Filomat 35:1 (2021), 271–286
https://doi.org/10.2298/FIL2101271R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Multiple Interpolation in the Privalov Classes in a Disk

Eugenia Gennad'evna Rodikova[a]

[a]*Bryansk State University named after I. G. Petrovsky, str. Bezhitskaya, 14, 241036 Bryansk, Russia*

Abstract. For all $0 < q < +\infty$ the Privalov class Π_q consists of all analytic functions f in a unit disk such that

$$\sup_{0\le r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^q d\theta < +\infty.$$

In this paper we solve a multiple interpolation problem in the class Π_q for all $0 < q < 1$. Namely, we find the sufficient conditions for the explicit construction of the function that solves the interpolation problem in the Privalov class. In addition, we discuss the necessity of these conditions.

1. Introduction

Let $\mathbb{C}$ be the complex plane, D be the unit disk on $\mathbb{C}$, $H(D)$ be the set of all functions, holomorphic in D. For all $0 < q < +\infty$ we define the Privalov class of function Π_q as follows:

$$\Pi_q = \left\{ f \in H(D) : \sup_{0\le r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^q d\theta < +\infty \right\}.$$

Here, as usual, $\ln^+ |a| = \max(\ln |a|, 0)$, $\forall a \in \mathbb{C}$.

The classes Π_q were first considered by I. I. Privalov in [11]. If $q = 1$ the Privalov class coincides with the Nevanlinna class N, well-known in scientific literature (see [10]). Using Holder's inequality, it is easy to prove the inclusion chain:

$$\Pi_q\,(q > 1) \subset N \subset \Pi_q\,(0 < q < 1).$$

In the case of $1 \le q < +\infty$ the Privalov spaces were studied by M. Stoll, V. I. Gavrilov, A. V. Subbotin, D. A. Efimov, R. Mestrovic, Z. Pavicevic, etc. The monograph [7] contains a brief overview of their results. Certain results were distributed to case $0 < q < 1$ by author of this paper (see [14]). Notice that the case $0 < q < 1$ was studied little in the scientific literature. Apparently, the Privalov classes $\Pi_q\,(0 < q < 1)$ were studied only in papers [14], [15], [18] and [19]. Factorization representation is not obtained for these classes, which makes it difficult to solve many existing problems. Motivated by the recent results of F. A. Shamoyan given in [18] and related investigations for the $\Pi_q\,(q > 1)$, here we consider the interpolation questions on

2010 *Mathematics Subject Classification*. Primary 30E05; Secondary 30H15, 30H50
Keywords. Interpolation, multiple nodes, analytic functions, the Privalov class, the Stolz angles.
Received: 18 December 2019; Revised: 15 March 2020; Accepted: 18 April 2020
Communicated by Miodrag Mateljević
The work was financially supported by Russian Foundation for Fundamental Research, project number 18-31-00180.
Email address: evheny@yandex.ru (Eugenia Gennad'evna Rodikova)

the set of multiple nodes in the Privalov classes Π_q $(0 < q < 1)$. Our methods are similar to those used for the area Nevanlinna type spaces in [2] and [13].

State the problem of multiple interpolation for the class Π_q: let $\{\alpha_k\}_1^\infty$ and $\{w_k\}_1^\infty$ be the arbitrary sequences of complex numbers, $\{\alpha_k\}_1^\infty \subset D$; we put p_j be the multiplicity of the number α_j in the sequence $\{\alpha_k\}_1^\infty$, $s_j \geq 1$ be the multiplicity of the number α_j on the interval $\{\alpha_k\}_1^j$. Obviously, $1 \leq s_j \leq p_j \leq +\infty$. We need to find conditions for growth of α_k and distribution of w_k under which one can construct a function $f \in \Pi_q$ such that the following task is solvable:

$$f^{(s_k-1)}(\alpha_k) = w_k,\ k = 1, 2, \ldots. \tag{1}$$

In this case $\{\alpha_k\}_1^\infty$ is called *interpolating sequence.* For $s_k = 1$ we say that the interpolation is performed on a set of simple nodes $\{\alpha_k\}_1^\infty$.

