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On Refined Generalised Quasi-Adequate Transversals

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Abstract. Some properties and characterizations for abundant semigroups with generalised quasi-adequate transversals are explored. In such semigroups, an interesting property $[\forall a, b \in RegS,\ V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)]$ is investigated and thus the concept of *refined generalised quasi-adequate transversals*, for short, *RGQA transversals* is introduced. It is shown that RGQA transversals are the *real* common generalisations of both orthodox transversals and adequate transversals in the abundant case. Finally, by means of two abundant semigroups R and L, a spined product structure theorem for an abundant semigroup with a quasi-ideal RGQA transversal is established.

1. Introduction and preliminaries

Suppose that S is a regular semigroup with a subsemigroup S^o. We denote the intersection of $V(a)$ and S^o by $V_{S^o}(a)$ and that $I = \{aa^o : a \in S, a^o \in V_{S^o}(a)\}$ and $\Lambda = \{a^o a : a \in S, a^o \in V_{S^o}(a)\}$. An *inverse transversal* of the semigroup S is a subsemigroup S^o that contains exactly one inverse of every element of S, that is, S^o is an inverse semigroup with $|V_{S^o}(a)| = 1$. This important concept was introduced by Blyth and McFadden [1]. Thereafter, this class of regular semigroups excited many semigroup researchers' attention and a good deal of important results were obtained (see [1-4] and their references). Tang [4] shown that for S a regular semigroup with an inverse transversal S^o, then I and Λ are both bands with I left regular and Λ right regular. These two bands play a key role in the study of regular semigroups with inverse transversals. Other important subsets of S are $R = \{x \in S : x^o x = x^o x^{oo}\}$ and $L = \{x \in S : xx^o = x^{oo} x^o\}$. Both R and L are subsemigroups with R left inverse (i.e. R an orthodox semigroup with a left regular band of idempotents) and L right inverse (i.e. L an orthodox semigroup with a right regular band of idempotents). The concept of *orthodox transversals* was introduced by Chen [5] as a generalisation of inverse transversals.

Definition 1.1 [5] Let S be a regular semigroup with an orthodox subsemigroup of S^o. Then S^o is said to be an *orthodox transversal* of S, if the following two conditions are satisfied:
(1) $(\forall\, a \in S)\quad V_{S^o}(a) \neq \emptyset$;
(2) For any $a, b \in S$, if $\{a, b\} \cap S^o \neq \emptyset$, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$.

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Some elementary properties associated with orthodox transversals were obtained by Chen and Guo [6]. In [7,8], Kong and Zhao introduced two interesting sets R and L and established the structure theorems for regular semigroups with quasi-ideal orthodox transversals. In 2014, Kong [9] introduced the concept of *generalised orthodox transversals* and obtained some basic properties associated with them. Kong and Meng [10] acquired the characterization for generalised orthodox transversals to be orthodox transversals.

Lemma 1.2 [10, **Theorem2.1**] *Let S be a regular semigroup and S^o a subsemigroup of S with $V_{S^o}(a) \neq \emptyset$ for each $a \in S$. Then S^o is an orthodox transversal of S if and only if*

$$(\forall a, b \in S) \quad [V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)].$$

More recently, Kong [11] investigated the weakly simplistic orthodox transversal and obtained the equivalent conditions for sets I and Λ to be bands.

On a semigroup S the relation $\mathcal{L}^*$ is defined by $a\ \mathcal{L}^*\ b$ if and only if the elements a, b of S are related by Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is dually defined. Certainly, $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ a left congruence with $\mathcal{L}\subseteq\mathcal{L}^*, \mathcal{R}\subseteq\mathcal{R}^*$ and if a, b are regular elements of S, then $a\ \mathcal{L}^*\ b$ ($a\ \mathcal{R}^*\ b$) if and only if $a\ \mathcal{L}\ b$ ($a\ \mathcal{R}\ b$). A semigroup is called *abundant* [12] if each $\mathcal{L}^*$- class and each $\mathcal{R}^*$- class contains at least one idempotent. An abundant semigroup S is called *quasi-adequate* [13] (*adequate*) if its idempotents form a subsemigroup (semilattice). We list some basic results as follows which are frequently used in this paper. The following two Lemmas are due to Fountain [12] and provides alternative descriptions for $\mathcal{L}^*$ ($\mathcal{R}^*$).

Lemma 1.3 **[12]** *Let S be a semigroup and $a, b \in S$. Then the following conditions are equivalent:*

(1) *$a\ \mathcal{L}^*\ b$ ($a\ \mathcal{R}^*\ b$);*

(2) *For all $x, y \in S^1$, $ax = ay$ ($xa = ya$) if and only if $bx = by$ ($xb = yb$).*

Lemma 1.4 **[12]** *Let a be an element of a semigroup S and e be an idempotent of S. Then the following conditions are equivalent:*

(1) *$a\ \mathcal{L}^*\ e$ ($a\ \mathcal{R}^*\ e$);*

(2) *$a = ae$ ($ea = a$) and for all $x, y \in S^1$, $ax = ay$ ($xa = ya$) implies $ex = ey$ ($xe = ye$).*

Let S be an abundant semigroup and U an abundant subsemigroup of S. U is called a $*$-subsemigroup of S if for any $a \in U$, there exist idempotents $e \in L^*_a(S)\cap U$ and $f \in R^*_a(S)\cap U$. As pointed out in [14], an abundant subsemigroup U of an abundant semigroup S is a $*$-subsemigroup of S if and only if $\mathcal{L}^*(U) = \mathcal{L}^*(S)\cap(U\times U)$ and $\mathcal{R}^*(U) = \mathcal{R}^*(S)\cap(U\times U)$. The concept of *adequate transversals*, was introduced for abundant semigroups by El-Qallali [14] as an analogue of the concept of inverse transversals.

Definition 1.5 [14] Let S be an abundant semigroup, S^o a $*$-adequate subsemigroup of S. S^o is called *an adequate transversal* of S if for each $x \in S$, there are a unique element $\overline{x} \in S^o$ and idempotents $e, f \in E$ such that $x = e\overline{x}f$, where $e\ \mathcal{L}\ \overline{x}^+$ and $f\ \mathcal{R}\ \overline{x}^*$. It can easily be shown that e and f are uniquely determined by x and S^o (see [14]).

Chen, Guo and Shum [15, 16] obtained some important results about quasi-ideal adequate transversals. Afterwards, Kong [17] considered some properties associated with adequate transversals. The authors [18] explored the product of quasi-ideal adequate transversals and proposed the open problem of the isomorphism of adequate transversals.

Lemma 1.6 **[19]** *Let S be an abundant semigroup with set of idempotents E and $x, y \in S$. If there exist $e, f \in E$ such that $x = eyf$ and $e\ \mathcal{L}\ y^+$, $f\ \mathcal{R}\ y^*$ for some $y^+, y^* \in E$, then $e\ \mathcal{R}^*\ x$ and $f\ \mathcal{L}^*\ x$.*

Let S be an abundant semigroup with set of idempotents E and S^o a quasi-adequate $*$−subsemigroup of S with set of idempotents E^o. The semigroup S^o is called a *generalised quasi-adequate transversal* of S, if for any $x \in S$,

$$C_{S^o}(x) = \{\overline{x} \in S^o \mid x = i_x\overline{x}\lambda_x,\ i_x, \lambda_x \in E, i_x\ \mathcal{L}\ \overline{x}^+, \lambda_x\ \mathcal{R}\ \overline{x}^* \text{ for some } \overline{x}^+, \overline{x}^* \in E^o\} \neq \emptyset.$$

Let

$$I_x = \{i_x \in E \mid (\exists \overline{x} \in C_{S^o}(x))\, x = i_x\overline{x}\lambda_x,\ i_x, \lambda_x \in E, i_x \,\mathcal{L}\, \overline{x}^+, \lambda_x \,\mathcal{R}\, \overline{x}^* \text{ for some } \overline{x}^+, \overline{x}^* \in E^o\},$$

$$\Lambda_x = \{\lambda_x \in E \mid (\exists \overline{x} \in C_{S^o}(x))\, x = i_x\overline{x}\lambda_x,\ i_x, \lambda_x \in E, i_x \,\mathcal{L}\, \overline{x}^+, \lambda_x \,\mathcal{R}\, \overline{x}^* \text{ for some } \overline{x}^+, \overline{x}^* \in E^o\},$$

$$I = \bigcup_{x\in S} I_x, \quad \Lambda = \bigcup_{x\in S} \Lambda_x.$$

The generalised quasi-adequate transversal S^o is called a *quasi-adequate transversal* of S if it satisfies a further condition

$$(QA2): \ (\forall e \in E)\ (\forall g \in E^o), C_{S^o}(e)C_{S^o}(g) \subseteq C_{S^o}(ge) \text{ and } C_{S^o}(g)C_{S^o}(e) \subseteq C_{S^o}(eg).$$

The concept of *quasi-adequate transversals* was introduced by Ni [19] and followed by Luo, Kong and Wang [20, 21], their work mainly focused on the properties and the structure of multiplicative quasi-adequate transversals. Unfortunately, quasi-adequate transversals are neither the generalisations of orthodox transversals nor the generalisations of adequate transversals. Wang gave an example (see [22] Example 5.2) to show that an orthodox transversal S^o of a regular semigroup S may not be a quasi-adequate transversal of S. An example given by Chen [15] demonstrating that in general, an adequate transversal S^o of an abundant semigroup S is not a quasi-adequate transversal of S. Let $S = \{e, g, h, w, f\}$ with set of idempotents $E = \{e, g, h, f\}$ and $S^o = \{w, e, f, g\}$ with set of idempotents $E^o = \{e, f, g\}$. Then by the multiplication table (for detail, see [15] Example 2.7), S^o is a quasi-ideal adequate transversal of S. But S^o is not a quasi-adequate transversal of S, since $C_{S^o}(h) = \{e\}, C_{S^o}(f) = \{f\}$, while $fe = g \notin C_{S^o}(fh) = C_{S^o}(w) = \{w\}$.

To achieve the *real* common generalisations of both orthodox transversals and adequate transversals in the abundant case, inspired by the essential characterization of orthodox transversals (see Lemma 1.2), in this paper, we introduce the concept of refined generalised quasi-adequate transversals. A generalised quasi-adequate transversal S^o of an abundant semigroup S is called a *refined generalised quasi-adequate transversal*, if it satisfies

$$(\forall a, b \in RegS), \ [V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)].$$

Then we obtain the connection of refined generalised quasi-adequate transversals with orthodox transversals and adequate transversals (see the following Theorem 3.2 in this paper).

Theorem 3.2 *Let S^o be a refined generalised quasi-adequate transversal of the abundant semigroup S. Then*
(i) S^o is an orthodox transversal of S if and only if S is a regular semigroup.
(ii) S^o is an adequate transversal of S if and only if S^o is an adequate semigroup.

Therefore, in the class of abundant semigroups, refined generalised quasi-adequate transversals are the generalisation of both orthodox transversals and adequate transversals. Two significant components R and L are introduced in this paper and described by Green's $*$-relations. By means of R and L, a spined product structure theorem is established for abundant semigroups with quasi-ideal refined generalised quasi-adequate transversals. Followed this paper, the product of quasi-ideal refined generalised quasi-adequate transversals [23] and quasi-Ehresmann transversals [24] was considered.

