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A New Quaternion Valued Frame of Curves with an Application

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Abstract. The aim of the paper is to obtain a new version of Serret-Frenet formulae for a quaternionic curve in $\mathbb{R}^4$ by using the method given by Bharathi and Nagaraj. Then, we define quaternionic helices in $\mathbb{H}$ named as quaternionic right and left $X-$helix with the help of given a unit vector field X. Since the quaternion product is not commutative, the authors ([4], [7]) have used by one-sided multiplication to find a space curve related to a given quaternionic curve in previous studies. Firstly, we obtain new expressions by using the right product and the left product for quaternions. Then, we generalized the construction of Serret-Frenet formulae of quaternionic curves. Finally, as an application, we obtain an example that supports the theory of this paper.

1. Introduction

The quaternion number system was first defined by Irısh Mathematician Hamilton in 1843 and it was applied to mathematics in $\mathbb{R}^3$. At first, quaternions were considered problematic because the commutative property for multiplication in quaternions was not provided. Then, they were used to represent the rotation and translation movements in geometry with their unique structures. Quaternions are used in classical Newton physics, quantum physics, computer applications, and Astrophysics. Especially in recent years, they have been used to obtain kinematic and dynamic expressions used in robotic applications and animations.

The set of real quaternions corresponds to the 4-dimensional vector space $\mathbb{R}^4$ and its characteristic greater than 2. Each element of the set is expressed $q = ae_1 + be_2 + ce_3 + de_4$ where a, b, c, d are ordinary numbers and $e_i (1 \leq i \leq 4)$ is the standard orthonormal bases for $\mathbb{R}^4$ and satisfy the relations given in Eq.(1). The Clifford algebra on $\mathbb{R}^4$ is denoted by $\mathbb{H}$.

K. Baharatti and M. Nagaraj studied quaternionic curves in three-dimensional and four-dimensional space and their Frenet formulas [7]. Then A.C. Coken and A. Tuna worked the differential geometry of quaternionic curves in 4-dimensional semi-Euclidean space [1].

The velocity vector of a general helix in Euclidean space with a fixed direction makes a constant angle along the curve. Quaternionic helices are helices defined using quaternions. Yoon characterized helices in $\mathbb{R}^4$ by a constant function

$$\left(\frac{K}{k}\right)^2 + \left[\frac{1}{r-K}\frac{d}{ds}\left(\frac{K}{k}\right)\right]^2$$

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with the help of curvatures of quaternionic curves [3]. In the interesting paper, Aksoyak has obtained a new version of Serret-Frenet formulae for the quaternionic curve in $\mathbb{R}^4$ by using a method similar to the method given in [7] and called it *Type 2-Quaternionic Frame* [4]. Also, she has given an application of this new type of the quaternionic frame by an example. Except for them, many authors have reported on quaternionic curves by using the quaternionic Frenet formulas and defined some new special quaternionic curves in ([5], [6], [8], [9], [11], [12]).

In this study, we give a new perspective finding the Frenet formulas of quaternionic curves. In this perspective, first of all, we consider a smooth unit quaternion X along to quaternionic curve β and we define a new unit quaternion $Y(s) = \frac{1}{\|X'(s)\|}X'(s)$ in terms of X. Although the quaternion product does not provide the commutative property for the multiplication, in the previous studies, were used by one-sided multiplication by some researchers. Then we obtain two space curves with their frames $\{\xi_i, \eta_i, \varrho_i\}$ for $i = R$ or $i = L$, and they are related to the quaternionic curve β. Furthermore, we want to emphasize that $\xi_R(s) = Y(s) \times \bar{X}(s)$ and $\xi_L(s) = \bar{X}(s) \times Y(s)$ along to the paper. As a result, we construct two new expressions of the Serret-Frenet frame of the quaternionic curve β using the right product and the left product. Finally, we introduce new quaternionic helices in $\mathbb{H}$ named as *quaternionic right and left $X-$helices* with the help of a general quaternionic frame. It is possible to say that the paper is the generalization of the theory in ([3], [4], [7]).

2. Basic concepts and background

We now mention basic concepts on real quaternion algebra. Denote the algebra of real quaternions by $\mathbb{H}$ and its natural basis by $\{e_1, e_2, e_3, e_4\}$. The multiplication of $\mathbb{H}$ is defined as follows:

$$\begin{aligned} e_i \times e_i &= -e_4,\ e_4 = 1\ (1 \le i \le 3) \\ e_i \times e_j &= -e_j \times e_i = e_k\ (1 \le i, j \le 3) \end{aligned} \tag{1}$$

where (ijk) is even permutation of (123). An element of $\mathbb{H}$ is called a real quaternion and is denoted by $q = ae_1 + be_2 + ce_3 + de_4$. A real quaternion q can also expressed as $q = S_q + V_q$ where $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$ are named as the scalar and vector part of $q \in \mathbb{H}$, respectively. If $S_q = 0$, then real quaternion q is called a *pure real quaternion*.

The conjugate of $q = S_q + V_q = ae_1 + be_2 + ce_3 + de_4 \in \mathbb{H}$ is denoted by $\bar{q}$ and given by

$$\bar{q} = S_q - V_q = de_4 - ae_1 - be_2 - ce_3. \tag{2}$$

Let $p = S_p + V_p$ and $q = S_q + V_q$ be any two elements of $\mathbb{H}$ and the product of p and q is defined by

$$p \times q = S_pS_q - \langle V_p, V_q \rangle + S_pV_q + S_qV_p + V_p \wedge V_q \tag{3}$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and cross product in $\mathbb{R}^3$, respectively.

We give quaternionic inner product h (see [7]) as follows:

$$\begin{aligned} h: \quad \mathbb{H} \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (p, q) \quad &\longrightarrow h(p, q) = \frac{1}{2}(p \times \bar{q} + q \times \bar{p}) \end{aligned} \tag{4}$$

and the norm of any real quaternion $q = ae_1 + be_2 + ce_3 + de_4$ is denoted by the equality

$$N_q = \left\|q\right\|^2 = h(q, q) = q \times \bar{q} = \bar{q} \times q = a^2 + b^2 + c^2 + d^2.$$

If $N_q = 1$, then q is called a unit real quaternion.

On the other hand, a curve $\alpha : I \to \mathbb{R}^n$ is said to ccr-curve (that is to say, it has constant curvature ratios) if all the ratios $\frac{k_{i+1}}{ki}$ are constant for $1 \le i \le n-2$, [10].

3. A Generalization of Quaternion Valued Frame and an Application in $\mathbb{R}^4$

First of all, we give Serret Frenet Formulas for quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ given by Bharathi and Nagaraj [7]. Then we construct a general quaternionic frame for a quaternionic curve in $\mathbb{H}$ by using a similar method of Bharathi and Nagaraj. As an application, we obtain the same helical curves that generalized the papers [3] and [9]. The definitions and characterizations of quaternionic helices via quaternionic right or left frame are similar. Hence in the following section, we consider the quaternionic right frame.

