).$$

As a consequence of Theorem 2.2, we give a generalization of [7, Theorem 3.1(ii), Theorem 3.2] and characterize biprojectivity of $X \rtimes A$ without any extra condition as follows.

**Corollary 2.4.** *Let $A$ and $X$ be two Banach algebras and let $X$ be a unital algebraic Banach $A$-bimodule. Then $X \rtimes A$ is biprojective if and only if $A$ and $X$ are biprojective.*

In the sequel, we characterize biprojectivity of $\theta$-Lau product of Banach algebras. To this end, we need the following lemma.

**Lemma 2.5.** *Suppose that $A$ and $B$ are two Banach algebras and $\theta \in \sigma(B)$. If $A \times_\theta B$ is biprojective, then $A$ is unital.*

*Proof.* Let $\Phi : (A \times_\theta B) \hat{\otimes} (A \times_\theta B) \longrightarrow A \times_\theta B$ be the linear map defined by

$$\langle \Phi, (a, b) \otimes (a', b') \rangle = \theta(b')(a, b), \qquad (a, a' \in A, b, b' \in B).$$

Then it is easy to check that $\|\Phi\| \leq \|\theta\|$ and so $\Phi$ is bounded. Furthermore, it is straightforward to check that for all $a, a_1, a_2 \in A$ and $b_1, b_2 \in B$ the map $\Phi$ has the following properties:

$$\langle \Phi, (a, 0) \odot (a_1, b_1) \otimes (a_2, b_2) \rangle = (a, 0) \odot \langle \Phi, (a_1, b_1) \otimes (a_2, b_2) \rangle, \tag{2}$$

and

$$\langle \Phi, (a_1, b_1) \otimes (a_2, b_2) \odot (a, 0) \rangle = (0, 0). \tag{3}$$

Since $A \times_\theta B$ is biprojective, $\Delta_{A \times_\theta B}$ has a bounded linear right inverse map

$$\rho : A \times_\theta B \longrightarrow (A \times_\theta B) \hat{\otimes} (A \times_\theta B),$$

which is an $A \times_\theta B$-bimodule homomorphism. Choose $d \in B$ such that $\theta(d) \neq 0$ and set $Q = \Phi(\rho((0, d)))$. Suppose that $\rho((0, d)) = \sum_{i=1}^{\infty} (a_i, b_i) \otimes (c_i, d_i)$, for some sequences $((a_i, b_i))$ and $((c_i, d_i))$ in $A \times_\theta B$ such that

$$\sum_{i=1}^{\infty} \|(a_i, b_i)\| \, \|(c_i, d_i)\| < \infty.$$

On the other hand, we can write $Q = (e_0, f_0)$ where $e_0 \in A$ and $f_0 \in B$. Therefore, we have

$$
\begin{aligned}
\theta(f_0) &= \langle (0, \theta), Q \rangle \\
&= \langle (0, \theta), \sum_{i=1}^{\infty} \theta(d_i)(a_i, b_i) \rangle \\
&= \sum_{i=1}^{\infty} \theta(b_i d_i) \\
&= \langle (0, \theta), \Delta_{A \times_\theta B}(\rho((0, d))) \rangle \\
&= \theta(d) \neq 0.
\end{aligned}
$$

Moreover, by equations (2) and (3) for each $a \in A$ we conclude that

$$
\begin{aligned}
(a, 0) \odot (e_0, f_0) &= (a, 0) \odot Q = (a, 0) \odot \Phi(\rho((0, d))) \\
&= \Phi((a, 0) \cdot \rho((0, d))) - \Phi(\rho((0, d)) \cdot (a, 0)) \\
&= \Phi(\rho((a, 0) \odot (0, d) - (0, d) \odot (a, 0))) \\
&= \Phi(\rho((\theta(d)a, 0) - (\theta(d)a, 0))) = (0, 0).
\end{aligned}
$$