A subsemigroup S^o of S is called a *quasi-ideal* of S if $S^oSS^o \subseteq S^o$. The so called Miller-Clifford theorem will be used frequently.

Lemma 1.7 [25] (1) *Let e and f be $\mathcal{D}$-equivalent idempotents of a semigroup S. Then each element a of $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$, such that $aa' = e$ and $a'a = f$;*

(2) *Let a, b be elements of a semigroup S. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

2. Generalised quasi-adequate transversals

The objective in this section is to investigate some elementary properties associated with abundant semigroups with generalised quasi-adequate transversals. Also in this section we introduce two sets R and L

and use them to obtain some equivalence conditions for a generalised quasi-adequate transversal to be a quasi-ideal. For any result concerning R there is a dual result for L which we list but omit its proof.

Proposition 2.1 *Let S be an abundant semigroup with a generalised quasi-adequate transversal S^o. Then*
(1) $I = \{e \in E : (\exists e^* \in E^o)\ e\ \mathcal{L}\ e^*\}$ *and* $\Lambda = \{f \in E : (\exists f^+ \in E^o)\ f\ \mathcal{R}\ f^+\}$;
(2) $I \cap \Lambda = E^o$.

Proof. This is evident since in the proof of Lemma 2.1 in [21] the condition (QA2) was not used. □

Proposition 2.2 *Let S^o be a generalised quasi-adequate transversal of an abundant semigroup S. Then $\mathcal{D}^{*S^o} = \mathcal{D}^{*S} \cap (S^o \times S^o)$.*

Proof. If $a^o, b^o \in S^o$ are such that $a^o\ \mathcal{D}^{*S}\ b^o$, then $R^*_{a^o} \cap L^*_{b^o} \neq \emptyset$. For any $d \in R^*_{a^o} \cap L^*_{b^o}$, there exist $a^{o+}, b^{o*} \in E^o$ such that $a^{o+}\ \mathcal{R}^*\ a^o\ \mathcal{R}^*\ d\ \mathcal{L}^*\ b^o\ \mathcal{L}^*\ b^{o*}$ by $a^o, b^o \in S^o$ and S^o is quasi-adequate. By means of the definition of a generalised quasi-adequate transversal, $d = i_d\overline{d}\lambda_d$, where $i_d\ \mathcal{L}\ \overline{d}^+, \lambda_d\ \mathcal{R}\ \overline{d}^*$ for some $\overline{d}^+, \overline{d}^* \in E^o$. Furthermore $i_d\ \mathcal{R}^*\ d\ \mathcal{L}^*\ \lambda_d$, and so $a^{o+}\ \mathcal{R}^*\ d\ \mathcal{R}^*\ i_d\ \mathcal{L}\ \overline{d}^+$. It follows from Proposition 2.1 that $i_d \in I \cap \Lambda = E^o$ and similarly $\lambda_d \in E^o$. Thus $d = i_d\overline{d}\lambda_d \in E^o \cdot S^o \cdot E^o \subseteq S^o$, and so $a^o\ \mathcal{D}^{*S^o}\ b^o$. Therefore $\mathcal{D}^{*S} \cap (S^o \times S^o) \subseteq \mathcal{D}^{*S^o}$ and the reverse inclusion is obvious. □

Proposition 2.3 *Let S^o be a generalised quasi-adequate transversal of an abundant semigroup S. Then for every regular element a of S, a has an inverse a' in S^o. In this case, $V_{S^o}(a') \subseteq C_{S^o}(a)$.*

Proof. Take any regular element $a \in S$, then $a = i_a\overline{a}\lambda_a$ for some $i_a \in I_a, \overline{a} \in C_{S^o}(a), \lambda_a \in \Lambda_a$, where $e_a\ \mathcal{L}\ \overline{a}^+, \lambda_a\ \mathcal{R}\ \overline{a}^*$ for some $\overline{a}^+, \overline{a}^* \in E^o$. It follows from a, i_a, λ_a are all regular and $i_a\ \mathcal{R}^*\ a\ \mathcal{L}^*\ \lambda_a$ that $i_a\ \mathcal{R}\ a\ \mathcal{L}\ \lambda_a$, so by Lemma 1.7 a has an inverse a' in $R_{\lambda_a} \cap Li_a$. Thus $\overline{a}^*\ \mathcal{R}\ \lambda_a\ \mathcal{R}\ a'\ \mathcal{L}\ i_a\ \mathcal{L}\ \overline{a}^+$ and so by Proposition 2.2, $a' \in S^o$. □

Proposition 2.4 *Let S^o be a generalised quasi-adequate transversal of an abundant semigroup S. Then the relation $\mathcal{H}^*$ on S saturates S^o, that is, S^o is the union of some $\mathcal{H}^*$-classes on S.*

Proof. Let H^* be an $\mathcal{H}^*$-class of S, $H^* \cap S^o \neq \emptyset$. We shall prove that $H^* \subseteq S^o$. Take $x^o \in H^* \cap S^o$, since S^o is quasi-adequate, there exist $x^{o+}, x^{o*} \in E^o$ such that $x^{o+}\ \mathcal{R}^*(S^o)\ x^o\ \mathcal{L}^*(S^o)\ x^{o*}$. Since S^o is a $*$-subsemigroup, then we can assume that $x^{o+}\ \mathcal{R}^*(S)\ x^o\ \mathcal{L}^*(S)\ x^{o*}$. Now take $h \in H^*$, then $h = i_h\overline{h}\lambda_h$ with $i_h \in I_h, \lambda_h \in \Lambda_h$ and $i_h\ \mathcal{R}^*\ h\ \mathcal{L}^*\ \lambda_h$. It follows that $i_h\ \mathcal{R}^*\ h\ \mathcal{R}^*\ x^{o+}$ and so by Proposition 2.1, $i_h \in \Lambda$. Consequently, $i_h \in I \cap \Lambda = E^o$. Similarly, $\lambda_h \in E^o$. Therefore, $h = i_h\overline{h}\lambda_h \in E^oS^oE^o \subseteq S^o$. □

Proposition 2.5 *Let S be an abundant semigroup with a generalised quasi-adequate transversal S^o. Let*

$$R = \{x \in S : (\exists \lambda_x \in \Lambda_x)\ \lambda_x \in E^o\} \ \text{ and } \ L = \{p \in S : (\exists i_p \in I_p)\ i_p \in E^o\}.$$

Then

$$R = \{x \in S : (\exists l \in E^o)\ x\ \mathcal{L}^*\ l\} \ \text{ and } \ L = \{p \in S : (\exists h \in E^o)\ p\ \mathcal{R}^*\ h\}$$

with $R \cap L = S^o, E(R) = I$ and $E(L) = \Lambda$.

Proof. If $x \in R$, there exists $\lambda_x \in E^o$ such that $x\ \mathcal{L}^*\ \lambda_x$.

Conversely, for $x \in S$ if $x\ \mathcal{L}^*\ l$ for some $l \in E^o$, then $\lambda_x\ \mathcal{L}^*\ x\ \mathcal{L}^*\ l$. It follows from Proposition 2.1 that $\lambda_x \in I$ and so $\lambda_x \in I \cap \Lambda = E^o$. □

It is clear that if there exists $\lambda_x \in \Lambda_x$ such that $\lambda_x \in E^o$ ($i_\in I_a$ such that $i_a \in E^o$), then $\Lambda_x \subseteq E^o$ ($I_a \subseteq E^o$).

Proposition 2.6 *Let S^o be a generalised quasi-adequate transversal of an abundant semigroup S. The following statements are equivalent:*
(1) *S^o is a quasi-ideal of S;*
(2) $\Lambda I \subseteq S^o$;
(3) $E^o I \subseteq S^o$ and $\Lambda E^o \subseteq S^o$;
(4) $LR \subseteq S^o$;
(5) S^o is a left ideal of L and a right ideal of R;
(6) $S^o I \subseteq S^o$ and $\Lambda S^o \subseteq S^o$;
(7) $S^o I S^o \subseteq S^o$ and $S^o \Lambda S^o \subseteq S^o$.
(8) $SS^o \subseteq R, S^o S \subseteq L$;
(9) *R is a left ideal and L is a right ideal of S.*

Proof. (1) $\Longrightarrow$ (2). For any $\lambda \in \Lambda$ and $i \in I$, there exist $\lambda^+, i^* \in E^o$ such that $\lambda\ \mathcal{R}\ \lambda^+$ and $i\ \mathcal{L}\ i^*$. So we have $\lambda i = \lambda^+ \cdot \lambda i \cdot i^* \in E^o S E^o \subseteq S^o S S^o \subseteq S^o$.

(2) $\Longrightarrow$ (3). This is trivial.

(3) $\Longrightarrow$ (4). For any $p \in L$ and $x \in R$, there exist $h, l \in E^o$ such that $p\mathcal{R}^*h$ and $x\mathcal{L}^*l$. Thus

$$px = h(px)l = hi_{px}\overline{px}\lambda_{px}l \in E^o I \overline{px} \Lambda E^o \subseteq S^o \overline{px} S^o \subseteq S^o.$$

(4) $\Longrightarrow$ (5). This is clear since $S^o \subseteq L, R$.

(5) $\Longrightarrow$ (6). This is clear since $I = E(R)$ and $\Lambda = E(L)$.

(6) $\Longrightarrow$ (7). This is obvious.

(7) $\Longrightarrow$ (8). If (7) holds, then for any $a \in S, x^o \in S^o$, we have

$$ax^o = i_a \cdot \overline{a}\lambda_a \cdot x^o\ \mathcal{L}^*\ \overline{a}^+\overline{a}\lambda_a x^o = \overline{a}\lambda_a x^o\ \mathcal{L}^*\ (\overline{a}\lambda_a x^o)^* \in E^o,$$

since $\overline{a}\lambda_a x^o \in S^o \Lambda S^o \subseteq S^o$. Hence $ax^o \in R$ by Proposition 2.5 and $SS^o \subseteq R$. Dually $S^o S \subseteq L$.

(8) $\Longrightarrow$ (9). For any $a \in S, x \in R$, $x = i_x x^o \lambda_x$ with $\lambda_x \in E^o$, we have $ax = ai_x \cdot x^o \lambda_x \in SS^o \subseteq R$ and R is a left ideal of S. Dually, $S^o S \subseteq L$ implies that L is a right ideal of S.

(9) $\Longrightarrow$ (1). For any $s, t \in S^o$ and $a \in S$, we have

$$sat = (sa)t \in SS^o \subseteq SR \subseteq R \text{ and } sat = s(at) \in S^o S \subseteq LS \subseteq L.$$

Consequently $sat \in R \cap L = S^o$ and S^o is a quasi-ideal of S. □

Proposition 2.7 *Suppose that S is an abundant semigroup with a quasi-ideal generalised quasi-adequate transversal S^o. Let R and L be defined as in* Proposition 2.5. *Then R and L are abundant semigroups sharing a common generalised quasi-adequate transversal S^o which is a right ideal of R and a left ideal of L. In particular, since S^o is a right ideal of S, then $\Lambda_x \subseteq E^o$ for every $x \in S$ and $E = I$, and there is a dual result for S^o being a left ideal of S.*

Proof. Since both a left ideal and a right ideal are subsemigroups, if one of the conditions in Proposition 2.6 is satisfied, then R and L are subsemigroups. From Proposition 2.5 we deduce that R and L are abundant semigroups with $E(R) = I$ and $E(L) = \Lambda$. Let $x \in R$ and $y^o \in S^o$, then $y^o x = y^o x \lambda_x \in S^o$ for some $\lambda_x \in E^o$ since S^o is a quasi-ideal of S. Thus S^o is a right ideal of R, and dually S^o is a left ideal of L.