Theorem 3.1. *Let $\gamma = \gamma(s)$ be a unit speed spatial quaternionic curve in $\mathbb{R}^3$ with its Frenet frame $\{t(s), n(s), b(s)\}$ and curvatures $\{k(s), r(s)\}$. Then, the Frenet formulas are denoted by*

$$\begin{aligned} t'(s) &= k(s)n(s) \\ n'(s) &= -k(s)t(s) + r(s)b(s) \\ b'(s) &= -r(s)n(s) \end{aligned} \tag{5}$$

where $t(s) = \gamma'(s)$ is unit tangent, $k(s) = \|t'(s)\|$ is the principal curvature, $n(s)$ is unit principal normal, $b(s) = t(s) \times n(s)$ is binormal, where $\times$ denotes the quaternion product and $r(s)$ is the torsion of $\gamma(s)$. Moreover these Frenet vectors hold $h(t,t) = h(n,n) = h(b,b) = 1$ and $h(t,n) = h(t,b) = h(n,b) = 0$ [7].

If we denote the vectors $n(s)$, $c(s) = \dfrac{n'(s)}{\|n'(s)\|} = -\dfrac{k}{f}t + \dfrac{r}{f}b$ and $w(s) = n(s) \times c(s) = \dfrac{r}{f}t + \dfrac{k}{f}b$ along to the spatial quaternionic curve γ in $\mathbb{R}^3$. Then, we can construct an alternative moving quaternionic frame along the curve γ in $\mathbb{R}^3$ with the help of a similar method in [2]. Then, we can find a new alternative moving quaternionic frame $\{n(s), c(s), w(s)\}$ and new curvature functions $\{f(s), g(s)\}$ of γ. It is easily calculate that $f = \sqrt{k^2 + r^2}$ and $g = \dfrac{k^2}{k^2 + r^2}\left(\dfrac{r}{k}\right)'$ which are named as curvatures of the curve γ in terms of the alternative moving quaternionic frame. Moreover these new alternative moving quaternionic frame vectors hold $h(n,n) = h(c,c) = h(w,w) = 1$ and $h(n,c) = h(n,w) = h(c,w) = 0$. Consequently, the above expressions give that the Frenet formulas of alternative moving quaternionic frame with the following theorem.

Theorem 3.2. *Let $\gamma = \gamma(s)$ be a unit speed spatial quaternionic curve in $\mathbb{R}^3$ with its new Frenet apparatus $\{n(s), c(s), w(s); f(s), g(s)\}$. Then, the derivatives of alternative moving quaternionic frame are given by*

$$\begin{aligned} n'(s) &= f(s)c(s), \\ c'(s) &= -f(s)n(s) + g(s)w(s), \\ w'(s) &= -g(s)c(s). \end{aligned} \tag{6}$$

Theorem 3.3. *The four-dimensional Euclidean space $\mathbb{R}^4$ is identified with the space of unit quaternions. Let $I = [0,1]$ be an interval in the real line $\mathbb{R}$ and $s \in I$ be the arc-length parameter along the curve*

$$\begin{aligned} \beta : \ I \subset \mathbb{R} &\longrightarrow \mathbb{H} \\ s &\longrightarrow \beta(s) = \beta_0(s)e_1 + \beta_1(s)e_2 + \beta_2(s)e_3 + \beta_3(s)e_4 \end{aligned}$$

where $\beta = \beta(s)$ is called a quaternionic curve *in $\mathbb{H}$. Considering the Frenet vectors $\{T(s), N_1(s), N_2(s), N_3(s)\}$ and non-zero curvatures $\{K(s), k(s), (r-K)(s)\}$, the Frenet formulas are denoted by*

$$\begin{aligned} T'(s) &= K(s)N_1(s), \\ N_1'(s) &= -K(s)T(s) + k(s)N_2(s), \\ N_2'(s) &= -k(s)N_1(s) + (r-K)(s)N_3(s), \\ N_3'(s) &= -(r-K)(s)N_2(s). \end{aligned} \tag{7}$$

where $T(s) = \beta'(s)$ is a unit tangent vector, $N_1(s), N_2(s), N_3(s)$ are unit normal vectors of the curve $\beta(s)$. Moreover, $K(s) = \|T'(s)\|$, $k(s)$ and $(r-K)(s)$ denote the principal curvature, the torsion and the bitorsion of the curve $\beta(s)$, respectively [7].

Remark 3.4. *Bharathi and Nagaraj [7] introduced the Serret-Frenet formulae and Frenet apparatus of the curve β in $\mathbb{R}^4$ by using the Serret-Frenet formulae of the curve γ in $\mathbb{R}^3$ with the following theory:*

Assume that γ is a spatial quaternionic curve with Frenet apparatus

$$\{t(s), n(s), b(s); k(s), r(s)\}$$

and β is a quaternionic curve with Frenet apparatus

$$\{T(s), N_1(s), N_2(s), N_3(s); K(s)\, k(s), (r-K)(s)\}.$$

The quaternionic curve β is related to the spatial quaternionic curve γ because that the unit spatial quaternion $N_1 \times \bar{T}$ is equal to the unit tangent vector t of the curve γ in [7]. And then, by using the Frenet vectors and curvatures functions of the curve γ in $\mathbb{R}^3$, the other Frenet vectors and curvatures of the curve β in $\mathbb{H}$ are obtained. The Frenet vectors β satisfying the following equalities:

$$\begin{aligned} N_1 &= t \times T,\ t = N_1 \times \bar{T},\ N_2 = n \times T,\ N_3 = b \times T, \\ h(T,T) &= h(N_1,N_1) = h(N_2,N_2) = h(N_3,N_3) = 1, \\ h(T,N_1) &= h(T,N_2) = h(T,N_3) = h(N_1,N_2) = h(N_2,N_3) = 0. \end{aligned} \tag{8}$$

We want to emphasize that the torsion of β is the principal curvature of the curve γ and the bitorsion of β is $(r-K)$, where r is the torsion of γ and K is the principal curvature of β.

Theorem 3.5. *Let $I = [0,1]$ be an interval in the real line $\mathbb{R}$ and $s \in I$ be the arc-length parameter along the quaternionic curve*

$$\begin{array}{llll} \beta: & I \subset \mathbb{R} & \longrightarrow & \mathbb{H} \\ & s & \longrightarrow & \beta(s) = \beta_0(s)e_1 + \beta_1(s)e_2 + \beta_2(s)e_3 + \beta_3(s)e_4 \end{array}$$

Considering the Frenet vectors $\{T(s), N_1(s), N_2(s), N_3(s)\}$ and non-zero curvatures $\{K(s), r(s), (K-k)(s)\}$, the Frenet formulas are denoted by

$$\begin{aligned} T'(s) &= K(s)N_1(s), \\ N_1'(s) &= -K(s)T(s) - r(s)N_2(s), \\ N_2'(s) &= r(s)N_1(s) + (K-k)(s)N_3(s), \\ N_3'(s) &= -(K-k)(s)N_2(s). \end{aligned} \tag{9}$$

where $T(s) = \beta'(s)$ is a unit tangent vector, $N_1(s), N_2(s), N_3(s)$ are unit normal vectors of the curve $\beta(s)$. Moreover, $K(s) = \|T'(s)\|$, $-r(s)$ and $(K-k)(s)$ denote the principal curvature, the torsion and the bitorsion of the curve $\beta(s)$, respectively [4].