It follows that $ae_0 + \theta(f_0)a = 0$ and so $e := -\theta(f_0)^{-1}e_0$ is a right identity for $A$. Similarly, by considering the linear map $\Phi' : (A \times_\theta B) \hat{\otimes} (A \times_\theta B) \longrightarrow A \times_\theta B$ by

$$
\langle \Phi', (a, b) \otimes (a', b') \rangle = \theta(b)(a', b'), \qquad (a, a' \in A, b, b' \in B),
$$

we deduce that $A$ has a left identity and so $A$ is unital. $\square$

Khoddami and Vishki in [14] characterized biprojectivity $A \times_\theta B$, in the special case where $A$ is unital. Indeed, it is shown that in the case where $A$ is a unital Banach algebra, $A \times_\theta B$ is biprojective if and only if $B$ is biprojective and $A$ is contractible. In the following, we exhibit a characterization of biprojectivity of $\theta$-Lau product of Banach algebras in the general case.

**Theorem 2.6.** *Let $A$ and $B$ be two Banach algebras and $\theta \in \sigma(B)$. Then the following statements are equivalent.*

*(i) $A \times_\theta B$ is biprojective.*
*(ii) $B$ is biprojective and $A$ is contractible.*

*Proof.* (i)$\Rightarrow$(ii) By the hypothesis and using Lemma 2.5, we conclude that $A$ is unital. Now, by Theorem 2.2, we have

$$
A \times_\theta B = A \rtimes B \cong A \oplus_1 B.
$$

Therefore, $A \oplus_1 B$ is biprojective and so $A$ and $B$ are biprojective. On the other hand, since $A$ has an identity by [3, Theorem 2.8.48 ], it follows that $A$ is contractible.

(ii)$\Rightarrow$(i) Since $A$ is contractible, by [3, Theorem 2.8.48 ], we deduce that $A$ is unital and biprojective. Now, by Corollary 2.4, it follows that $A \times_\theta B$ is biprojective. $\square$

As a consequence of Theorem 2.6, we have the following:

**Corollary 2.7.** *Let $A$ and $B$ be two Banach algebras with $\theta \in \sigma(B)$. Then the following assertions are equivalent.*

*(i) $A \times_\theta B$ is biprojective and $B$ is unital.*
*(ii) $A$ and $B$ are contractible.*

Let $G$ be a locally compact Hausdorff group with the left Haar measure $\lambda$. Then $L^1(G)$ consisting of complex-valued measurable functions $f$ on $G$ with

$$
\|f\|_1 = \int_G |f(x)| \, d\lambda(x) < \infty
$$

becomes a Banach algebra equipped with the convolution product

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\, d\lambda(y) \qquad (f, g \in L^1(G)).$$

Moreover, $C_0(G)$ denotes the Banach algebra of all complex-valued continuous functions on $G$ such that the functions vanish at infinity, equipped with the supremum norm $\| \cdot \|_\infty$ and the pointwise product.

**Corollary 2.8.** *Let $A$ be a Banach algebra and $G$ be a locally compact Hausdorff group. Moreover, suppose that $\rho \in \sigma(A)$, $\theta \in \sigma(L^1(G))$ and $\phi \in \sigma(C_0(G))$. Then,*

(i) *$L^1(G) \times_\rho A$ is biprojective if and only if $G$ is finite and $A$ is biprojective.*

(ii) *$A \times_\theta L^1(G)$ is biprojective if and only if $G$ is compact and $A$ is contractible.*

(iii) *$C_0(G) \times_\rho A$ is biprojective if and only if $G$ is finite and $A$ is biprojective.*

(iv) *$A \times_\phi C_0(G)$ is biprojective if and only if $G$ is discrete and $A$ is contractible.*

*Proof.* By Theorem 2.6, the clauses (i) and (ii) follow from [19, Exercise 4.1.7] and [3, Theorem 3.3.32(ii)], respectively. Moreover, the clauses (iii) and (iv) immediately follow from [19, Corollary 4.1.3] and [3, Proposition 4.2.31], respectively. $\square$