For any $\lambda_x \in \Lambda_x$, by Proposition 2.1, $\lambda_x\ \mathcal{R}\ l \in E^o$ and so $\lambda_x = l\lambda_x \in E^o$ since S^o is a right ideal of S. Consequently, for any $h \in E$, $h\ \mathcal{L}\ \lambda_h \in E^o$ and thus $h \in I$. □

3. Refined generalised quasi-adequate transversals

In an abundant semigroup S with a generalised quasi-adequate transversal S^o, the property [$\forall a, b \in$

$RegS$, $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)]$ is investigated and thus the concept of *refined generalised quasi-adequate transversals*, for short, *RGQA transversals* is introduced. It is shown that RGQA transversals are the *real* common generalisations of both orthodox transversals and adequate transversals in the abundant case. Also, for $a \in RegS$, a new description of $C_{S^o}(a)$ is established, thus providing a sufficient condition for $C_{S^o}(a) = V_{S^o}(a)$.

Theorem 3.1 *Let S be an abundant semigroup with a generalised quasi-adequate transversal S^o. Then*

$$(\forall a, b \in RegS), \;\; [V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)]$$

if and only if $IE^o, E^o\Lambda \subseteq E$ and for all $i \in I, \lambda \in \Lambda, e^o \in E^o$, if $e^o i, \lambda e^o$ are regular, then they are idempotent.

Proof. (Necessity) Suppose $e^o \in E^o, i \in I$ and $i \mathcal{L} i^* \in E^o$. From $ie^o \mathcal{L}^* i^*e^o$ and E^o is a band, we deduce that $ie^o \cdot e^o i^* \cdot ie^o = i \cdot e^o e^o \cdot i^* i \cdot e^o = ie^o \cdot i^* e^o = ie^o$ and $e^o i^* \cdot ie^o \cdot e^o i^* = e^o \cdot i^* i \cdot e^o e^o \cdot i^* = e^o i^* \cdot e^o i^* = e^o i^*$. Thus $e^o i^* \in V_{S^o}(ie^o)$ and $e^o i^* \in V_{S^o}(e^o i^*) \cap V_{S^o}(ie^o)$. By the assumption, we have $V_{S^o}(e^o i^*) = V_{S^o}(ie^o)$. By S^o is quasi-adequate, E^o is a band and thus is the semilattice Y of rectangular bands $E_\alpha (\alpha \in Y)$. Certainly, i^*e^o and $e^o i^*$ are in the same rectangular band, and so they are inverses of each other. Thus $i^*e^o \in V_{S^o}(e^o i^*) = V_{S^o}(ie^o)$. Consequently, $ie^o = ie^o \cdot i^*e^o \cdot ie^o = (ie^o)^2$ and ie^o is an idempotent. Therefore $IE^o \subseteq E$.

If $e^o i$ is a regular element, by Proposition 2.3 we can take $x \in V_{S^o}(e^o i)$ and $x^o \in V_{S^o}(x)$. It is easy to see that ixe^o is idempotent and $ixe^o \in V(e^o i)$. It follows from $\mathcal{L}^*$ is a right congruence that $ixe^o \mathcal{L}^* i^*xe^o \in S^o$, and so there exists $(i^*xe^o)^* \in E^o$ such that $(i^*xe^o)^* \mathcal{L}^* i^*xe^o$. Thus $ixe^o \mathcal{L} (i^*xe^o)^* \in E^o$ and so $(i^*xe^o)^* \in V_{S^o}(ixe^o) \cap V_{S^o}((i^*xe^o)^*)$. By hypothesis we have $V_{S^o}(ixe^o) = V_{S^o}((i^*xe^o)^*)$ and S^o beging quasi-adequate, gives $V_{S^o}(ixe^o) = V_{S^o}((i^*xe^o)^*) \subseteq E^o$. Since S^o is quasi-adequate, the regular elements of S^o form an orthodox subsemigroup, and consequently $e^o x^o i^* \in V_{S^o}(i^*xe^o)$. It follows from $i^* \mathcal{L} i$ and i, i^* are idempotent that

$$e^o x^o i^* \cdot ixe^o \cdot e^o x^o e^* = e^o x^o i^* \cdot i^*xe^o \cdot e^o x^o i^* = e^o x^o i^*$$

and

$$ixe^o \cdot e^o x^o i^* \cdot ixe^o = i \cdot i^*xe^o \cdot e^o x^o i^* \cdot i^*xe^o = i \cdot i^*xe^o = ixe^o.$$

Thus $e^o x^o i^* \in V_{S^o}(ixe^o)$ and by a similar proof we have $i^*xe^o \in V_{S^o}(e^o i) \cap V_{S^o}(e^o x^o i^*)$ and $V_{S^o}(e^o i) = V_{S^o}(e^o x^o i^*) \subseteq E^o$. Therefore $x \in E^o$ and $e^*xe^o \in E^o$. Consequently,

$$e^o i = e^o i \cdot i^*xe^o \cdot e^o i = e^o i \cdot ixe^o \cdot i^*xe^o \cdot e^o i = e^o ix \cdot e^o i^* \cdot xe^o i.$$

Premultiplying and postmultiplying by x, we have $x = xe^o ix = xe^o ix \cdot e^o i^* \cdot xe^o ix = xe^o i^* x$, and so $e^o i^* x \mathcal{L} x$. Notice that $e^o i^* x \in E^o$ and so $e^o i^* xe^o = e^o i^* x \cdot xe^o \mathcal{L} xe^o$. It follows from $e^o i^* xe^o, xe^o \in E^o$ that $e^o i^* xe^o \in V_{S^o}(xe^o)$. Clearly $xe^o \in V(e^o i)$ and $xe^o \in E^o$ gives $V_{S^o}(e^o i) = V_{S^o}(xe^o)$. Therefore $e^o i^* xe^o \in V_{S^o}(e^o i)$ and by ie^o is idempotent, we have $e^o i = e^o i \cdot e^o i^* xe^o \cdot e^o i = e^o(ie^o)(ie^o)i^* xe^o e^o i = e^o i(e^o ie^o i^* xe^o e^o i) = e^o ie^o i$. Thus $e^o i$ is idempotent and we have proved that if $e^o i$ is regular, then it is idempotent. In a similar way, we may show that $E^o \Lambda \subseteq E$ and if for all $\lambda \in \Lambda, e^o \in E^o$, if λe^o is regular, then it is idempotent.

(Sufficiency) Let $e \in I$ with $e \mathcal{L} a^{o+} \in E^o$. For any $x \in V_{S^o}(e), x^o \in V_{S^o}(x)$, we have $x^o xe \cdot x \cdot x^o xe = x^o x(exe) = x^o xe$, that is, $x^o xe$ is regular and so by the condition $x^o xe \in E$. Thus $e \mathcal{L} x^o xe \mathcal{R} x^o x$ and $e \mathcal{R} exx^o \mathcal{L} xx^o$ with $exx^o \in IE^o \subseteq E$.

Similarly, $a^{o+} \mathcal{L} e \mathcal{R} exx^o$ implies $a^{o+} \mathcal{R} a^{o+}xx^o \mathcal{L} xx^o$ with $a^{o+}xx^o \in E^o E^o \subseteq E^o$, and so $a^{o+} \mathcal{L} xx^o a^{o+} \mathcal{R} xx^o$, thus $xx^o a^{o+} = xe$. Therefore,

$$x^o a^{o+} x^o = x^o \cdot xe(xx^o a^{o+} x^o) = x^o(xe)(xe)x^o = (x^o xe)x^o = x^o$$

$$a^{o+} x^o a^{o+} = a^{o+}(x^o xe) \cdot (exx^o)a^{o+} = a^{o+} \cdot exe = a^{o+}e = a^{o+}.$$

Thus $V_{S^o}(V_{S^o}(e)) \subseteq V(a^{o+}) = E(a^{o+})$.

From the above proof, $x^o \in V_{S^o}(a^{o+}) \subseteq E^o$ and consequently, $x \in V_{S^o}(x^o) \subseteq E^o$ since S^o is quasi-adequate. It is easy to check that $E(a^{o+}) \subseteq V_{S^o}(V_{S^o}(e))$ and so $V_{S^o}(V_{S^o}(e)) = E(a^{o+})$. Since S^o is quasi-adequate, this implies that $V_{S^o}(e) = E(a^{o+})$. Hence, if $e, f \in I$ with $e \mathcal{L} f$, then $V_{S^o}(e) = V_{S^o}(f)$.

Dually, if $i, j \in \Lambda$ with $i \mathcal{R} j$, then $V_{S^o}(i) = V_{S^o}(j)$.

Similar to the proof of Theorem 2.1 of [10], we have that if a is regular, for any $a^o \in V_{S^o}(a)$, then $V_{S^o}(a) = V_{S^o}(a^o a)a^o V_{S^o}(aa^o)$.

For $a, b \in RegS$, if $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset$, then we can take $x^o \in V_{S^o}(a) \cap V_{S^o}(b)$ and so $V_{S^o}(a) = V_{S^o}(x^o a)x^o V_{S^o}(ax^o)$ and $V_{S^o}(b) = V_{S^o}(x^o b)x^o V_{S^o}(bx^o)$. Obviously, $ax^o, bx^o \in I$ and $ax^o \mathcal{L} bx^o$, thus $V_{S^o}(ax^o) = V_{S^o}(bx^o)$. Similarly, $V_{S^o}(x^o a) = V_{S^o}(x^o b)$. Therefore $V_{S^o}(a) = V_{S^o}(b)$. □

A generalised quasi-adequate transversal S^o of an abundant semigroup S is called a *refined generalised quasi-adequate transversal*, for short, an *RGQA transversal* of S, if it satisfies

$$(\forall a, b \in RegS), \ \ [V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)].$$

Obviously, a regular semigroup with an orthodox transversal is an abundant semigroup with a generalised quasi-adequate transversal. By means of Lemma 1.2, the transversal is refined. Thus, RGQA transversals are the generalisation of orthodox transversals in the abundant case.

By means of the properties of adequate transversal [17, *Theorem* 3.3], one can easily observe that an abundant semigroup with an adequate transversal is an abundant semigroup with an RGQA transversal.

In the following, we will investigate when an RGQA transversal is an orthodox transversal and when an RGQA transversal is an adequate transversal, respectively. We have the following result.

Theorem 3.2 *Let S^o be a refined generalised quasi-adequate transversal of the abundant semigroup S. Then*
(i) S^o is a regular subsemigroup of S, if and only if S is a regular semigroup. In this case, S^o is an orthodox transversal of S.
(ii) S^o is an adequate transversal of S if and only if S^o is an adequate semigroup.