Remark 3.6. *Aksoyak [4] introduced a new approach to constructing the Serret-Frenet formulae and Frenet apparatus of the curve β in $\mathbb{R}^4$ by using the Serret-Frenet formulae of the curve γ in $\mathbb{R}^3$ with the following theory:*

Assume that γ is a spatial quaternionic curve with Frenet apparatus

$$\{t(s), n(s), b(s); k(s), r(s)\}$$

and β is a quaternionic curve with Frenet apparatus

$$\{T(s), N_1(s), N_2(s), N_3(s); K(s)\, k(s), (r-K)(s)\}.$$

The quaternionic curve β is related to the spatial quaternionic curve γ because that the unit spatial quaternion $N_1 \times \bar{T}$ is equal to the unit binormal vector b of the curve γ in [4]. And then by using the Frenet vectors and curvatures functions of the curve γ in $\mathbb{R}^3$, the other Frenet vectors and curvatures of β in $\mathbb{H}$ are obtained. The Frenet vectors β satisfying the following equalities:

$$\begin{aligned} N_1 &= b \times T,\ b = N_1 \times \bar{T},\ N_2 = n \times T,\ N_3 = t \times T, \\ h(T,T) &= h(N_1,N_1) = h(N_2,N_2) = h(N_3,N_3) = 1 \\ h(T,N_1) &= h(T,N_2) = h(T,N_3) = h(N_1,N_2) = h(N_2,N_3) = 0. \end{aligned} \tag{10}$$

We want to emphasize that the torsion of β is the torsion of the curve γ and the bitorsion of β is $(K - k)$, where k is the principal curvature of γ and K is the principal curvature of β .

Now, in the present paper, we define a new alternative moving quaternionic frame of the quaternionic curve of β in $\mathbb{H}$ with the help of a similar method given in [4]. This new quaternionic frame is a generalization of the frames given in [4] and [7], and it is given with the following theorems. Furthermore, we will give a Corollary which is one of the special cases of our theory but different from the theories in [4] and [7].

Theorem 3.7. *(Main Theorem) Let X be a smooth unit quaternion function of s in $\mathbb{H}$ and Y be a unit quaternion function constructed by X with the following equality $Y(s) = \frac{1}{\|X'(s)\|}X'(s)$. Then considering the quaternionic curve*

$$\begin{aligned} \beta :\ & I \subset R \longrightarrow \mathbb{H} \\ & s \longrightarrow \beta(s) = \beta_0(s)e_1 + \beta_1(s)e_2 + \beta_2(s)e_3 + \beta_3(s)e_4 \end{aligned}$$

related to pure quaternionic curve $\gamma_i = \gamma_i(s)$ (for $i = R$ and $i = L$ we get $\xi_R(s) = Y(s) \times \bar{X}(s)$ and $\xi_L(s) = \bar{X}(s) \times Y(s)$, respectively.) we can introduce two different types Frenet formulas with the help of $X(s)$ along the curve β. Here the most important point is that $\left\{\xi_i(s), \eta_i = \frac{\xi_i'}{\|\xi_i'\|}, \varrho_i(s) = \xi_i(s) \times \eta_i\ ;\ r_1(s), r_2(s)\right\}$ is the general moving orthonormal quaternionic frame of the curve γ_i. And we can note that it is constructed by the quaternionic curve β.

Type I (Right Frenet Frame):

$$\begin{aligned} X'(s) &= r_3(s)Y(s), \\ Y'(s) &= -r_3(s)X(s) + r_1(s)Z(s), \\ Z'(s) &= -r_1(s)Y(s) + (r_2 - r_3)(s)W(s), \\ W'(s) &= -(r_2 - r_3)(s)Zs). \end{aligned} \tag{11}$$

where $\{X(s), Y(s), Z(s) = \eta_R(s) \times X(s), W(s) = \varrho_R(s) \times X(s)\}$ define an orthonormal frame along the curve β with condition $\xi_R = Y \times \bar{X}$. And then, $r_3(s) = \|X'(s)\|$, $r_1(s)$ and $(r_2 - r_3)(s)$ are non-zero principal curvature, torsion and bitorsion of the curve β, respectively.

On the other hand, we should say a critical point that r_1 and r_2 are general curvatures and torsion of the curve γ_R, respectively.

Type II (Left Frenet Frame):

$$\begin{aligned} X'(s) &= r_3(s)Y(s), \\ Y'(s) &= -r_3(s)X(s) + r_1(s)Z(s), \\ Z'(s) &= -r_1(s)Y(s) + (r_2 + r_3)(s)W(s), \\ W'(s) &= -(r_2 + r_3)(s)Z(s). \end{aligned} \tag{12}$$

where $\left\{X(s), Y(s), Z(s) = X(s) \times \eta_L(s), W(s) = X(s) \times \varrho_L(s)\right\}$ define an orthonormal frame along the curve β with condition $\xi_L = \bar{X} \times Y$, $r_3(s) = \|X'(s)\|$, $r_1(s)$ and $(r_2 + r_3)(s)$ are non-zero principal curvature, torsion and bitorsion of the curve β, respectively.

On the other hand, we should say an important point that r_1 and r_2 are general curvatures and torsion of the curve γ_L, respectively.

Proof. Let β be a smooth quaternionic curve with the arc-length parameter s and X be a smooth unit quaternion function of s in $\mathbb{R}^4$. Since $\|X(s)\| = 1$, the curve β can be determined by $X(s) = \beta'(s)$. Then by using the definition of quaternionic inner product given in Eq.(4) we can write

$$X \times \bar{X} = 1. \tag{13}$$

The differantiation of the Eq.(13) with respect to parameter s, we obtained

$$X' \times \bar{X} + X \times \bar{X}' = 0. \tag{14}$$

The equality (14) implies two interesting and important results that first one is $X'(s)$ is h-orthogonal to X, that is, $h(X(s), X'(s)) = 0$ and second one is $X'(s) \times \bar{X}(s)$ is a smooth unit spatial quaternion. Furthermore, the principal curvature of the curve β is written by $r_3(s) = \|X'(s)\|$ and then we can construct a new smooth unit quaternion function $Y = \dfrac{1}{r_3(s)} X'(s)$ of the parameter s in $\mathbb{R}^4$. It gives us

$$X'(s) = r_3(s)Y(s), \ \ \|Y(s)\| = 1. \tag{15}$$

Then the equations (14) and (15) imply that

$$Y \times \bar{X} + X \times \bar{Y} = 0. \tag{16}$$

The equality (16) gives us that $Y(s)$ is h-orthogonal to $X(s)$, that is, $h(Y(s), X(s)) = 0$ and $Y(s) \times \bar{X}(s)$ or $\bar{X}(s) \times Y(s)$ is a spatial quaternion. Hence we can determine two-unit spatial quaternions such that $\xi_R(s) = Y(s) \times \bar{X}(s)$ or $\xi_L(s) = \bar{X}(s) \times Y(s)$. We can construct two different frames of the curve β in terms of them. Let's consider the unit and smooth spatial quaternion ξ_R because that the other one can be easily obtained using a similar method. It is clear that ξ_R is a unit spatial quaternion since both X and Y are unit quaternions in $\mathbb{R}^4$.