Let $S$ be a semigroup and $\ell^1(S) = \{f : S \longrightarrow \mathbb{C} : \|f\|_1 = \sum_{s \in S} |f(s)| < \infty\}$. We define the convolution of two elements $f, g \in \ell^1(S)$ by

$$(f * g)(s) = \sum_{uv=s} f(u)g(v),$$

where $\sum_{uv=s} f(u)g(v) = 0$, when there are no elements $u, v \in S$ with $uv = s$. Then $(\ell^1(S), *, \| \cdot \|_1)$ is also a Banach algebra, called *the semigroup algebra* of $S$, for more details see [4].

**Example 2.9.** *Let $B$ be an arbitrary Banach algebra and $\theta \in \sigma(B)$.*

(i) *If $S = (\mathbb{N}, \max)$, then it is well-known that the semigroup algebra $\ell^1(\mathbb{N}, \max)$ has an identity but it is not contractible, see [5, Theorem 10]. So by Theorem 2.6, $\ell^1(\mathbb{N}, \max) \times_\theta B$ is not biprojective. Note that this example shows that the converse of Lemma 2.5 is not true in general.*

(ii) *If $S = (\mathbb{N}, \min)$, then the semigroup algebra $\ell^1(\mathbb{N}, \min)$ is a non-unital Banach algebra with a bounded approximate identity. Furthermore, $\ell^1(\mathbb{N}, \min)$ is not contractible and so by Theorem 2.6, we conclude that $\ell^1(\mathbb{N}, \min) \times_\theta B$ is not biprojective.*

(iii) *Let $A$ be an arbitrary infinite-dimensional $C^*$-algebra. By [19, Corollary 4.1.6], we conclude that $A$ is not contractible. Hence, $A \times_\theta B$ is not biprojective.*

## 3. Biflatness of $\theta$-Lau Product of Banach Algebras

In this section, we study biflatness of $\theta$-Lau product of Banach algebras and give a characterization under some conditions. First, we need the following result about biflatness of the generalized module extension Banach algebras. In [6, Theorem 4.1], it is shown that biflatness of $X \rtimes A$ implies that $A$ is biflat. In addition, if $X$ has a non-zero idempotent element in $Z(X)$ and $A^2$ is dense in $A$, it is proved that $X$ is biflat. Indeed, in the proof of [6, Theorem 4.1], the authors used the fact that the map $\phi$ defined in (1) is well-defined, which has the same gap mentioned in Remark 2.3.

In the following, we remove this gap and improve this result with a short proof.

We note that if $X$, $Y$ and $Z$ are Banach $A$-bimodules, $T_1 : X \longrightarrow Y$ and $T_2 : Y \longrightarrow Z$ are bounded $A$-bimodule morphisms, then we say the short sequence

$$0 \to X \xrightarrow{T_1} Y \xrightarrow{T_2} Z \to 0,$$

is *exact* if $T_1$ is one-to-one, $T_2$ is surjective and $Im T_1 = ker T_2$. In the case where, $I$ is a closed two-sided ideal of $A$, we can regard the following short exact sequence

$$0 \to I \xrightarrow{i} A \xrightarrow{q} \frac{A}{I} \to 0,$$

where $i : I \longrightarrow A$ is the inclusion map and $q : A \longrightarrow \frac{A}{I}$ is the quotient map.