Proof. (i) (Sufficiency) Suppose that S is a regular semigroup, then every element in S is regular. It follows from Proposition 2.3 that every element in S has an inverse in S^o, that is $V_{S^o}(a) \neq \emptyset$ for each $a \in S$. From Theorem 3.1 we deduce that for any $a, b \in S$, $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset$ implies that $V_{S^o}(a) = V_{S^o}(b)$. Notice that in this case S^o is an orthodox subsemigroup of S, and it follows from Lemma 1.2 that S^o is an orthodox transversal of S.

(Necessity) Let $a \in S, a = i\overline{a}\lambda$, where $i, \lambda \in E, i \mathcal{L} \overline{a}^+ \in E^o, \lambda \mathcal{R} \overline{a}^* \in E^o$. If S^o is a regular subsemigroup of S, then $\overline{a}$ is a regular. From $\overline{a}^* \mathcal{L} \overline{a} \mathcal{R} \overline{a}^+$, we have $\overline{a}$ has a unique inverse $x \in R_{\overline{a}^*} \cap L_{\overline{a}^+}$, with the property that $\overline{a}x = \overline{a}^+, x\overline{a} = \overline{a}^*$. Notice that $\lambda \mathcal{R} \overline{a}^* = x\overline{a} \mathcal{L} i \mathcal{L} \overline{a}^+ = \overline{a}^*$ and so $axa = i\overline{a}\lambda \cdot x\overline{a}x \cdot i\overline{a}\lambda = i\overline{a} \cdot x\overline{a}x \cdot \overline{a}\lambda = i(\overline{a}x\overline{a}x\overline{a})\lambda = e\overline{a}\lambda = a$. Thus a is regular and the semigroup S is regular.

(ii) It is clear that the necessary condition is true.

(Sufficiency) Suppose that $x \in S, x = i\overline{x}\lambda$ with $i, \lambda \in E, i \mathcal{L} \overline{x}^+ \in E^o, \lambda \mathcal{R} \overline{x}^* \in E^o$, in the following we will show if S^o is an adequate semigroup, then $\overline{x}$ is unique. If $x = i'\overline{x}'\lambda'$ with $i', \lambda' \in E, i' \mathcal{L} (\overline{x}')^+ \in E^o, \lambda' \mathcal{R} (\overline{x}')^* \in E^o$, then $i \mathcal{R}^* x \mathcal{R}^* i' \mathcal{L} (\overline{x}')^+$. Thus $(\overline{x}')^+ i \cdot i' \cdot (\overline{x}')^+ i = (\overline{x}')^+ i' (\overline{x}')^+ i = (\overline{x}')^+ i$ and so $(\overline{x}')^+ i$ is regular. It follows from the Theorem 3.1 that $(\overline{x}')^+ i \in E$ and $(\overline{x}')^+ \mathcal{R} (\overline{x}')^+ i \mathcal{L} i \mathcal{L} \overline{x}^+$. From Lemma 1.7 we have $(\overline{x}')^+ \mathcal{R} (\overline{x}')^+ \overline{x}^+ \mathcal{L} \overline{x}^+ \mathcal{R} \overline{x}^+ (\overline{x}')^+ \mathcal{L} (\overline{x}')^+$. If S^o is adequate, the E^o is a semilattice and $\overline{x}^+ (\overline{x}')^+ = (\overline{x}')^+ \overline{x}^+$. Thus $\overline{x}^+, (\overline{x}')^+$ are in the same $\mathcal{H}$-class and so $\overline{x}^+ = (\overline{x}')^+$ and similarly $\overline{x}^* = (\overline{x}')^*$. Consequently $\overline{x} = \overline{x}^+ x \overline{x}^* = (\overline{x}')^+ x (\overline{x}')^* = \overline{x}'$. Therefore S^o is the adequate transversal of S. □

Corollary 3.3 *In the class of abundant semigroups, refined generalised quasi-adequate transversals are a generalisation of both orthodox transversals and adequate transversals.*

Theorem 3.4 *Let S^o be an RGQA transversal of S, $S^\diamond$ be an RGQA transversal of S^o. Then $S^\diamond$ is an RGQA transversal of S.*

Proof. For any $a \in S$, since S^o is an RGQA transversal of S, there exists $\overline{a} \in C_{S^o}(a)$ such that $a = e\overline{a}f$, where $e, f \in E, e \mathcal{L} \overline{a}^+ \in E^o, f \mathcal{R} \overline{a}^* \in E^o$. By means of $S^\diamond$ is an RGQA transversal of S^o, there exists $\overline{a}^\diamond \in C_{S^\diamond}(\overline{a})$ such that $\overline{a} = i\overline{a}^\diamond j$, where $i, j \in E, i \mathcal{L} (\overline{a}^\diamond)^+ \in E^\diamond, j \mathcal{R} (\overline{a}^\diamond)^* \in E^\diamond$. Thus $a = e\overline{a}f = e \cdot i\overline{a}^\diamond j \cdot f = (ei)\overline{a}^\diamond (jf)$. Since

$i \mathcal{R}^* \overline{a} \mathcal{R}^* \overline{a}^+$ and $i, \overline{a}^+ \in E$, we have $i \mathcal{R} \overline{a}^+ \in E^o$. Combining this with $i \mathcal{L} (\overline{a}^\diamond)^+ \in E^\diamond \subseteq E^o$, we obtain that $i \in S^o \cap E = E^o$. Thus $ei \in IE^o \subseteq E$ and similarly $jf \in E^o\Lambda \subseteq E$. Also, from $e \mathcal{L} \overline{a}^+ \mathcal{R}^* \overline{a} \mathcal{R}^* i$, by Lemma 1.7, we deduce that $e \mathcal{R} ei \mathcal{L} i$, and so $ei \mathcal{L} i \mathcal{L} (\overline{a}^\diamond)^+ \in E^\diamond$. Similarly, $jf \mathcal{R} j \mathcal{R} (\overline{a}^\diamond)^* \in E^\diamond$. Therefore $\overline{a}^\diamond \in C_{S^\diamond}(a)$ and $S^\diamond$ is a generalised quasi-adequate transversal of S.

For any regular element $a \in S, V_{S^\diamond}(a) \neq \emptyset$ by Proposition 2.3. If $a, b \in RegS, V_{S^\diamond}(a) \cap V_{S^\diamond}(b) \neq \emptyset$, since $S^\diamond \subseteq S^o, V_{S^\diamond}(a) \subseteq V_{S^o}(a)$ then $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset$ and so $V_{S^o}(a) = V_{S^o}(b)$ by Theorem 3.1. Meanwhile $S^\diamond$ is an RGQA transversal of S^o, it is easy to see that $V_{S^\diamond}(a) = V_{S^\diamond}(b)$. Therefore $S^\diamond$ is an RGQA transversal of S. □

Theorem 3.5 *Let S be an abundant semigroup with an RGQA transversal S^o. Then $C_{S^o}(a) = V_{S^o}(\overline{a}^+)\overline{a}V_{S^o}(\overline{a}^+)$, where $a = e\overline{a}f$ with $e, f \in E, e \mathcal{L} \overline{a}^+, f \mathcal{R} \overline{a}^*$ and consequently,*

$$C_{S^o}(a) \cap C_{S^o}(b) \neq \emptyset \Rightarrow C_{S^o}(a) = C_{S^o}(b).$$

Proof. Notice that $C_{S^o}(a) = \{\overline{a} \in S^o | a = e\overline{a}f, e \mathcal{L} \overline{a}^+, f \mathcal{R} \overline{a}^* \text{ for some } e, f \in E, \overline{a}^+, \overline{a}^* \in E^o\}$, we first show $C_{S^o}(a) \subseteq V_{S^o}(\overline{a}^+)\overline{a}V_{S^o}(\overline{a}^*)$. Let $\overline{b} \in C_{S^o}(a)$, that is, let $a = i\overline{b}j, i \mathcal{L} \overline{b}^+, j \mathcal{R} \overline{b}^*$ for some $\overline{b}^+, \overline{b}^* \in E^o$. Then $\overline{a} = \overline{a}^+ a \overline{a}^*$ and $\overline{b} = \overline{b}^+ a \overline{b}^*$ with $\overline{a}^+ \mathcal{L} e \mathcal{R}^* a \mathcal{L}^* f \mathcal{R} \overline{a}^*$ and $\overline{b}^+ \mathcal{L} i \mathcal{R}^* a \mathcal{L}^* j \mathcal{R} \overline{b}^*$. It is easy to see $\overline{b}^+ e \cdot i \cdot \overline{b}^+ e = \overline{b}^+ e$ and so $\overline{b}^+ e \in E$. Similarly $f\overline{b}^* \in E$. From $\overline{a}^+ \mathcal{L} e \mathcal{R} i \mathcal{L} \overline{b}^+$ and $\overline{a}^* \mathcal{R} f \mathcal{L} j \mathcal{R} \overline{b}^*$, we deduce that $\overline{b}^+ e = \overline{b}^+ \overline{a}^+ \in E^o$ and $f\overline{b}^* = \overline{a}^* \overline{b}^* \in E^o$. Hence we have $\overline{b} = \overline{b}^+ (e\overline{a}f)\overline{b}^* = (\overline{b}^+ e)\overline{a}(f\overline{b}^*) = (\overline{b}^+\overline{a}^+)\overline{a}(\overline{a}^*\overline{b}^*)$ with $\overline{b}^+\overline{a}^+ \in V_{S^o}(\overline{a}^+), \overline{a}^*\overline{b}^* \in V_{S^o}(\overline{a}^*)$. This shows that $C_{S^o}(a) \subseteq V_{S^o}(\overline{a}^+)\overline{a}V_{S^o}(\overline{a}^*)$.

We now show that $V_{S^o}(\overline{a}^+)\overline{a}V_{S^o}(\overline{a}^*) \subseteq C_{S^o}(a)$. Certainly, the regular elements of S^o form an orthodox subsemigroup of S^o since S^o is a quasi-adequate semigroup. Then for any $\alpha \in V_{S^o}(\overline{a}^+), \beta \in V_{S^o}(\overline{a}^*)$, we have that α and β are both idempotents of S^o.