Since ξ_R a unit spatial quaternion, we can write $Y(s)$ as follows:

$$Y = \xi_R \times X \text{ where } \xi_R = Y \times \bar{X} \text{ along the curve } \beta. \tag{17}$$

The point that ξ_R is determined uniquely as a smooth unit spatial quaternion via the Eq.(17). Hence we can determine a new spatial quaternionic curve $\gamma = \gamma(s)$ in $\mathbb{R}^3$ with an orthonormal frame $\left\{ \xi_R, \eta_R = \frac{\xi_R}{\|\xi_R\|}, \varrho_R = \xi_R \times \eta_R \right\}$ on $\gamma(s)$. By using a similar method to construct Frenet formulae for a spatial quaternionic curve in $\mathbb{R}^3$ given in [7], we can easily construct general spatial quaternionic Frenet formulae as follows:

$$\begin{bmatrix} \xi_R'(s) \\ \eta_R'(s) \\ \varrho_R'(s) \end{bmatrix} = \begin{bmatrix} 0 & r_1(s) & 0 \\ -r_1(s) & 0 & r_2(s) \\ 0 & -r_2(s) & 0 \end{bmatrix} \begin{bmatrix} \xi_R(s) \\ \eta_R(s) \\ \varrho_R(s) \end{bmatrix} \tag{18}$$

where $r_1(s) = \left\| \xi_R(s) \right\|$ and $r_2(s)$ are the curvature and torsion of the curve $\beta(s)$ in the 3−dimensional space, respectively.

By differentiating the (17) with respect to s and using the Eqs.(15) and (18), we have

$$Y'(s) = r_1(s)\eta_R(s) \times X(s) + r_3(s)\xi_R(s) \times Y(s). \tag{19}$$

If we define a unit and smooth quaternion

$$Z(s) = \eta_R(s) \times X(s) \tag{20}$$

then using the Eq.(17), the Eq.(19) can be rewritten by

$$Y'(s) = -r_3(s)X(s) + r_1(s)Z(s). \tag{21}$$

About the quaternion $Z(s)$, it is easy to see that

$$h(X,Z) = h(Y,Z) = 0 \text{ and } h(Z,Z) = 1,$$

hence the quaternion $Z(s)$ is unit and h-orthogonal to the unit quaternions $X(s)$ and $Y(s)$.

Similarly, by taking derivatives of the Eq.(20) with respect to s and using the Eqs.(15) and (18), we get

$$Z'(s) = -r_1(s)\xi_{_R}(s) \times X(s) + r_2(s)\varrho_{_R}(s) \times X(s) + \eta_{_R}(s) \times \xi_{_R}(s) \times Y(s)$$

or defining a unit and smooth quaternion

$$W(s) = \varrho_{_R}(s) \times X(s) \tag{22}$$

and using the Eq.(17) we have

$$Z'(s) = -r_1(s)Y(s) + (r_2 - r_3)(s)W(s) \tag{23}$$

Then it can be easily calculated that

$$h(X,W) = h(Y,W) = h(Z,W) = 0 \text{ and } h(W,W) = 1,$$

hence the quaternion $W(s)$ is unit and h-orthoganal to the unit quaternions $X(s)$, $Y(s)$ and $Z(s)$.

Finally, by taking derivatives of the Eq.(22) with respect to s and using the Eqs.(15), (17) and (18), we obtain

$$W'(s) = -(r_2 - r_3)(s)Z(s). \tag{24}$$

With the help of the Eqs.(15), (19), (23) and (24) we have the Eq.(11).

The new orthonormal Frenet elements $\{X(s), Y(s), Z(s), W(s)\}$ in the 4-dimensional space of the quaternionic curve $\beta(s)$ is named as general **Type I** Frenet formulas of β.

With the help of $\xi_{_L}(s) = \bar{\mathsf{X}}(s) \times \mathsf{Y}(s)$ and similar operations of **Type I**, the new Frenet elements $\{\mathsf{X}(s), \mathsf{Y}(s), \mathsf{Z}(s), \mathsf{W}(s)\}$ in the 4-dimensional space of the quaternionic curve $\beta(s)$ curve are obtained as **Type II** Frenet formulas in (12). Thus, the proof is completed.

In the following notes, we will introduce a new approach to constructing the Serret-Frenet formulae and Frenet apparatus of the curve β in $\mathbb{R}^4$ in terms of alternative Frenet frame of spatial quaternionic curve γ in $\mathbb{R}^3$.

Assume that γ is a spatial quaternionic curve with alternative Frenet apparatus

$$\{n(s), c(s), w(s); f(s), g(s)\}$$

and β is a quaternionic curve with Frenet apparatus

$$\{T(s), N_1(s), N_2(s), N_3(s); K(s)\, k(s), (r-K)(s)\}.$$

The quaternionic curve β is related to the spatial quaternionic curve γ because of the fact that the unit spatial quaternion $N_1 \times \bar{T}$ is equal to the unit normal vector n of the curve γ in $\mathbb{R}^3$. And then by using the Frenet vectors and curvatures functions of the curve γ in $\mathbb{R}^3$, the other Frenet vectors and curvatures of β in $\mathbb{H}$ are obtained. The Frenet vectors β satisfying the following equalities:

$$\begin{aligned} N_1 &= n \times T,\ n = N_1 \times \bar{T},\ N_2 = c \times T,\ N_3 = w \times T, \\ h(T,T) &= h(N_1,N_1) = h(N_2,N_2) = h(N_3,N_3) = 1 \\ h(T,N_1) &= h(T,N_2) = h(T,N_3) = h(N_1,N_2) = h(N_2,N_3) = 0. \end{aligned} \tag{25}$$

Considering the above construction, we can give the following corollary for the quaternionic curve β in $\mathbb{H}$.

Corollary 3.8. *Let $\beta = \beta(s)$ be a unit speed quaternionic curve in $\mathbb{H}$ with an alternative Frenet apparatus $\{T(s), N_1(s), N_2(s), N_3(s), K(s), f(s), (g-K)(s)\}$. Then the new alternative Frenet formulas of the curve β are denoted by*

$$\begin{aligned} T^{'}(s) &= K(s)N_1(s), \\ N_1'(s) &= -K(s)T(s) + f(s)N_2(s), \\ N_2'(s) &= -f(s)N_1(s) + (g-K)(s)N_3(s), \\ N_3'(s) &= -(g-K)(s)N_2(s). \end{aligned} \tag{26}$$

where $f(s) = \left(k^2 + r^2\right)^{1/2}(s)$, $g(s) = \dfrac{k^2}{(k^2+r^2)^{3/2}}\left(\dfrac{r}{k}\right)'$, $T(s) = \beta'(s)$ *is unit tangent vector,* $N_1(s), N_2(s), N_3(s)$ *are unit normal vectors of the curve* $\beta(s)$. *Moreover* $K(s) = \|T'(s)\|$, $f(s)$ *and* $(g-K)(s)$ *denote the principal curvature, the torsion and the bitorsion of the curve* $\beta(s)$*, respectively.*

Definition 3.9. *Let $\beta(s) : I \subset \mathbb{R} \to \mathbb{H}$ be a quaternionic curve with non-zero curvatures in $\mathbb{H}$ and $\{X(s), Y(s), Z(s), W(s)\}$ be an orthonormal frame on along curve $\beta(s)$. We call the curve $\beta(s)$ as a* quaternionic right $X-$helix *if the unit vector fields X makes a constant angle θ with a fixed and unit direction U_R, that is,*

$$h(X, U_R) = \cos\theta,\ \theta = constant \neq \frac{\pi}{2}$$

where $\left\|U_R(s)\right\| = 1$ *and* U_R *is a constant reel quaternion named as the axis of the curve* β *for all* $s \in I$ *with the unit vector space on* $X(s)$.