**Proposition 3.1.** *Let $A$ and $X$ be two Banach algebras and let $X$ be an algebraic Banach $A$-bimodule such that $X \rtimes A$ is biflat. Then*

 (i) *$A$ is biflat.*
 (ii) *If $X$ is essential, then $X$ is biflat.*
 (iii) *If $X$ has a bounded approximate identity, then $X$ is amenable.*

*Proof.* (i) Since $X \rtimes A$ is biflat, it is known that $X \rtimes A$ is essential [11, Proposition 6]. This follows that

$$(\overline{AX + XA + X^2}, \overline{A^2}) = X \rtimes A.$$

Therefore, we have $\overline{AX + XA + X^2} = X$ and

$$\overline{(X \rtimes A)X + X(X \rtimes A)} = \overline{(AX + X^2, 0) + (XA + X^2, 0)}$$
$$= (\overline{AX + XA + X^2}, 0) = X.$$

Now, since $X$ is a closed two-sided ideal of $X \rtimes A$ and by considering the short exact sequence

$$0 \to X \xrightarrow{i} X \rtimes A \xrightarrow{q} \frac{X \rtimes A}{X} \to 0,$$

we deduce that $\frac{X \rtimes A}{X} \cong A$ is biflat [10, Proposition 3.3].
   (ii) By assumption $X$ is an essential two-sided ideal of $X \rtimes A$ and by [11, Proposition 8], it follows that $X$ is biflat.
   (iii) This follows immediately from clause (ii) and [3, Theorem 2.9.65]. □

Now, we obtain a similar version of Lemma 2.5 for biflatness of $\theta$-Lau product of Banach algebras.

**Lemma 3.2.** *Assume that $A$ and $B$ are two Banach algebras and $\theta \in \sigma(B)$. If $A \times_\theta B$ is biflat, then $A$ has a bounded approximate identity.*

*Proof.* Let $\Phi$ be the same linear map defined in Lemma 2.5. It is easy to see that bounded linear maps $\Phi^* : (A \times_\theta B)^* \longrightarrow ((A \times_\theta B)\hat{\otimes}(A \times_\theta B))^*$ and

$$\Phi^{**} : ((A \times_\theta B)\hat{\otimes}(A \times_\theta B))^{**} \longrightarrow (A \times_\theta B)^{**},$$

for all $a \in A$, $f \in (A \times_\theta B)^*$ and $G \in ((A \times_\theta B)\hat{\otimes}(A \times_\theta B))^{**}$ satisfy in the following equations:

$$\Phi^*(f \cdot (a, 0)) = \Phi^*(f) \cdot (a, 0) \quad \text{and} \quad (a, 0) \cdot \Phi^*(f) = (0, 0),$$

and so

$$(\hat{a}, 0) \square \Phi^{**}(G) = \Phi^{**}((a, 0) \cdot G) \quad \text{and} \quad \Phi^{**}(G \cdot (a, 0)) = (0, 0), \tag{4}$$

where $\square$ denotes the first Arens product on $(A \times_\theta B)^{**}$. Since $A \times_\theta B$ is biflat, there exists an $(A \times_\theta B)$-bimodule homomorphism

$$\rho : A \times_\theta B \longrightarrow ((A \times_\theta B)\hat{\otimes}(A \times_\theta B))^{**},$$

such that $\Delta^{**}_{A\times_\theta B} \circ \rho = \kappa_{A\times_\theta B}$. Choose $d \in B$ such that $\theta(d) \neq 0$ and set $Q = \Phi^{**}(\rho(0,d))$. First, note that for all $a, a' \in A$ and $b, b' \in B$ we have,

$$
\begin{aligned}
\langle \Phi^*((0,\theta)), (a,b) \otimes (a',b') \rangle &= \langle (0,\theta), \theta(b')(a,b) \rangle = \theta(bb') \\
&= \langle (0,\theta), \Delta_{A\times_\theta B}((a,b) \otimes (a',b')) \rangle \\
&= \langle \Delta^*_{A\times_\theta B}((0,\theta)), (a,b) \otimes (a',b') \rangle.
\end{aligned}
$$

Therefore, $\Phi^*((0,\theta)) = \Delta^*_{A\times_\theta B}((0,\theta))$. It follows that

$$
\begin{aligned}
\langle Q, (0,\theta) \rangle &= \langle \Phi^{**}(\rho((0,d))), (0,\theta) \rangle \\
&= \langle \rho((0,d)), \Phi^*((0,\theta)) \rangle \\
&= \langle \kappa_{A\times_\theta B}((0,d)), (0,\theta) \rangle \\
&= \langle \widehat{(0,d)}, (0,\theta) \rangle \\
&= \langle (0,\theta), (0,d) \rangle = \theta(d) \neq 0.
\end{aligned}
$$