So, $a = e\overline{a}f = e(\alpha\overline{a}^+\overline{a}(\overline{a}^*\beta)f = e(\alpha\overline{a}\beta)f$ with $e \mathcal{L} \overline{a}^+ \mathcal{L} \alpha\overline{a}^+ \mathcal{R}^* \alpha\overline{a}\beta, f \mathcal{R} \overline{a}^* \mathcal{R} \overline{a}^*\beta \mathcal{L}^* \alpha\overline{a}\beta$, that is, $\alpha\overline{a}^+$ is the typical element of $(\alpha\overline{a}\beta)^+$, $\overline{a}^*\beta$ is the typical element of $(\alpha\overline{a}\beta)^*$. Therefore $\alpha\overline{a}\beta \in C_{S^o}(a)$ and we have in fact shown that $C_{S^o}(a) = V_{S^o}(\overline{a}^+)\overline{a}V_{S^o}(\overline{a}^*)$. Consequently, $C_{S^o}(a) \cap C_{S^o}(b) \neq \emptyset \Rightarrow C_{S^o}(a) = C_{S^o}(b)$. □

Theorem 3.6 *Let S be an abundant semigroup with an RGQA transversal S^o. If $C_{S^o}(a) \cap E^o \neq \emptyset$ or $V_{S^o}(a) \cap E^o \neq \emptyset$, then $C_{S^o}(a) = V_{S^o}(a) \subseteq E^o$.*

Proof. If $C_{S^o}(a) \cap E^o \neq \emptyset$, take $x \in C_{S^o}(a) \cap E^o$, for any $\bar{x} \in C_{S^o}(x)$, we have $\bar{x} = \bar{x}^+ x \bar{x}^*$, and so $\bar{x} \in E^o$. It is clear that $x \in C_{S^o}(a) \cap C_{S^o}(x)$, by Theorem 3.5, $C_{S^o}(a) = C_{S^o}(x) \subseteq E^o$. Similarly, if $V_{S^o}(a) \cap E^o \neq \emptyset$, for any $a' \in V_{S^o}(a) \cap E^o$, it follows from $a' \in V_{S^o}(a) \cap V_{S^o}(a')$ and the definition of RGQA transversals that $V_{S^o}(a) = V_{S^o}(a') \subseteq E^o$.

If $V_{S^o}(a) \cap E^o \neq \emptyset$, for any $a' \in V_{S^o}(a) \cap E^o$, $a = aa' \cdot a' \cdot a'a$, $aa' \in E$ with $aa' \mathcal{L} a' \in E^o$ and $a'a \in E$ with $a'a \mathcal{R} a' \in E^o$. Thus $a' \in C_{S^o}(a)$ and so $V_{S^o}(a) \subseteq C_{S^o}(a)$. Conversely, let $a^o \in C_{S^o}(a)$, from the above proof $a' \in C_{S^o}(a) \cap E^o$ and so $a^o \subseteq E^o$. From $ea^o \in IE^o \subseteq E, a^o f \in E^o\Lambda \subseteq E$, we deduce that

$$aa^o a = e(a^o f) \cdot a^o \cdot (ea^o)f = ea^o(ea^o)f = ea^o f$$

and

$$a^o aa^o = (a^o e)a^o(fa^o) = a^o(ea^o)fa^o = (a^o f)a^o = a^o,$$

since $ea^o \mathcal{L} a^o$ and $a^o f \mathcal{R} a^o$. Thus $a^o \in V_{S^o}(a)$, that is $C_{S^o}(a) \subseteq V_{S^o}(a)$. Therefore $C_{S^o}(a) = V_{S^o}(a)$ and so $C_{S^o}(a) = V_{S^o}(a) \subseteq E^o$.

Similarly, if $C_{S^o}(a) \cap E^o \neq \emptyset$, then $V_{S^o}(a) = C_{S^o}(a)$ and so $V_{S^o}(a) = C_{S^o}(a) \subseteq E^o$. □

Theorem 3.7 *Let S be an abundant semigroup with an RGQA transversal S^o. Then $I\Lambda = \{x \in S : (\exists x^o \in V_{S^o}(x))x^o \in E^o\} = \{x \in S : V_{S^o}(x) \subseteq E^o\}$ and consequently, $S^o \cap I\Lambda = E^o$.*

Proof. By Theorem 3.6, the second equality certainly holds. If $x \in S$ with some(any) $x^o \in V_{S^o}(x) \cap E^o$, then $x = xx^o x = xx^o \cdot x^o x \in I\Lambda$. Conversely, for any $i \in I, j \in \Lambda$, there exist $i^o, j^o \in E^o$ such that $i^o \mathcal{L}\ i, j^o \mathcal{R}\ j$. Let $(i^o j^o)^o \in V_{S^o}(i^o j^o)$. Then $(i^o j^o)^o \in E^o$ and so

$$j^o(i^o j^o)^o i^o ij j^o(i^o j^o)^o i^o = (j^o(i^o j^o)^o i^o)^2 = j^o(i^o j^o)^o i^o,$$

$$ij \cdot j^o(i^o j^o)^o i^o \cdot ij = i \cdot j^o(i^o j^o)^o i^o \cdot j = i \cdot i^o j^o(i^o j^o)^o i^o j^o \cdot j = i \cdot i^o j^o \cdot j = ij.$$

Therefore $j^o(i^o j^o)^o i^o \in V_{S^o}(ij) \cap E^o$ and the equlities hold. □

In the following, we will consider the case when $I\Lambda$ is closed.

Theorem 3.8 *Let S be an abundant semigroup with an RGQA transversal S^o. Then the following statements are equivalent: (1) $I\Lambda$ is a subsemigroup of S; (2) $\Lambda I \subseteq I\Lambda$; in this case, we have $V_{S^o}(E) \subseteq E^o$; (3) $I\Lambda = \langle E \rangle$.*

Proof. (1) $\Longrightarrow$ (2). For any $i \in I, j \in \Lambda$, there exist $i^o, j^o \in E^o$ such that $i^o \mathcal{L}\ i, j^o \mathcal{R}\ j$. If (1) holds, then $ji = j^o jii^o \in E^o \Lambda I E^o \subseteq I\Lambda I\Lambda \subseteq I\Lambda$. Therefore $\Lambda I \subseteq I\Lambda$. Now if $x \in E$, then for any $x^o \in V_{S^o}(x)$, $x^o = x^o x x^o = x^o x \cdot x x^o \in \Lambda I \subseteq I\Lambda$. By Theorem 3.7, $x^o \in E^o$. Hence (2) holds.

(2) $\Longrightarrow$ (3). Let $x, y \in I\Lambda$. Then by Theorem 3.7, there exist $x^o \in V_{S^o}(x), y^o \in V_{S^o}(y)$ with $x^o, y^o \in E^o$. If (2) holds, then there exists $(x^o xyy^o)^o \in V_{S^o}(x^o xyy^o)$ with $(x^o xyy^o)^o \in E^o$ and so $y^o(x^o xyy^o)^o x^o \in E^o$. It is easy to check that $y^o(x^o xyy^o)^o x^o \in V_{S^o}(xy)$. Hence $xy \in I\Lambda$ by Theorem 3.7 and $I\Lambda$ is a subsemigroup. By the proof of (1) $\Longrightarrow$ (2), we have $V_{S^o}(E) \subseteq E^o$ and so $E \subseteq I\Lambda$ by Theorem 3.7. It is obvious that $I\Lambda \subseteq \langle E \rangle$. Therefore $I\Lambda = \langle E \rangle$.

(3) $\Longrightarrow$ (1). This is trivial. □

Theorem 3.9 *Let S be an abundant semigroup with an RGQA transversal S^o. If $C_{S^o}(a) \cap C_{S^o}(b) \neq \emptyset$ and $a \mathcal{L}^*\ b, a \mathcal{R}^*\ b$, then $a = b$.*

Proof. Let $\overline{x} \in C_{S^o}(a) \cap C_{S^o}(b)$. Then $a = e_a \overline{x} f_a, e_a \mathcal{L}\ \overline{x}^+, f_a \mathcal{R}\ \overline{x}^*, b = e_b \overline{x} f_b, e_b \mathcal{L}\ \overline{x}', f_b \mathcal{R}\ \overline{x}^{*'}$, for some $\overline{x}^+, \overline{x}^*, \overline{x}^{+'}, \overline{x}^{*'} \in E^o$ with $\overline{x}^+ \mathcal{R}^*\ \overline{x} \mathcal{R}^*\ \overline{x}^{+'}$, $\overline{x}^* \mathcal{L}^*\ \overline{x} \mathcal{L}^*\ \overline{x}^{*'}$. From $a \mathcal{L}^*\ b, a \mathcal{R}^*\ b$ and Lemma 1.7, we deduce that $e_a \overline{x}^{+'} = e_b$ and $\overline{x}^{*'} f_a = f_b$. Thus

$$a = e_a \overline{x} f_a = e_a(\overline{x}^{+'} b \overline{x}^{*'}) f_a = (e_a \overline{x}^{+'}) b (\overline{x}^{*'} f_a) = e_b b f_b = b.$$

□

4. The main theorem

The main purpose in this section is to establish a structure theorem for abundant semigroups with quasi-ideal RGQA transversals. In the following R denotes an abundant semigroup with a right ideal RGQA transversal S^o. Then by Proposition 2.7, $\Lambda_x \subseteq E^o$ for each $x \in R$ and $E(R) = I$. For $x \in R$, the $\mathcal{R}^*$-class of R containing x will be denoted by R^*_x and we define $K(x) = K(y)$ if $R^*_x = R^*_y$ and $C_{S^o}(x) = C_{S^o}(y)$ for $x, y \in R$. The relation $\mathcal{K}$, defined on R by $(x, y) \in \mathcal{K}$ if and only if $K(x) = K(y)$, is an equivalence relation on R. If L denotes an abundant semigroup with a left ideal RGQA transversal S^o, then by Proposition 2.7, $Ia \subseteq E^o$ for each $a \in L$ and $E(L) = \Lambda$.

Theorem 4.1 *Let L and R be a pair of abundant semigroups with a common RGQA transversal S^o. Let S^o be a left ideal of L and a right ideal of R. Let $L \times R \longrightarrow S^o$ described by $(p, x) \longmapsto p * x$ be a mapping such that for any $p, q \in L$ and for any $x, y \in R$:*

(1) $(p * x)y = p * (xy)$ and $p(q * x) = (pq) * x$;

(2) if $\{x, p\} \cap S^o \neq \emptyset$, then $p * x = px$;

(3) For any $q_1, q_2 \in L^1, x_1, x_2 \in R^1$, if $y_1 \mathcal{R}^*\ y_2$ in R, then $x_1(q_1 * y_1) = x_2(q_2 * y_1)$ if and only if $x_1(q_1 * y_2) = x_2(q_2 * y_2)$;

if $p_1 \mathcal{L}^* p_2$ *in* L, *then* $(p_1 * x_1)q_1 = (p_1 * x_2)q_2$ *if and only if* $(p_2 * x_1)q_1 = (p_2 * x_2)q_2$.

Define a multiplication on the set

$$\Gamma \equiv R/\mathcal{K} \ |\times|\ L/\mathcal{L}^* = \{(K(x), L_p^*) \in R/\mathcal{K} \times L/\mathcal{L}^* : \exists z \in C_{S^o}(x) \cap C_{S^o}(p)\}$$

by

$$(K(x), L_p^*)\,(K(y), L_q^*) = (K(i_x(p * y)), L^*_{(p*y)\lambda_q}).$$

Then Γ *is an abundant semigroup with a quasi-ideal RGQA transversal that is isomorphic to* S^o.

Conversely, every abundant semigroup with a quasi-ideal RGQA transversal can be obtained in this manner.

To prove this theorem, we give a sequence of Lemmas as follows.