Theorem 3.10. *Let $\beta(s) : I \subset \mathbb{R} \to \mathbb{H}$ be an arc-lengthed parameter real quaternionic curve and X be a unit real quaternion of $\mathbb{H}$ such that*

$$\{X(s), Y(s), Z(s), W(s); r_3(s), r_1(s), (r_2 - r_3)(s)\}$$

is Frenet apparatus of the quaternionic curve along the curve β. If the curve β is a real quaternionic right $X-$helix, then the axis of the β is given by

$$U_R(s) = (Q(s) \times X(s))\cos\theta$$

where $\rho(s) = \dfrac{r_3(s)}{r_1(s)}$, $\sigma(s) = \dfrac{\rho'(s)}{(r_2 - r_3)(s)}$ *and* $Q(s) = 1 + \rho(s)\eta_R(s) + \sigma(s)\rho_R(s)$ *is a real quaternion.*

Proof. Assume that $\beta(s)$ is a quaternionic right X-helix with non-zero curvatures in $\mathbb{H}$. Then the Definition (3.9) gives us

$$h(X, U_R) = \cos\theta,\ \theta \neq \frac{\pi}{2},\ \theta = constant \tag{27}$$

If we differantiate (27) with respect to s this equation, we obtain that

$$h(X', U_R) = 0$$

and considering the Eq.(11) we have

$$h(Y, U_R) = 0. \tag{28}$$

Again if we differantiate the last equation and applying the equation (11) we get

$$h(Z, U_R) = \rho(s)h(X, U_R) = \frac{r_3(s)}{r_1(s)}\cos\theta. \tag{29}$$

The differantiation of the (29) via the equation (11) gives us

$$h(Z', U_R) = \rho'(s)\cos\theta.$$

Since $h(Y, U_R) = 0$ the last equality can be written as

$$h(W(s), U_R) = \sigma(s)\cos\theta \tag{30}$$

Consequently, the axis of the right X-helix can be easily given by

$$U_R(s) = (Q(s) \times X(s))\cos\theta$$

where $\rho(s) = \dfrac{r_3(s)}{r_1(s)}$, $\sigma(s) = \dfrac{\rho'(s)}{(r_2 - r_3)(s)}$ and $Q(s) = 1 + \rho(s)\eta_R(s) + \sigma(s)\rho_R(s)$ is a real quaternion. Hence the proof is completed.

Definition 3.11. *Let $\beta : I \subset R \to \mathbb{H}$ be a regular real quaternionic curve given by arc-lengthed parameter s and $\{X(s), Y(s), Z(s), W(s); r_3(s), r_1(s), (r_2 - r_3)(s)\}$ be the Frenet apparatus of the curve β. If we consider $\rho(s) = \dfrac{r_3(s)}{r_1(s)}$ and $\sigma(s) = \dfrac{\rho'(s)}{(r_2 - r_3)(s)}$ then the real quaternion,*

$$\begin{aligned}\mathcal{D} &= Q(s) \times X(s)\\ &= X(s) + \rho(s)Z(s) + \sigma(s)W(s)\end{aligned}$$

is called the Darboux quaternion *of real quaternionic right X-helix β.*

Corollary 3.12. *Let $\beta : I \subset R \to \mathbb{H}$ be an arc-lengthed real quaternionic curve given and $\{X(s), Y(s), Z(s), W(s); r_3(s), r_1(s), (r_2 - r_3)(s)\}$ be the Frenet apparatus of the curve β. Then β is a real quaternionic right X-helix if and only if $\mathcal{D}$ is a constant real quaternion.*

Proof. It can be easily proved with the help of Theorem 3.10.

Theorem 3.13. *Let $\beta : I \subset R \to \mathbb{H}$ be an arc-lengthed real quaternionic curve given and $\{X(s), Y(s), Z(s), W(s); r_3(s), r_1(s), (r_2 - r_3)(s)\}$ be the Frenet apparatus of the curve β. Then β is a real quaternionic right X-helix if and only if*

$$\rho^2(s) + \sigma^2(s) \tag{31}$$

is a constant function where $\rho(s) = \dfrac{r_3(s)}{r_1(s)}$ and $\sigma(s) = \dfrac{\rho'(s)}{(r_2 - r_3)(s)}$.

Proof. Let $\beta(s)$ be a real quaternionic X-helix in $\mathbb{H}$ and the axis of the curve $\beta(s)$ be the unit vector $U_R(s)$. Then, we have $h(X(s), U_R(s))$ is a constant along to the curve. By differentiating this constant equation with the respect to s and using the right helix frame formulas in (11), we have

$$\begin{aligned}0 &= \frac{d}{ds}h(X(s), U_R(s)) = \frac{1}{2}\frac{d}{ds}(X(s) \times \bar{U}_R(s) + U_R(s) \times \bar{X}(s))\\ &= \frac{1}{2}(X'(s) \times \bar{U}_R(s) + U_R(s) \times \bar{X}'(s))\\ &= h(X'(s), U_R(s))\\ &= r_3(s)h(Y(s), U_R(s)).\end{aligned}$$

Therefore, the unit vector $U_R(s)$ can be written as follows:

$$U_R(s) = a_1X(s) + a_2(s)Z(s) + a_3(s)W(s) \ . \tag{32}$$

where $h(X(s), U_R(s)) = a_1 =$ constant, $h(Z(s), U_R(s)) = a_2(s)$, $h(W(s), U_R(s)) = a_3(s)$ and $a_1^2 + a_2(s)^2 + a_3(s)^2 = 1$.
If we differentiate the Eq.(32), we have

$$(a_1 r_3 - a_2 r_1)(s) Y(s) + a_2' - a_3)(s)(r_2 - r_3)(s) Z(s) + a_3' + a_2)(s)(r_2 - r_3)(s) W(s) = 0,$$

and then it implies that

$$\begin{aligned} a_1(s) r_3(s) - a_2(s) r_1(s) &= 0, \\ a_2'(s) - a_3(s)(r_2 - r_3)(s) &= 0, \\ \left(a_3'(s) + a_2(s)(r_2 - r_3)(s)\right) &= 0. \end{aligned}$$

The above equalities gives us

$$\begin{aligned} a_2(s) = &\quad \rho(s) a_1(s) = -\tfrac{a_3'(s)}{(r_2 - r_3)(s)}, \\ a_2'(s) = &\quad a_3(s)(r_2 - r_3)(s). \end{aligned} \tag{33}$$

By differentiating the first equation of (33) and using the second equation of (33), we obtain the ODE for $a_3(s)$ as follows

$$a_3''(s) - \frac{(r_2 - r_3)'(s)}{(r_2 - r_3)(s)} a_3' + (r_2 - r_3)^2(s) a_3 = 0. \tag{34}$$

If we change variable in (34) as $t = \int_0^s (r_2 - r_3)(s) ds$, then the equation (34) becomes

$$\frac{d^2}{dt^2} a_3(s) + a_3(s) = 0. \tag{35}$$

Thus, the solution of the differential equation (35) is given by

$$a_3(s) = A \cos t(s) + B \sin t(s), \tag{36}$$

for some constants A and B. From the first equation of (33) and (36), we find

$$\begin{aligned} a_2(s) = &\quad a_1(s) \rho(s) = A \sin t(s) - B \cot t(s), \\ a_3(s) = &\quad \sigma(s) a_1(s) = A \cos t(s) + B \sin t(s). \end{aligned}$$

From the above equations, we obtain

$$\begin{aligned} A = &\quad a_1(s)(\rho(s) \sin t(s) + \sigma(s) \cos t(s)), \\ B = &\quad a_1(s)(\sigma(s) \sin t(s) - \rho(s) \cos t(s)) \end{aligned}$$

which imply that

$$A^{2+} B^2 = a_1^2(s)\left(\rho^2(s) + \sigma^2(s)\right).$$

Thus, we have the equation (31).