By using the equations (4), for each $a \in A$ we have

$$
\begin{aligned}
(\hat{a},0)\square Q &= (\hat{a},0)\square\Phi^{**}(\rho((0,d))) \\
&= \Phi^{**}((a,0) \cdot \rho((0,d))) - \Phi^{**}(\rho((0,d)) \cdot (a,0)) \\
&= \Phi^{**}(\rho((a,0) \odot (0,d) - (0,d) \odot (a,0))) \\
&= \Phi^{**}(\rho((0,0))) = (0,0).
\end{aligned}
$$

Pick $Q = (E_0, F_0)$, where $E_0 \in A^{**}$ and $F_0 \in B^{**}$. Using [21, Proposition 2.12], we conclude that $a \cdot E = \hat{a}$, where $E := -F_0(\theta)^{-1}E_0$. By Goldstine's theorem, there exists a bounded net $(e_\alpha)_{\alpha \in I}$ in $A$ such that $ae_\alpha \longrightarrow a$ in the weak topology. It follows that $A$ has a bounded right approximate identity [1, Proposition I.11.4]. By the same argument, $A$ has a bounded left approximate identity and this completes the proof. $\square$

**Theorem 3.3.** *Let $A$ and $B$ be two Banach algebras with $\theta \in \sigma(B)$. If $A \times_\theta B$ is biflat, then the following statements hold.*

(i) *$B$ is biflat.*

(ii) *$A$ is amenable.*

*Proof.* Since $A \times_\theta B = A \rtimes B$, the clause (i), follows from Proposition 3.1(i). Also, by Lemma 3.2, we deduce that $A$ has a bounded approximate identity and by using Proposition 3.1(iii) the proof is complete. $\square$

**Example 3.4.** *Let $B$ be an arbitrary Banach algebra and $\theta \in \sigma(B)$.*

(i) *Suppose that $S$ is a right (left) zero semigroup with at least two elements, i.e.,*

$$ st = t \ (st = s) \qquad (s,t \in S). $$

*Then the semigroup algebra $\ell^1(S)$ does not have any bounded approximate identity and so is not amenable. By Lemma 3.2, we conclude that $\ell^1(S) \times_\theta B$ is not biflat.*

(ii) *It is well-known that the semigroup algebras $\ell^1(\mathbb{N}, \max)$ and $\ell^1(\mathbb{N}, \min)$ are not amenable [5, Theorem 10]. So by Theorem 3.3, $\ell^1(\mathbb{N}, \max) \times_\theta B$ and $\ell^1(\mathbb{N}, \min) \times_\theta B$ are not biflat. These examples also show that the converse of Lemma 3.2 is not true in general because $\ell^1(\mathbb{N}, \max)$ is unital and $\ell^1(\mathbb{N}, \min)$ has a bounded approximate identity.*

**Corollary 3.5.** *Let $A$ and $B$ be two Banach algebras with $\theta \in \sigma(B)$. Then the following statements are equivalent.*

(i) *$A \times_\theta B$ is amenable.*

  *(ii)* $A \times_\theta B$ *is biflat and B has a bounded approximate identity.*
  *(iii)* *A and B are amenable.*

*Proof.* (i)⇒(ii) Since $A \times_\theta B$ is amenable, it follows that $A \times_\theta B$ is biflat and has a bounded approximate identity. In this case, it is clear that $B$ has a bounded approximate identity [21, Proposition 2.3(iv)].

    (ii)⇒(iii) This follows from Theorem 3.3.