Lemma 4.2 *The multiplication on* Γ *is well-defined.*

Proof. We first prove that $(K(i_x(p * y)), Ł^*_{(p*y)\lambda_q}) \in \Gamma$, by means of Lemma 4.3 in [23], we have

$$i_x(p * y) = i_x\overline{x}^+(p * y) = i_x\overline{x}(\lambda_p * i_y)\overline{y}\lambda_y = i_x[\overline{x}(\lambda_p * i_y)]^+ \cdot \overline{x}(\lambda_p * i_y)\overline{y} \cdot [(\lambda_p * i_y)\overline{y}]^*\lambda_y,$$

and

$$(p * y)\lambda_q = (p * y)\overline{q}^*\lambda_q = i_p \cdot \overline{x}(\lambda_p * i_y)\overline{y}\lambda_q = i_p[\overline{x}(\lambda_p * i_y)]^+ \cdot \overline{x}(\lambda_p * i_y)\overline{y} \cdot [(\lambda_p * i_y)\overline{y}]^*\lambda_q.$$

If $i_x, i'_x \in I_x$, with $i_x \mathcal{L}\, \overline{x}^+, i'_x \mathcal{L}\, \overline{x}^{+'}$ for some $\overline{x} \in C_{S^o}(x) \cap C_{S^o}(a)$, then $R^*_{i_x(p*y)} = R^*_{i'_x(p*y)}$ and $C_{S^o}(i_x(p * y)) \cap C_{S^o}(i'_x(p * y)) \neq \emptyset$, and therefore the multiplication on Γ is not dependent on the choice of i_x. There is a dual result for λ_q.

We prove that for $(K(x), L_p^*) \in \Gamma$, we have $i_x \cdot p = x \cdot \lambda_p$. In fact, if $(K(x), L_p^*) \in \Gamma$, then there exists $\overline{x} \in C_{S^o}(x) \cap C_{S^o}(p)$ with $x = i_x\overline{x}\lambda_x, i_x \mathcal{L}\, \overline{x}^+, \lambda_x \mathcal{R}\, \overline{x}^*$ for some $\overline{x}^+, \overline{x}^* \in E^o$ and $a = i_p\overline{x}\lambda_p, i_p \mathcal{L}\, \overline{x}^{+'}, \lambda_p \mathcal{R}\, \overline{x}^{*'}$ for some $\overline{x}^{+'}, \overline{x}^{*'} \in E^o$. Hence

$$i_xp = i_xi_p\overline{x}\lambda_p = i_xi_p \cdot \overline{x} \cdot \overline{x}^*\lambda_p$$

and

$$x\lambda_p = i_x\overline{x}\lambda_x\lambda_p = i_x\overline{x}^{+'} \cdot \overline{x} \cdot \lambda_x\lambda_p.$$

It is easy to see that $i_xi_p = i_x\overline{x}^{+'}$ and $\overline{x}^*\lambda_p = \lambda_x\lambda_p$ since $i_p, \lambda_x \in E^o$ and so $i_xp = x\lambda_p$.

Next we show that if $(K(x), L_p^*)$ and $(K(x'), L^*_{p'})$ in Γ are such that $(K(x), L_p^*) = (K(x'), L^*_{p'})$, then $i_xp = i_{x'}p'$. From $K(x) = K(x')$, that is, from $x \mathcal{R}^* x'$ and $C_{S^o}(x) \cap C_{S^o}(x') \neq \emptyset$ with $x, x' \in R$, we deduce that $x = i_x\overline{x}\lambda_x, x' = i_{x'}\overline{x}\lambda_{x'}$ with $\overline{x}^+ \mathcal{L}\, i_x \mathcal{R}^* x \mathcal{R}^* x' \mathcal{R}^* i_{x'} \mathcal{L}\, \overline{x}^{+'}, \overline{x}^+ \mathcal{R}^* \overline{x} \mathcal{R}^* \overline{x}^{+'}$ and $\lambda_x, \lambda_{x'} \in E^o$. Thus

$$x' = i_{x'}\overline{x}\lambda_{x'} = i_{x'}(\overline{x}^+x\overline{x}^*)\lambda_{x'} = i_{x'}(\overline{x}^+x\overline{x}^*)\lambda_{x'} = i_xx\overline{x}^*\lambda_{x'} = x\overline{x}^*\lambda_{x'}.$$

Meanwhile, $h = \overline{x}^*\lambda_{x'} \in E^oE^o \subseteq E^o$ with $\overline{x}^* \mathcal{R}\, h \mathcal{L}^* x'$. Since this result will be frequently mentioned in this section, it is worth to denote it by a remark.

Remark 1 *If* $K(x) = K(x')$ *in* R, *then* $x' = xh$ *with* $h \in E^o$ *and* $\overline{x}^* \mathcal{R}\, h \mathcal{L}^* x'$.

Thus $x\lambda_p = i_x\overline{x}\lambda_x\lambda_p$ and $x'\lambda_{p'} = xh\lambda_{p'} = i_x\overline{x}\lambda_xh\lambda_{p'}$. Since $x \in R$ we have $\lambda_x \in E^o$ and consequently $\lambda_xh\lambda_{p'} \in E^oE^o\Lambda \subseteq E^o\Lambda \subseteq E$ and $\lambda_x\lambda_p \in E^o\Lambda \subseteq E$. It is easy to check that $\lambda_xh\lambda_{p'}$ and $\lambda_x\lambda_p$ are in the same $\mathcal{H}^*$-class and hence $\lambda_xh\lambda_{p'} = \lambda_x\lambda_p$. Therefore $x\lambda_p = x'\lambda_{p'}$ and consequently $i_xp = i_{x'}p'$.

Finally we shall show that the multiplication on Γ is not dependent on the choice of x, p, y and q. Let

$$(K(x), L_p^*) = (K(x'), L^*_{p'}) \text{ and } (K(y), L_q^*) = (K(y'), L^*_{q'}).$$

Then

$$(K(x), L_p^*)\,(K(y), L_q^*) = (K(i_x(p * y)), Ł^*_{(p*y)\lambda_q})$$

and

$$(K(x'), L^*_{p'})\,(K(y'), L^*_{q'}) = (K(i_{x'}(p' * y')), L^*_{(p'*y')\lambda_{q'}}).$$

In the following part, we will prove that $C_{S^o}(i_x(p*y)) \cap C_{S^o}(i_{x'}(p'*y')) \neq \emptyset$. Since $K(x) = K(x'), K(y) = K(y')$, by Remark 1, $x = x'l$ and $y = y'h$ with $l, h \in E^o$ and $l \ \mathcal{L}^* \ x, h \ \mathcal{L}^* \ y$. Similarly, we may show that

$$i_x(p*y) = i_{x'}(p'*y) = i_{x'}(p'*y')h = i_{x'}(p'*y')[i_{x'(p'*y')}]^*h$$

and $[i_{x'}(p'*y')]^*h, i_{x'}(p'*y'))^*$ are in the same $\mathcal{D}$-class. It follows that $C_{S^o}(i_x(p*y)) \cap C_{S^o}(i_{x'}(p'*y')) \neq \emptyset$.

We then show that $i_x(p*y) \ \mathcal{R}^* \ i_{x'}(p'*y')$. By the above proof of $C_{S^o}(i_x(p*y)) \cap C_{S^o}(i_{x'}(p'*y')) \neq \emptyset$, we have $i_x(p*y) = i_{x'}(p'*y')h$. Similarly, we have $i_{x'}(p'*y') = i_x(p*y)h'$ for some $h' \in E^o$. Thus $i_x(p*y) \ \mathcal{R}^* \ i_{x'}(p'*y')$ and dually, $(p*y)\lambda_q \ \mathcal{L}^* \ (p'*y')\lambda_{q'}$. □

Lemma 4.3 *The set* Γ *is an abundant semigroup.*

Proof. For any $e, f, g \in \Gamma$, where $e = (K(x), L^*_p),\ f = (K(x_1), L^*_{p_1}),\ g = (K(x_2), L^*_{p_2})$, we have

$$\begin{aligned}
(ef)g &= (K(i_x(p*x_1)), L^*_{(p*x_1)\lambda_{p_1}})\,(K(x_2), L^*_{p_2})\\
&= (K(i_{i_x(p*x_1)}\,(((p*x_1)\lambda_{p_1})*x_2)), L^*_{(((p*x_1)\lambda_{p_1})*x_2)\lambda_{p_2}})\\
&= (K(i_x(p*x_1)^+\,(p*x_1)(\lambda_{p_1}*x_2)), L^*_{(p*x_1)(\lambda_{p_1}*x_2)\lambda_{p_2}})\\
&= (K(i_x(p*x_1)\,(\lambda_{p_1}*x_2)), L^*_{(p*x_1)(\lambda_{p_1}*x_2)\lambda_{p_2}})
\end{aligned}$$

and

$$\begin{aligned}
e(fg) &= (K(x), L^*_p)\,(K(i_{x_1}(p_1*x_2)), L^*_{(p_1*x_2)\lambda_{p_2}})\\
&= (K(i_x(p*(i_{x_1}(p_1*x_2))), L^*_{(p*(i_{x_1}(p_1*x_2)))\lambda_{p_2}})\\
&= (K(i_x(p*(x_1(\lambda_{p_1}*x_2))), L^*_{(p*(x_1(\lambda_{p_1}*x_2)))\lambda_{p_2}})\ \ (i_{x_1}p_1 = x_1\lambda_{p_1})\\
&= (K(i_x(p*x_1)(\lambda_{p_1}*x_2)), L^*_{(p*x_1)(\lambda_{p_1}*x_2)\lambda_{p_2}}).
\end{aligned}$$

Hence $(ef)g = e(fg)$ and so Γ is a semigroup.

Let $(K(x), L^*_p) \in \Gamma$. We will prove that $(K(x), L^*_p) \in E(\Gamma)$ if and only if $p*x = i_px (= p\lambda_x)$. Since $(K(x), L^*_p)(K(x), L^*_p) = (K(i_x(p*x)), L^*_{(p*x)\lambda_p})$, if $p*x = i_p \cdot x = p \cdot \lambda_x$, then we have

$$(K(i_x(p*x)), L^*_{(p*x)\lambda_p}) = (K(i_x \cdot i_p \cdot x), L^*_{p\lambda_x\lambda_p}) = (K(x), L^*_p).$$

Thus $(K(x), L^*_p) \in E(\Gamma)$. Conversely, if $(K(x), L^*_p) \in E(\Gamma)$, then $K(i_x(p*x)) = K(x)$ and so by Remark 1, $i_x(p*x)l = x$ for some $l \in E^o$. So,

$$x = i_x(p*x)l = i_x(p*xl) = i_x(p*x).$$

Hence $p*x = i_px$.

Suppose that $(K(x), L^*_p) \in \Gamma$, let $u = (K(i_x), L^*_{\overline{x}^+})$ and $v = (K(\overline{x}^*), L^*_{\lambda_p})$, where $x = i_x\overline{x}\lambda_x, p = i_p\overline{x}\lambda_p$ and $i_x \ \mathcal{L} \ \overline{x}^+, \lambda_p \ \mathcal{R} \ \overline{x}^*$ for some $\overline{x}^+, \overline{x}^* \in E^o$. Then we have $u, v \in E(\Gamma)$ and $u \ \mathcal{R}^* \ (K(x), L^*_p) \ \mathcal{L}^* \ v$.