Conversely, if the condition (31) holds, then we can always find a constant unit vector U_R satisfying $h(X, U_R)$ =constant. We consider the unit vector defined by

$$U_R(s) = X(s) + \rho(s) Z(s) + \sigma(s) W(s). \tag{37}$$

Differentiation of U_R is equal to zero, that is, $U_R' = 0$ means that U_R is a constant vector. Consequently, the curve $\beta(s)$ is a general right X-helix in $\mathbb{H}$.

Corollary 3.14. *Let $\beta : I \subset R \to \mathbb{H}$ be an arc-lengthed quaternionic curve given and $\{X(s), Y(s), Z(s), W(s); r_3(s), r_1(s), (r_2 - r_3)(s)\}$ be the Frenet apparatus of the curve β. Then β is a real quaternionic right X-helix if and only if*

$$\rho(s)(r_2 - r_3)(s) + \sigma'(s) = 0.$$

Proof. Assume that β is a real quaternionic right X-helix. Then the Corollary 3.12 gives us $\mathcal{D}$ is a constant real quaternion. If we differentiate $\mathcal{D}$ along the curve β and use the Eq. (11) we get $\rho(s)(r_2 - r_3)(s) + \sigma'(s) = 0$. Conversely if the equation $\rho(s)(r_2 - r_3)(s) + \sigma'(s) = 0$ holds it is easy to obtain that $\mathcal{D}' = 0$ or $\mathcal{D}$ is a constant real quaternion. Hence, the curve β is a real quaternionic right X-helix. This completes the proof.

Corollary 3.15. *Let $\beta : I \subset R \to \mathbb{H}$ be an arc-lengthed real quaternionic curve with quaternionic Frenet apparatus $\{X(s), Y(s), Z(s), W(s); r_3(s), r_1(s), (r_2 - r_3)(s)\}$ and the curve $\gamma(s) = \int t(s)ds$ be the spatial quternionic curve related to the curve β. If the curve β is a quaternionic CCR curve (see [5] and [10]) then the curve γ is a helix in $\mathbb{R}^3$.*

Proof. Let β be an arc-lengthed real quaternionic CCR curve with its curvatures $r_3, r_1, (r_2 - r_3)$ and $\gamma(s) = \int \xi_R(s)ds$ be the spatial quaternionic curve whose curvatures r_1, r_2. Since β is real quaternionic CCR curve $\frac{r_3}{r_1}$ and $\frac{r_2 - r_3}{r_1}$ are constants. Then it is easy to show that $\frac{r_2}{r_1}$ is constant. So, the curve γ is a helix in $\mathbb{R}^3$.

Example 3.16. *Let $\beta = \beta(s) = \cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}\,\vec{e}_1 + \frac{s}{\sqrt{3}}\vec{e}_2 + \frac{s}{\sqrt{3}}\vec{e}_3$ be a quaternionic helix curve in $\mathbb{H}$. Now, we will determine the spatial quaternionic curves $\gamma = \gamma(s)$ in $\mathbb{R}^3$ associated with the quaternionic curve β in $\mathbb{H}$.*

The quaternionic frame of the quaternionic helix curve β is calculated in terms of the theory given in [7].

The unit tangent vector β is given by

$$X(s) = T(s) = \beta'(s) = \frac{1}{\sqrt{3}}\left(-\sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}}, 1, 1\right). \tag{38}$$

The principal curvature of β is $r_3 = \|X'(s)\| = \frac{1}{3}$. Then the principal normal vector of β is

$$Y(s) = N_1(s) = \left(-\cos\frac{s}{\sqrt{3}}, -\sin\frac{s}{\sqrt{3}}, 0, 0\right). \tag{39}$$

Since $N_1 \times \bar{T}$ is a spatial quaternion, Bharathi and Nagaraj considered that $N_1 \times \bar{T} = t$ where t is a unit tangent vector of a space curve γ. It is given by

$$\xi_R(s) = t(s) = N_1 \times \bar{T} = \frac{1}{\sqrt{3}}\left(1, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}\right)$$

and integration of the last equation gives us

$$\gamma_R(s) = \int t(s)ds = \left(\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}} + \cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}} - \cos\frac{s}{\sqrt{3}}\right).$$

By applying the Frenet formulas in three-dimension real space, we obtain the other Frenet vectors of γ and we obtain curvatures of γ as follows:

$$\begin{aligned}
\eta_R(s) &= n(s) = \frac{1}{\sqrt{2}}\left(0, -\sin\frac{s}{\sqrt{3}} - \cos\frac{s}{\sqrt{3}}, -\sin\frac{s}{\sqrt{3}} + \cos\frac{s}{\sqrt{3}}\right),\\
\varrho_R(s) &= b(s) = \frac{1}{\sqrt{6}}\left(2, -\cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}, -\cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}\right),\\
r_1(s) &= k(s) = \|t'\| = \frac{\sqrt{2}}{3},\\
r_2(s) &= r(s) = \frac{1}{3},\\
(r_2 - r_3)(s) &= r(s) - K(s) = 0.
\end{aligned}$$

From the following equalities $Z(s) = N_2(s) = n(s) \times T(s)$ and $W(s) = N_3(s) = b(s) \times T(s)$ we can calculate

$$\begin{aligned}
Z(s) &= N_2(s) = \frac{1}{\sqrt{6}}\left(2\sin\frac{s}{\sqrt{3}}, -2\cos\frac{s}{\sqrt{3}}, 1, 1\right),\\
W(s) &= N_3(s) = (0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).
\end{aligned}$$

Similarly considering we can easily find the principal normal vector of the spatial quaternionic curve γ_L as:

$$\xi_L(s) = \mathsf{t}(s) = \bar{\mathsf{T}} \times \mathsf{N}_1 = \frac{1}{\sqrt{3}}\left(1, \sin\frac{s}{\sqrt{3}} + \cos\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}\right)$$

and integration of the last equation gives us

$$\gamma_L(s) = \int \mathsf{t}(s)ds = \left(\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}} - \cos\frac{s}{\sqrt{3}} +, \sin\frac{s}{\sqrt{3}} + \cos\frac{s}{\sqrt{3}}\right).$$