The fundamental result in the theory of interpolation belongs to L. Carleson. In [3] he fully described interpolating sequences for the class of bounded analytic functions. The interpolation problem in classes of functions with bounded characteristic was solved by A. G. Naftalevic in [9]; a so-called free interpolation problem in these spaces was investigated by A. Hartmann, X. Massaneda, A. Nicolau, P. Thomas in [6]. The same problems in the Hardy spaces were studied by H. Shapiro, A. Shields in [23] and by K. Seip in [24]; multiple interpolation problem for Hardy's spaces was solved by M. M. Djrbashian in [5]. The questions of interpolation on the sets of simple nodes in the Smirnov classes were investigated by N. Yanagihara in [26], the same questions on the sets of simple and multiple nodes in the classes of analytic functions with the restrictions on the Nevanlinna characteristic were studied by V. A. Bednazh, F. A. Shamoyan and E. G. Rodikova (see [1], [2], [12], [13], [21]; for a detailed description of the mentioned classes see the monograph [22]). Overview of early results on interpolation theory is contained in the monograph of S. A. Vinogradov, V. P. Havin (see [25]).

So-called free interpolation problem on the Carleson sets (i.e. in the case of uniformly separated sequence) in the Π_q - classes was solved by R. Mestrovic, J. Susic in [8] for $q > 1$, and by the author of this paper and V. A. Bednazh in [15] for $0 < q < 1$. Recall that a sequence $\{\alpha_k\}_1^\infty \subset D$ is said to be uniformly separated if

$$\prod_{k \neq n} \left| \frac{\alpha_k - \alpha_n}{1 - \overline{\alpha}_k \alpha_n} \right| \geq \delta > 0,\ \forall k \in \mathbb{N}. \tag{2}$$

In [15] the authors also investigated the questions of multiple interpolation in Π_q - classes for $q > 1$ provided that the nodes are in the Stoltz angles and satisfy the condition of so-called weak separation instead of (2).

Thus, in this work we continue the study of interpolation issues in the Privalov classes.

The paper is organized as follows: in the second part of the paper we prove auxiliary assertions and present the main result, and in the third part of the article we prove the main result, namely, we construct an explicit solution to the interpolation problem (1).

2. Formulation of main result and proof of auxiliary assertions

To formulate and prove the results of the work we introduce some more notations and definitions.

For any $\beta > -1$ we denote by $\pi_\beta(z, \alpha_k)$ the M. M. Djrbashian's infinite product with zeros at points of the sequence $\{\alpha_k\}_1^\infty \subset D$ (see [4]):

$$\pi_\beta(z, \alpha_k) = \prod_{k=1}^{+\infty} \left(1 - \frac{z}{\alpha_k}\right) \exp(-U_\beta(z, \alpha_k)), \tag{3}$$

where

$$U_\beta(z, \alpha_k) = \frac{2(\beta+1)}{\pi} \int_0^1 \int_{-\pi}^{\pi} \frac{(1-\rho^2)^\beta \ln|1 - \frac{\rho e^{i\theta}}{\alpha_k}|}{(1 - z\rho e^{-i\theta})^{\beta+2}} d\theta \rho d\rho.$$

We denote by $\pi_{\beta,n}(z,\alpha_k)$ the infinite product $\pi_\beta(z,\alpha_k)$ without n-th factor.

As stated in [4], the infinite product $\pi_\beta(z,\alpha_k)$ is absolutely and uniformly convergent in the unit disk D if and only if the following series converges:

$$\sum_{k=1}^{+\infty}(1-|\alpha_k|)^{\beta+2} < +\infty.$$

If $\beta+1 = p \in \mathbb{Z}_+$, then product (3) takes a form (see [4]):

$$\pi_p(z,\alpha_k) = \prod_{k=1}^{+\infty} \frac{\overline{\alpha}_k(\alpha_k - z)}{1-\overline{\alpha}_k z} \exp \sum_{j=1}^{p+1} \frac{1}{j}\left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha}_k z}\right)^j.$$

Definition 1. *The Stolz angle $\Gamma_\delta(\theta)$ with the vertex at the point $e^{i\theta}$ is the angle of the solution $\pi\delta$, $0<\delta<1$, whose bisector coincides with the segment $re^{i\theta}$, $0 \le r < 1$, that is the set of points $z \in D$ for which the following inequalities hold:*

$$\left|\arg\left(e^{i\theta}-z\right)-\theta\right| \le \frac{\pi\delta}{2},$$
$$\left|e^{i\theta}-z\right| < \cos\frac{\pi\delta}{2}.$$

Everywhere below, unless otherwise specified, we assume that $0<q<1$.