In fact, by the above result, $u, v \in E(\Gamma)$ is clear. It follows from $\overline{x}^+ \in E^o$ and $\overline{x}\lambda_p \ \mathcal{L}^* \ i_p\overline{x}\lambda_p = p$ that

$$\begin{aligned}
(K(i_x), L^*_{\overline{x}^+})(K(x), L^*_p) &= (K(i_x(\overline{x}^+ * x)), L^*_{(\overline{x}^+ * x)\lambda_p}) = (K(i_x\overline{x}^+x), L^*_{\overline{x}^+x\lambda_p})\\
&= (K(x), L^*_{\overline{x}\lambda_p}) = (K(x), L^*_p).
\end{aligned}$$

If $(K(y), L^*_q), (K(z), L^*_c) \in \Gamma^1$ are such that $(K(y), L^*_q)(K(x), L^*_p) = (K(z), L^*_c)(K(x), L^*_p)$, then $(K(i_y(q*x)), L^*_{(q*x)\lambda_p}) = (K(i_z(c*x)), L^*_{(c*x)\lambda_p})$. Thus

$$i_y(q*x) \ \mathcal{R}^* \ i_z(c*x),\ \ C_{S^o}(i_y(q*x)) \cap C_{S^o}(i_z(c*x)) \neq \emptyset \ \text{ and } \ (q*x)\lambda_p \ \mathcal{L}^* \ (c*x)\lambda_p.$$

By $(q*x)\lambda_p \mathcal{L}^* (c*x)\lambda_p$, we have $(q*x)\lambda_p\lambda_x \mathcal{L}^* (c*x)\lambda_p\lambda_x$ and so $(q*x) \mathcal{L}^* (c*x)$. Therefore, $i_y(q*x) \mathcal{L}^* i_z(c*x)$ since $i_qi_yi_q = i_q$ and $i_ci_zi_c = i_c$. Hence by Theorem 3.9, $i_y(q*x) = i_z(c*x)$. From $x \mathcal{R}^* i_x$ and (3) we deduce that $i_y(q*i_x) = i_z(c*i_x)$, and so

$$q*i_x = i_q(q*i_x) \mathcal{L}^* i_yi_q(q*i_x) = i_zi_c(c*i_x) \mathcal{L}^* i_c(c*i_x) = c*i_x.$$

Therefore,

$$\begin{aligned}(K(y), L^*_q)(K(i_x), L^*_{\overline{x}^+}) &= (K(i_y(q*i_x)), L^*_{(q*i_x)\overline{x}^+})\\ &= (K(i_z(c*i_x)), L^*_{(c*i_x)\overline{x}^+})\\ &= (K(z), L^*_c)(K(i_x), L^*_{\overline{x}^+}).\end{aligned}$$

By Lemma 1.4, $u \mathcal{R}^* (K(x), L^*_p)$.

Dually, we have $v \mathcal{L}^* (K(x), L^*_p)$. Therefore Γ is an abundant semigroup. □

Lemma 4.4 *Let $W = \{(K(x^o), L^*_{x^o}) : x^o \in S^o\}$. Then W is a quasi-adequate $*$-subsemigroup of Γ, which is isomorphic to S^o.*

Proof. Obviously $W \subseteq \Gamma$. For $(K(x^o), L^*_{x^o}), (K(y^o), L^*_{y^o}) \in W$, we have

$$(K(x^o), L^*_{x^o})\,(K(y^o), L^*_{y^o}) = (K(i_{x^o}x^oy^o), L^*_{x^oy^o\lambda_{y^o}}) = (K(x^oy^o), L^*_{x^oy^o}) \in W,$$

and so W is a subsemigroup. For any $x^o \in S^o$, define $x^o\varphi = (K(x^o), L^*_{x^o})$, it is clear that φ is an isomorphism. Thus $S^o \cong W$ and so $E(W) = \{(K(x^o), L^*_{x^o}) : x^o \in E^o\}$.

To prove that W is a $*$−subsemigroup, let $(K(x^o), L^*_{x^o}) \in W$. By the proof of Lemma 4.3, $u = (K(x^{o+}), L^*_{x^{o+}}) \in E(W)$ and $u \mathcal{R}^* (K(x^o), L^*_{x^o})$. Similarly, $v = (K(x^{o*}), L^*_{x^{o*}}) \in E(W)$ and $v \mathcal{L}^* (K(x^o), L^*_{x^o})$. □

Lemma 4.5 *Let $(K(x_1), L^*_{p_1}), (K(x_2), L^*_{p_2}) \in \Gamma$. Then*

(1) $(K(x_1), L^*_{p_1}) \mathcal{R}^* (K(x_2), L^*_{p_2})$ *if and only if* $x_1 \mathcal{R}^* x_2$.

(2) $(K(x_1), L^*_{p_1}) \mathcal{L}^* (K(x_2), L^*_{p_2})$ *if and only if* $p_1 \mathcal{L}^* p_2$.

Proof. (1). By Lemma 4.3, we need only show that

$$(K(i_{x_1}), L^*_{\overline{x_1}^+}) \mathcal{R}^* (K(i_{x_2}), L^*_{\overline{x_2}^+}) \text{ if and only if } x_1 \mathcal{R}^* x_2.$$

Then $u_1 = (K(i_{x_1}), L^*_{\overline{x_1}^+}) \mathcal{R}^* (K(i_{x_2}), L^*_{\overline{x_2}^+}) = u_2$

$\Longleftrightarrow u_1u_2 = u_2$ and $u_2u_1 = u_1$, that is $(K(i_{x_1}\overline{x_1}^+i_{x_2}), L^*_{\overline{x_1}^+i_{x_2}\overline{x_2}^+}) = (K(i_{x_2}), L^*_{\overline{x_2}^+})$ and $(K(i_{x_2}\overline{x_2}^+i_{x_1}), L^*_{\overline{x_2}^+i_{x_1}\overline{x_1}^+}) = (K(i_{x_1}), L^*_{\overline{x_1}^+})$

$\Longleftrightarrow (K(i_{x_1}i_{x_2}), L^*_{\overline{x_1}^+i_{x_2}}) = (K(i_{x_2}), L^*_{\overline{x_2}^+})$ and $(K(i_{x_2}i_{x_1}), L^*_{\overline{x_2}^+i_{x_1}}) = (K(i_{x_1}), L^*_{\overline{x_1}^+})$

since $i_{x_1} \mathcal{L} \overline{x_1}^+, i_{x_2} \mathcal{L} \overline{x_2}^+$ and $\overline{x_1}^+i_{x_2}, \overline{x_2}^+i_{x_1} \in E^o$.

$\Longleftrightarrow i_{x_1}i_{x_2} \mathcal{R}^* i_{x_2}, i_{x_2}i_{x_1} \mathcal{R}^* i_{x_1}$

$\Longleftrightarrow x_1 \mathcal{R}^* x_2$ since $x_1 \mathcal{R}^* i_{x_1}, x_2 \mathcal{R}^* i_{x_2}$.

(2) This is dual to (1). □

Lemma 4.6 *W is a generalised quasi-adequate transversal of Γ.*

Proof. Let $g = (K(x), L^*_p) \in \Gamma$. Then we have the following result:

$$C_W(g) = \{(K(z), L^*_z) \in W : z \in C_{S^o}(x) \cap C_{S^o}(p)\}.$$

Let $V = \{(K(z), L^*_z) \in W : z \in C_{S^o}(x) \cap C_{S^o}(p)\}$ and $(K(z), L^*_z) \in V$. For $z \in C_{S^o}(x) \cap C_{S^o}(p)$, there exist idempotents $i, \lambda, i_p, \lambda_p$ such that $x = iz\lambda$, $p = i_p z \lambda_p$ with $i \mathcal{L} z^+, \lambda \mathcal{R} z^*$ for some $z^+, z^* \in E^o$. Thus

$$(K(x), L^*_p) = (K(i), L^*_{z^+})(K(z), L^*_z)(K(z^*), L^*_{\lambda_p}).$$

Moreover, $(K(i), L^*_{z^+}) \mathcal{L} (K(z^+), L^*_{z^+}) \mathcal{R}^* (K(z), L^*_z)$ and $(K(z^*), L^*_{\lambda_p}) \mathcal{R} (K(z^*), L^*_{z^*}) \mathcal{L}^* (K(z), L^*_z)$. Therefore $(K(z), L^*_z) \in C_W(g)$, and $V \subseteq C_W(g)$.

Conversely, if $(K(z), L^*_z) \in C_W(g)$, then there exist $(K(y_1), L^*_{q_1}), (K(y_2), L^*_{q_2}) \in E(\Gamma)$ such that

$$(K(x), L^*_p) = (K(y_1), L^*_{q_1})(K(z), L^*_z)(K(y_2), L^*_{q_2}),$$

and $(K(y_1), L^*_{q_1}) \mathcal{L} (K(z), L^*_z)^+$, $(K(y_2), L^*_{q_2}) \mathcal{R} (K(z), L^*_z)^*$ for some $(K(z), L^*_z)^+, (K(z), L^*_z)^* \in E(W)$. It follows from Lemma 1.6 that $(K(y_1), L^*_{q_1}) \mathcal{R}^* g \mathcal{L}^* (K(y_2), L^*_{q_2})$. Hence $y_1 \mathcal{R}^* x$ and $p \mathcal{L}^* q_2$.

Again by Lemma 4.4, there exist $x', x'' \in E^o$ such that $(K(z), L^*_z)^+ = (K(x'), L^*_{x'})$ with $x' \mathcal{R}^* z$, and $(K(z), L^*_z)^* = (K(x''), L^*_{x''})$ with $x'' \mathcal{L}^* z$. It follows that

$$(K(x'), L^*_{x'})(K(x), L^*_p)(K(x''), L^*_{x''}) = (K(z), L^*_z).$$

Thus, $x'x\lambda_p x'' \mathcal{K} z$ and $x'x\lambda_p x'' \mathcal{L}^* z$, and so $z = x' \cdot x \cdot \lambda_p x''$ by Remark 1.

Since $(K(y_2), L^*_{q_2}) \mathcal{R} (K(z), L^*_z)^* = (K(x''), L^*_{x''})$ we have $y_2 \mathcal{R}^* x''$ and

$$(K(x''), L^*_{x''})(K(y_2), L^*_{q_2}) = (K(y_2), L^*_{q_2}).$$

Hence

$$(K(x''y_2), L^*_{x''y_2\lambda_{q_2}}) = (K(y_2), L^*_{q_2}).$$

From $y_2 \mathcal{R}^* x''$ we have $y_2 = x''y_2 \in S^o$ since S^o is a right ideal of R. It follows that $q_2 \mathcal{L}^* x''y_2\lambda_{q_2} = y_2\lambda_{q_2}$ and so

$$y_2\lambda_{q_2}y_2 = (y_2\lambda_{q_2}) * y_2 = y_2\lambda_{q_2}\lambda_{y_2} = y_2.$$

Thus y_2 is regular and $y_2 = y_2\lambda_{q_2}\ \lambda_{y_2} \in \Lambda E^o \subseteq E^o$. Therefore $\lambda_p \mathcal{R} \lambda_p x'' \mathcal{L} x''$ since λ_p and x'' are in the same rectangular band and $\lambda_p x'' \in \Lambda E^o \subseteq E^o$.