By applying the Frenet formulas in three-dimension real space, we obtain the other Frenet vectors of γ and we obtain curvatures of γ as follows:

$$\begin{aligned}
\eta_L(s) &= \mathsf{n}(s) = \frac{1}{\sqrt{2}}\left(0, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}, -\sin\frac{s}{\sqrt{3}} - \cos\frac{s}{\sqrt{3}}\right),\\
\varrho_L(s) &= \mathsf{b}(s) = \frac{1}{\sqrt{6}}\left(-2, \cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}\right),\\
\mathsf{k}(s) &= \mathsf{r}_1(s) = \|t'\| = \frac{\sqrt{2}}{3},\\
\mathsf{r}(s) &= \mathsf{r}_2(s) = \frac{1}{3}.
\end{aligned}$$

From the following equalities $\mathsf{Z}(s) = \mathsf{N}_2(s) = \mathsf{T}(s) \times \mathsf{n}(s)$ and $\mathsf{W}(s) = \mathsf{N}_3(s) = \mathsf{T}(s) \times \mathsf{b}(s)$ we can calculate

$$\begin{aligned}
\mathsf{Z}(s) &= \mathsf{N}_2(s) = \frac{1}{\sqrt{6}}\left(2\sin\frac{s}{\sqrt{3}}, -2\cos\frac{s}{\sqrt{3}}, 1, 1\right),\\
\mathsf{W}(s) &= \mathsf{N}_3(s) = (0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).
\end{aligned}$$

The quaternionic frame of the quaternionic helix curve β is calculated in terms of the theory given in [4](see for details Example 6).

Now we need to consider the quaternionic curve β given in [4] and determine the spatial quaternionic curves γ_R and γ_L whose principal normal vectors are equal to $N_1 \times \bar{T}$ and $\bar{T} \times N_1$, respectively. Furthermore, we know that the spatial quaternionic curves γ_R and γ_L are associated with the quaternionic curve $\beta(s)$.

With the help of (38) and (39), we can simply find the principal normal vector of the spatial quaternionic curve γ_R as

$$\xi_R(s) = n(s) = N_1 \times \bar{T} = \frac{1}{\sqrt{3}}\left(1, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}\right)$$

by applying the Theorem 3.2, we obtain the other alternative Frenet frame vectors of γ and we obtain curvatures of γ as follows:

$$\begin{aligned}
\eta_R(s) &= c(s) = \frac{1}{\sqrt{2}}\left(0, -\sin\frac{s}{\sqrt{3}} - \cos\frac{s}{\sqrt{3}}, -\sin\frac{s}{\sqrt{3}} + \cos\frac{s}{\sqrt{3}}\right),\\
\varrho_R(s) &= w(s) = \frac{1}{\sqrt{6}}\left(2, -\cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}, -\cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}\right),\\
f &= \sqrt{k^2 + r^2} = \|n'\| = \frac{\sqrt{2}}{3},\\
g &= 0
\end{aligned}$$

where we consider that $k = r = \frac{1}{3}$.

On the other hand, since there is $\vec{c} = -\frac{r_1}{f}\vec{t} + \frac{r_2}{f}\vec{b}$ and $w = \frac{r_2}{f}\vec{t} + \frac{r_1}{f}\vec{b}$, then there is $\frac{d\gamma}{ds} = t = \frac{r_2}{f}\vec{w} - \frac{r_1}{f}\vec{c}$. This question is $r_1^2 + r_2^2 = \frac{2}{9}$ because its $f = \sqrt{r_1^2 + r_2^2} = \frac{\sqrt{2}}{3}$. If $r_1 = \frac{1}{3}$ and $r_2 = \frac{1}{3}$ are selected then

$$\begin{aligned}
\frac{d\gamma}{ds} &= t = \left(\tfrac{1}{\sqrt{3}}, \tfrac{\sqrt{3}-1}{2\sqrt{3}}\cos\tfrac{s}{\sqrt{3}} + \tfrac{\sqrt{3}+1}{2\sqrt{3}}\sin\tfrac{s}{\sqrt{3}}, -\tfrac{\sqrt{3}+1}{2\sqrt{3}}\cos\tfrac{s}{\sqrt{3}} + \tfrac{\sqrt{3}-1}{2\sqrt{3}}\sin\tfrac{s}{\sqrt{3}}\right)\\
\gamma_R &= \left(\tfrac{s}{\sqrt{3}}, \tfrac{\sqrt{3}-1}{2}\sin\tfrac{s}{\sqrt{3}} - \tfrac{\sqrt{3}+1}{2}\cos\tfrac{s}{\sqrt{3}}, -\tfrac{\sqrt{3}+1}{2}\sin\tfrac{s}{\sqrt{3}} + \tfrac{1-\sqrt{3}}{2}\cos\tfrac{s}{\sqrt{3}})\right).
\end{aligned}$$

From the following equalities $Z(s) = N_2(s) = c(s) \times T(s)$ and $W(s) = N_3(s) = w(s) \times T(s)$ we can calculate

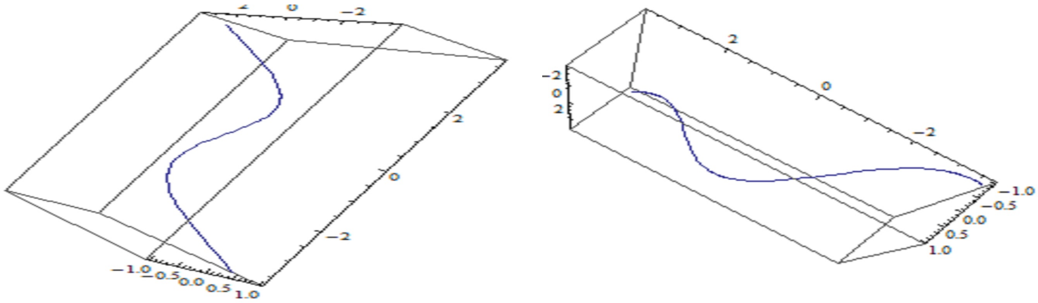
$$\begin{aligned}
Z(s) &= \frac{1}{\sqrt{6}}\left(2\sin\frac{s}{\sqrt{3}}, -2\cos\frac{s}{\sqrt{3}}, 1, 1\right),\\
W(s) &= \frac{1}{\sqrt{2}}(0, 0, -1, -1),\\
(r_2 - r_3)(s) &= r(s) - K(s) = 0.
\end{aligned}$$

Similarly considering we can easily find the principal normal vector of the spatial quaternionic curve γ_L as

$$\xi_L(s) = \mathrm{n(s)} = \bar{\mathrm{T}} \times \mathrm{N}_1 = \frac{1}{\sqrt{3}}\left(1, \cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}\right)$$