    (iii)⇒(i) Since $A$ and $\frac{A \times_\theta B}{A} \cong B$ are amenable, this immediately follows from [19, Theorem 2.3.10].   □

Ebadian and Jabbari in [6] arised an open question that whether biflatness of $A$ and $X$ implies biflatness of $X \rtimes A$. As an application of Corollary 3.5, with an example we give a negative answer to this question even if $A$ is amenable.

**Example 3.6.** *Suppose that X is a biflat Banach algebra which is not amenable, $A = \mathbb{C}$ and $\theta : \mathbb{C} \longrightarrow \mathbb{C}$ is the identity map. By Corollary 3.5, it follows that $X \times_\theta \mathbb{C} = X \rtimes \mathbb{C}$ is not biflat.*

We note that for some classes of Banach algebras the converse of Theorem 3.3 is true. For example, if $A$ is an amenable, Arens regular, weakly sequentially complete Banach algebra, $B$ is an arbitrary biflat Banach algebra and $\theta \in \sigma(B)$, then $A \times_\theta B$ is biflat. Indeed, this follows from the fact that $A$ is unital [3, Theorem 2.9.39] and so by Theorem 2.2, we have

$$A \times_\theta B = A \rtimes B \cong A \oplus_1 B.$$

However, it is worthwhile to mention that we do not know whether the converse of Theorem 3.3 holds in the general case. So, we remain the following interesting open problem:

**Problem 3.7.** *Let A be an amenable Banach algebra, B be a biflat Banach algebra and $\theta \in \sigma(B)$. Is $A \times_\theta B$ biflat?*

## 4. Pseudo-Contractibility of $\theta$-Lau Product of Banach Algebras

    We finish the paper with a result on pseudo-contractibility of $\theta$-Lau product of Banach algebras. According [9], a Banach algebra $A$ is called *pseudo-contractible* if $A$ has a central approximate diagonal, i.e. there is a net $(u_\alpha)_{\alpha \in I} \subseteq A \hat{\otimes} A$ such that for each $a \in A$,

$$a \cdot u_\alpha = u_\alpha \cdot a, \quad \Delta_A(u_\alpha)a \longrightarrow a.$$

Recently, Ghaderi et al. in [8] studied pseudo-contractibility of $\theta$-Lau product of Banach algebras. First, we give a simpler proof of [8, Theorem 3.2].

**Theorem 4.1.** *Let A and B be two Banach algebras with $\theta \in \sigma(B)$. If $A \times_\theta B$ is pseudo-contractible, then*

  *(i)* *B is pseudo-contractible,*
  *(ii)* *A is contractible.*

*Proof.* Since $A \times_\theta B$ is pseudo-contractible, by [8, Lemma 3.1], it follows that $A$ is unital. Now, by applying Theorem 2.2 we have,

$$A \times_\theta B = A \rtimes B \cong A \oplus_1 B.$$

On the other hand, by the same argument in [9, Proposition 2.2], we know that homomorphic image of a pseudo-contractible Banach algebra is again pseudo-contractible. Thus, $A$ and $B$ are pseudo-contractible. In addition, since $A$ is unital, we deduce that $A$ is contractible [9, Theorem 2.4].   □

    In [8, Question 1], the authors arised an open question that does the converse of the above theorem is true? Here, as an easy consequence of Theorem 2.2, we give an affirmative answer to this question.

**Theorem 4.2.** *Let A and B be two Banach algebras and $\theta \in \sigma(B)$. Then the following assertions are equivalent.*

  *(i)* $A \times_\theta B$ *is pseudo-contractible.*
  *(ii)* *B is pseudo-contractible and A is contractible.*

*Proof.* It suffices to show that the implication (ii)⇒(i) holds. Since $A$ is contractible, by [9, Theorem 2.4], $A$ is unital and pseudo-contractible. Using Theorem 2.2, we conclude that $A \times_\theta B = A \rtimes B \cong A \oplus_1 B$. Now, by [9, Proposition 2.1], it follows that $A \oplus_1 B$ is pseudo-contractible and the proof is complete.   □

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