Also by $c, c_1, ..., c_n(\alpha, \beta, ...)$ we denote arbitrary positive constants depending on α, $\beta, ...$, whose value immaterial.

For all $0 \le r < 1$ by definition, put $n(r) = \operatorname{card}\{\alpha_k : |\alpha_k| < r\}$. The sequence $\{\alpha_k\}_1^\infty \subset D$, satisfying the following conditions

$$\int_0^1 n^q(r)dr < +\infty, \tag{4}$$

$$\left|\pi_{p,k}(\alpha_k,\alpha_j)\right| \ge \exp\frac{-\mu(k)}{(1-|\alpha_k|)^{\frac{1}{q}}}, \tag{5}$$

where $p > \frac{1}{q}-1$, $\mu(k)>0$, $\mu(k)=o(1)$, $k\to+\infty$,

$$\sup_{k\ge1}\{p_k\} < +\infty,$$

we associate with a class $\widetilde{\Delta}$.

For given sequence $\{\alpha_k\}_1^\infty \subset D$ and fixed $0<q<1$ we denote by $\tilde{l}^q(\alpha_k)$ a sequence space $\{w_k\}_1^\infty$ such that

$$\ln^+|w_k| = o\left((1-|\alpha_k|)^{-1/q}\right),\ k\to+\infty,$$

i.e.

$$|w_k| = \exp\frac{\mu_1(k)}{(1-|\alpha_k|)^{1/q}}, \tag{6}$$

$\mu_1(k)>0$, $\mu_1(k)=o(1)$, $k\to+\infty$.

Notice, that class $\tilde{l}^q$ is natural for solving the interpolation problem in Privalov's spaces Π_q, because the following assertion is valid:

Theorem 1 (see [14]). *Let* $0 < q < 1$. *If* $f \in \Pi_q$, $M(r, f) = \max_{|z|=r} |f(z)|$, $z \in D$, *then*

$$\ln^+ M(r, f) = o\left((1-r)^{-1/q}\right), r \to 1-0, \tag{7}$$

and this estimate is exact.

The main result of this article is proof of the following theorem:

Theorem 2. *Let* $\{\alpha_k\}_1^\infty \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s)$ *for a certain* $0 < \delta < 1$.

If $\{\alpha_k\}_1^\infty \in \widetilde{\Delta}$, *then for any sequence* $\{w_k\}_1^\infty$ *from* $\tilde{l}^q(\alpha_k)$ *it is possible to construct explicit the function* $f \in \Pi_q$ *that solve the multiple interpolation problem* (1) *for all* $s_k \geq 1$.

Proof of main result are based on the following statements.

Theorem 3. *(see [18]) If* $\{\alpha_k\}_1^\infty$ *is a sequence of zeros for a function* $f \in \Pi_q$, *then*

$$n(r) \leq \frac{c}{(1-r)^{1/q}}.$$

Conversely: assume that $\{\alpha_k\}_1^\infty \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s)$ *for a certain* $0 < \delta < 1$; *if the following integral is convergent:*

$$\int_0^1 n^q(r)dr < +\infty,$$

then we can construct a nontrivial function f *from* Π_q *such that* $f(\alpha_k) = 0$, $k = 1, 2, \ldots$.

Remark 1. *In the recent work [16] the author established that the sufficient condition in Theorem 3 is also necessary.*

Remark 2. *Notice that zero set of function from the class* Π_q $(q > 1)$ *is completely characterized by the Blaschke condition due to inclusion this class in the Nevanlinna class* N.