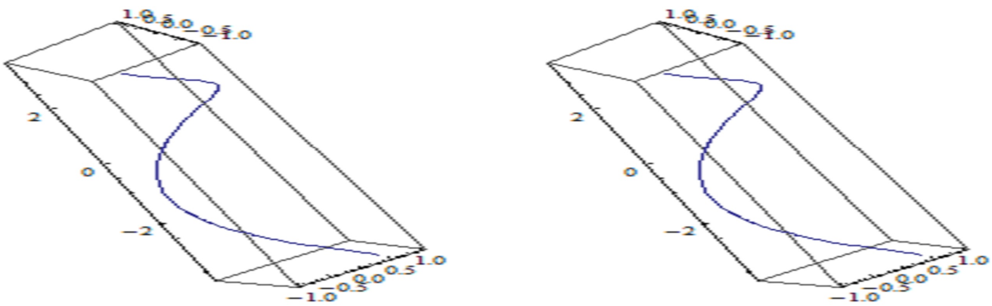
by applying the Theorem 3.2, we obtain the other alternative Frenet frame vectors of γ and we obtain curvatures of γ as follows:

$$\begin{aligned}
\eta_L(s) &= \mathrm{c(s)} = \frac{1}{\sqrt{2}}\left(0, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}, -\sin\frac{s}{\sqrt{3}} - \cos\frac{s}{\sqrt{3}}\right),\\
\varrho_L(s) &= \mathrm{w(s)} = \frac{1}{\sqrt{6}}\left(-2, \cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}\right),\\
f &= \sqrt{\mathrm{k}^2+\mathrm{r}^2} = \|\mathrm{n}'\| = \frac{\sqrt{2}}{3},\\
g &= 0
\end{aligned}$$



(a) (b)

Figure 1: (a) Orthogonal projection for $X = 0$, (b) Orthogonal projection for $Y = 0$.



(a) (b)

Figure 2: (a) Orthogonal projection for $Z = 0$, (b) Orthogonal projection for $W = 0$.

where we consider that $\mathsf{k} = \mathsf{r} = \frac{1}{3}$.

On the other hand, since there is $\tilde{\mathsf{c}} = -\frac{\mathsf{r}_1}{f}\tilde{\mathsf{t}}+\frac{\mathsf{r}_2}{f}\tilde{\mathsf{b}}$ and $\mathsf{w} = \frac{\mathsf{r}_2}{f}\tilde{\mathsf{t}}+\frac{\mathsf{r}_1}{f}\tilde{\mathsf{b}}$, then there is $\frac{d\gamma}{ds} = \mathsf{t} = \frac{r_2}{f}\tilde{\mathsf{w}}-\frac{r_1}{f}\tilde{\mathsf{c}}$. This question is $\mathsf{r}_1^2+\mathsf{r}_2^2=\frac{2}{9}$ because it's $\mathsf{f} = \sqrt{r_1^2 + r_2^2} = \frac{\sqrt{2}}{3}$. If $\mathsf{r}_1 = \frac{1}{3}$ and $\mathsf{r}_2 = \frac{1}{3}$ are selected then

$$\begin{aligned} \frac{d\gamma}{ds} &= \mathsf{t} = \left(-\frac{1}{\sqrt{3}}, \frac{1-\sqrt{3}}{2\sqrt{3}}\cos\frac{s}{\sqrt{3}} + \frac{1+\sqrt{3}}{2\sqrt{3}}\sin\frac{s}{\sqrt{3}}, \frac{1+\sqrt{3}}{2\sqrt{3}}\cos\frac{s}{\sqrt{3}} + \frac{\sqrt{3}-1}{2\sqrt{3}}\sin\frac{s}{\sqrt{3}}\right), \\ \gamma_L &= \left(-\frac{s}{\sqrt{3}}, \frac{1-\sqrt{3}}{2\sqrt{3}}\sin\frac{s}{\sqrt{3}} - \frac{1+\sqrt{3}}{2\sqrt{3}}\cos\frac{s}{\sqrt{3}}, \frac{1+\sqrt{3}}{2\sqrt{3}}\sin\frac{s}{\sqrt{3}} + \frac{1-\sqrt{3}}{2\sqrt{3}}\cos\frac{s}{\sqrt{3}}\right). \end{aligned}$$

From the following equalities $\mathsf{Z}(\mathsf{s}) = \mathsf{N}_2(\mathsf{s}) = \mathsf{T}(\mathsf{s}) \times \mathsf{c}(\mathsf{s})$ and $\mathsf{W}(\mathsf{s}) = \mathsf{N}_3(\mathsf{s}) = \mathsf{T}(\mathsf{s}) \times \mathsf{w}(\mathsf{s})$ we can calculate

$$\begin{aligned} \mathsf{Z}(\mathsf{s}) &= \mathsf{N}_2(\mathsf{s}) = \frac{1}{\sqrt{6}}\left(2\sin\frac{s}{\sqrt{3}}, -2\cos\frac{s}{\sqrt{3}}, 1, 1\right), \\ \mathsf{W}(\mathsf{s}) &= \mathsf{N}_3(\mathsf{s}) = \left(0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \end{aligned}$$

In the following pictures, we will give the orthogonal projection of the curve β on the hyperplanes $x = 0$ (`Figure 1`(a)), $y = 0$ (`Figure 1`(b)), $z = 0$ (`Figure 2`(a))and $w = 0$ (`Figure 2`(b)), respectively.

Finally, in the following figures, we will give the picture of the curves γ_R and γ_L (`Figure 3`) related to the curve β in terms of two pure quaternions $\xi_R(s)$ and $\xi_L(s)$.

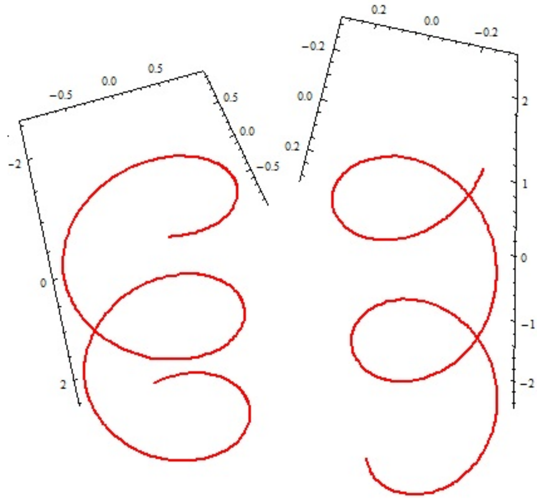


Figure 3: Left-oriented and right-oriented spatial quaternionic helices related to the curve β.

4. Conclusions

In this study, the general helix curves are given by a new perspective using quaternions. Since quaternions do not provide the commutative property for multiplication, the authors([3], [4], [7]) have been used by one-sided multiplication in previous studies. We have obtained new expressions for quaternionic curves using the right product and the left product for quaternions. It is also possible to see that the paper is a generalization of some papers published in the last. If we explain clearly that the authors Bharathi and Nagaraj [7] considered $\xi_R(s) = t(s)$ and the author Aksoyak [4] considered $\xi_R(s) = b(s)$ in terms of our theory. But we considered a general unit spatial quaternion $\xi_R(s)$. To prove this fact with as an example we consider the quaternionic curve

$$\beta(s) = \left(\cos\left(\frac{s}{\sqrt{3}}\right), \sin\left(\frac{s}{\sqrt{3}}\right), \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right)$$

given in [4] we obtain spatial quaternionic curves $\gamma_R = \gamma(s)$ and $\gamma_L = \gamma(s)$ whose principal normal vector is $n(s)$. Finally, we draw the orthogonal projection figures of the curve β and the curves $\gamma_R = \gamma(s)$ and $\gamma_L = \gamma(s)$.

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