Since $(K(x), L^*_p) \in \Gamma$, there exists $\overline{x} \in C_{S^o}(x) \cap C_{S^o}(p)$ such that $x = i_x \overline{x} \lambda_x$ and $p = i_p \overline{x} \lambda_p$ with $\lambda_x \mathcal{R} \overline{x}^{*'}, \lambda_p \mathcal{R} \overline{x}^*$ for some $\overline{x}^*, \overline{x}^{*'} \in E^o$. Thus $x \mathcal{L}^* \lambda_x \mathcal{L} \overline{x}^* \lambda_x \mathcal{R} \overline{x}^* \mathcal{R} \lambda_p \mathcal{R} \lambda_p x''$. Denote $\overline{x}^* \lambda_x = x^*$, then $x \mathcal{L}^* x^* \in E^o$ and $\lambda_p x'' \mathcal{R} x''$. Similarly, $i_{y_1} x' \in IE^o \subseteq E$ and $x \mathcal{R}^* y_1 \mathcal{R}^* i_{y_1} x' \mathcal{L} x'$. From $y = x'x\lambda_p x''$ we deduce that $y \in C_{S^o}(x)$. Similarly, we may show that $y \in C_{S^o}(p)$, and hence $C_W(g) \subseteq V$. We have in fact proved W is a generalised quasi-adequate transversal of Γ. □

Lemma 4.7 *The generalised quasi-adequate transversal W is refined and is a quasi-ideal of Γ.*

Proof. For any $(K(s), L^*_s) \in E(W), (K(e), L^*_{y^+}) \in I(\Gamma)$, with $s, y^+ \in E^o$ and $e \in I, e \mathcal{L} y^+$, we have

$$(K(e), L^*_{y^+})\ (K(s), L^*_s) = (K(i_e(y^+ * s)),\ L^*_{(y^+ * s)\lambda_s}) = (K(e(y^+ s)),\ L^*_{y^+ s \lambda_s}) = (K(es),\ L^*_{y^+ s})$$

with $p * x = (y^+ s) * (es) = y^+ ses = y^+ s$ and $i_p \cdot x = i_{y^+ s} es = y^+ s$. Thus $p * x = i_p \cdot x$ and $(K(e), L^*_{y^+})\ (K(s), L^*_s)$ is idempotent by the proof of Lemma 4.3.

Computing

$$(K(s),\ L^*_s)\ (K(e),\ L^*_{y^+}) = (K(i_s(s * e)),\ L^*_{(s*e)\lambda_{y^+}}) = (K(i_s se), L^*_{(s*e)\lambda_{y^+}}) = (K(se), L^*_{se})$$

since $i_s \mathcal{R} s$, $e \mathcal{L} y^+ \mathcal{L} \lambda_{y^+}$. From $se = sey^+ \in E^o IE^o \subseteq S^o RS^o \subseteq S^o$ we deduce that $(K(se), L^*_{se}) \in W$. By Lemma 4.4, $W \cong S^o$, thus $(K(se), Ł^*_{se})$ is regular if and only if se is regular. Since $se \in E^o I$, if se is regular, S^o being an RGQA transversal of R together with Theorem 3.1 give that se is idempotent, and so $se \in E^o$. By Lemma 4.4

again $(K(se), L^*_{se}) \in E(W)$. We have in fact obtained a stronger conclusion: if $(K(s),\ L^*_s)\ (K(e),\ L^*_{y^+})$ is regular, then it is an idempotent of W. By meas of Theorem 3.1, W is an RGQA transversal of Γ.

To show that W is a quasi-ideal, let $m = (K(m^o), L^*_{m^o}),\ n = (K(n^o), L^*_{n^o}) \in W$ and $g = (K(x), L^*_p) \in \Gamma$. Since S^o is a right ideal of R and a left ideal of L, we have $m^o x\lambda_p n^o \in S^o \cdot S^o \subseteq S^o$. Thus

$$\begin{aligned} mrn &= (K(m^o), L^*_{m^o})\ (K(x), L^*_p)\ (K(n^o), L^*_{n^o}) \\ &= (K(i_{m^o}(m^o x)(\lambda_p n^o)),\ L^*_{(m^o x)(\lambda_p n^o)\lambda_{n^o}}) \\ &= (K(m^o x \cdot \lambda_p n^o), L^*_{m^o x\cdot\lambda_p n^o}) \in W. \end{aligned}$$

Together with the above result, this implies that W is a quasi-ideal RGQA transversal of Γ. □

Now we prove the converse half of Theorem 4.1. Let S be an abundant semigroup with a quasi-ideal RGQA transversal S^o. Let

$$R = \{x \in S : (\exists\lambda_x \in \Lambda_x)\ \lambda_x \in E^o\} \text{ and } L = \{p \in S : (\exists i_p \in I_p)\ i_p \in E^o\}.$$

Then R and L are abundant semigroups with a common quasi-adequate transversal S^o and S^o is a right ideal of R and a left ideal of L.

For each $(p, x) \in L \times R$, define $p * x = px$. Then $p * x = px = i_p px\lambda_x \in S^o$ for some $i_p, \lambda_x \in E^o$ since S^o is a quasi-ideal of S. Clearly the map satisfies (1), (2) and (3). Therefore we acquire an abundant semigroup Γ in the way of the direct part of Theorem 4.1. Finally we prove that Γ is isomorphic to S.

For any $(K(x), L^*_p) \in \Gamma$, we define $\delta : \Gamma \longrightarrow S$ by $(K(x), L^*_p)\delta = i_x p$, where $i_x \in I_x$ and $i_x\ \mathcal{L}\ \overline{x}^+$ for some $\overline{x} \in C_{S^o}(x) \cap C_{S^o}(p)$ and some $\overline{x}^+ \in E^o$. Obviously the definition of δ is not dependent on the choice of i_x.

To show that δ is well-defined, if $(K(x), L^*_p) = (K(y), L^*_q)$, then $R^*_x = R^*_y, C_{S^o}(x) = C_{S^o}(y), L^*_p = L^*_q$. We notice that $p\ \mathcal{L}^*\ i_x p\ \mathcal{R}^*\ i_x$. In fact, for any $g, h \in S^1$, if $gi_x p = hi_x p$, then $gi_x i_p = hi_x i_p$ and consequently, $gi_x i_p\overline{x}^+ = hi_x i_p\overline{x}^+$, that is $gi_x = hi_p$ and so $i_x p\ \mathcal{L}^*\ i_x$. Since $\mathcal{L}^*$ is a right congruence, we have $p\ \mathcal{L}^*\ \lambda_p = \overline{x}^*\lambda_p\ \mathcal{L}^*\ \overline{x}\lambda_p = \overline{x}^+ p\ \mathcal{L}^*\ i_x p$. Thus

$$i_x p\ \mathcal{R}^*\ i_x\ \mathcal{R}^*\ x\ \mathcal{R}^*\ y\ \mathcal{R}^*\ i_y\ \mathcal{R}^*\ i_y q \text{ and } i_x p\ \mathcal{L}^*\ p\ \mathcal{L}^*\ q\ \mathcal{L}^*\ i_y q.$$

Consequently, $i_x p\ \mathcal{H}^*\ i_y q$.

From $x\ \mathcal{R}^*\ y$ and $C_{S^o}(x) = C_{S^o}(y)$, by Remark 1, there exists $h \in E^o$ such that $x = yh$, moreover $h\ \mathcal{L}^*\ x$. Thus $x\lambda_p = yh\lambda_p = i_y\overline{y}\lambda_y h\lambda_p$ and $y\lambda_q = i_y\overline{y}\lambda_y\lambda_q$. Since $y \in R$ we have $\lambda_y \in E^o$ and consequently $\lambda_y \cdot h \cdot \lambda_p \in E^o E^o \cdot \Lambda \subseteq E^o\Lambda \subseteq E$ and $\lambda_y\lambda_q \in E^o\Lambda \subseteq E$. It is easy to see that $\lambda_y h\lambda_p$ and $\lambda_y\lambda_q$ are in the same $\mathcal{H}^*$-class and hence $\lambda_y h\lambda_p = \lambda_y\lambda_q$. Therefore $x\lambda_p = y\lambda_q$ and so $i_x p = i_y q$, that is θ is well-defined.

Let $(K(x), L^*_p), (K(y), L^*_q) \in \Gamma$. Then from $y\lambda_q = i_y q$, we obtain

$$\begin{aligned} [(K(x), L^*_p)(K(y), L^*_q)]\delta &= (K(i_x py), L^*_{py\lambda_q})\delta = i_{i_x py} \cdot py\lambda_q = i_x i_{py} \cdot py\lambda_q \\ &= i_x py\lambda_q = i_x pi_y q = (K(x), L^*_p)\delta \cdot (K(y), L^*_q)\delta, \end{aligned}$$

and so δ is a homomorphism.

For each $x \in S$, it is easy to see that $x\overline{x}^* \in R$ and $\overline{x}^+ x \in L$, where $x = i_x\overline{x}\lambda_x, i_x\ \mathcal{L}\ \overline{x}^+,\ \lambda_x\ \mathcal{R}\ \overline{x}^*$ for some $\overline{x}^+, \overline{x}^* \in E^o$. It follows from $x\overline{x}^* = i_x\overline{x}\lambda_x\overline{x}^* = i_x\overline{x}^+\overline{x}^*$ and $\overline{x}^+ x = \overline{x}^+ i_x\overline{x}\lambda_x = \overline{x}^+\overline{x}\lambda_x$ that $\overline{x} \in C_{S^o}(x\overline{x}^*) \cap C_{S^o}(\overline{x}^+ x)$, and consequently $(K(x\overline{x}^*), L^*_{\overline{x}^+ x}) \in \Gamma$. It follows that

$$(K(x\overline{x}^*), L^*_{\overline{x}^+ x})\delta = i_{x\overline{x}^*} \cdot \overline{x}^+ x = i_x\overline{x}^+ x = i_x x = x,$$

and hence δ is surjective.

Suppose that $(K(x), L^*_p), (K(y), L^*_q) \in \Gamma$ are such that $(K(x), L^*_p)\delta = (K(y), L^*_q)\delta$, then $i_x p = i_y q$. So

$$x\ \mathcal{R}^*\ i_x\ \mathcal{R}^*\ i_x p = i_y q\ \mathcal{R}^*\ i_y\ \mathcal{R}^*\ y \text{ and } p\ \mathcal{L}^*\ i_x p = i_y q\ \mathcal{L}^*\ q.$$

That is $R^*_x = R^*_y$ and $L^*_p = L^*_q$. From $x\lambda_p = y\lambda_q$ we deduce that

$$y = y\lambda_q\lambda_y = x\lambda_p\lambda_y = i_x \cdot \overline{x} \cdot \lambda_x\lambda_p\lambda_y.$$

Since $\lambda_x, \lambda_y \in E^o$, we have $\lambda_x\lambda_p\lambda_y$ is idempotent in R and $\lambda_x\lambda_p\lambda_y \mathcal{R} \lambda_x\lambda_p \mathcal{R} \lambda_x \mathcal{R} \overline{x}^*$. Thus $\overline{x} \in C_{S^o}(y)$ and consequently $C_{S^o}(x) \cap C_{S^o}(y) \neq \emptyset$. Hence $K(x) = K(y)$ and $L_p^* = L_q^*$, that is, δ is injective. Therefore δ is an isomorphism.

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