We consider a function h associated with a sequence $\{\alpha_k\}_1^\infty$ in class Π_q:

$$h = h_k(z) = \exp \sum_{s=1}^{n} \sum_{m=1}^{+\infty} u_k^m \frac{(1-\rho_m{}^2)^\beta}{(1 - z\rho_m e^{-i\theta_s})^{\beta+\frac{1}{q}}}, z \in D, \tag{8}$$

where $0 < \beta < 1/q$, $0 < \rho_m < 1$, $m = 1, 2, ...$, $\{u_k\}_1^\infty$ is a positive infinitesimal sequence depending from the interpolation nodes $\{\alpha_k\}_1^\infty$.

We show that $h \in \Pi_q$. Without loss of generality, we assume that all interpolation nodes are in the angle $\Gamma_\delta(\theta)$. Denote for a brevity $\beta' = \beta + \frac{1}{q}$. We have:

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\ln^+ |h(re^{i\varphi})|)^q d\varphi = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\ln^+ \left| \exp \sum_{m=1}^{+\infty} u_k^m \frac{(1-\rho_m{}^2)^\beta}{(1 - re^{i\varphi}\rho_m e^{-i\theta})^{\beta'}} \right| \right)^q d\varphi \leq$$

$$\leq \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{m=1}^{+\infty} u_k^m \frac{(1-\rho_m{}^2)^\beta}{|1 - r\rho_m e^{i(\varphi-\theta)}|^{\beta'}} \right)^q d\varphi.$$

We continue the assessment:

$$\sup_{0\le r<1}\frac{1}{2\pi}\int_{-\pi}^{\pi}(\ln^+|h(re^{i\varphi})|)^q d\varphi \le \sup_{0\le r<1}\frac{1}{2\pi}\sum_{m=1}^{+\infty}\int_{-\pi}^{\pi}\left(u_k^m\frac{(1-\rho_m{}^2)^\beta}{|1-r\rho_m e^{i(\varphi-\theta)}|^{\beta'}}\right)^q d\varphi =$$
$$=\sup_{0\le r<1}\frac{1}{2\pi}\sum_{m=1}^{+\infty}\int_{-\pi}^{\pi}u_k^{mq}\frac{(1-\rho_m{}^2)^{\beta q}}{|1-r\rho_m e^{i(\varphi-\theta)}|^{\beta' q}}d\varphi \le \sup_{0\le r<1}\frac{1}{2\pi}\sum_{m=1}^{+\infty}\frac{u_k^{mq}(1-\rho_m{}^2)^{\beta q}}{(1-r\rho_m)^{(\beta' q-1)}} =$$
$$=\sup_{0\le r<1}\frac{1}{2\pi}\sum_{m=1}^{+\infty}\frac{u_k^{mq}(1-\rho_m{}^2)^{\beta q}}{(1-r\rho_m)^{\beta q}} \le \frac{2^{\beta q}}{2\pi}\sum_{m=1}^{+\infty}u_k^{mq} = \frac{2^{\beta q}}{2\pi}\cdot\frac{u_k^q}{1-u_k^q} < +\infty.$$

Thus we have $h_k \in \Pi_q$.

The following statement is valid.

Lemma 1. *Let $h(z)$ is defined by equality* (8) *under the following conditions:*

$$u_k = (\mu_1(k)+\mu(k))^{\frac{1}{2m_0^{(1)}}},\ k = 1,2,\dots,$$

here μ_1, μ are infinitesimals sequences from equations (6) *and* (5) *respectively, $m_0^{(1)} = \inf\limits_{\rho_m>r_k} m$,*

$$1-\rho_m = (2u_k^q)^m,\ m = 1,2,\dots.$$

If points of a sequence $\{\alpha_k\}_1^\infty$ are in the finite number of the Stolz angles, i.e. $\{\alpha_k\}_1^\infty \subset \bigcup\limits_{s=1}^{n}\Gamma_\delta(\theta_s)$, then for the function $h(z)$ the following estimate is valid:

$$|h(\alpha_k)| \ge \exp\frac{\mu_0(k)}{(1-|\alpha_k|)^{1/q}}, \tag{9}$$

$\mu_0(k) > 0$, $\mu_0(k) = o(1)$, $k \to +\infty$.

Proof. Without loss of generality, we assume that all interpolation nodes belong to the angle $\Gamma_\delta(\theta)$. For brevity we denote $\beta' = \beta + \frac{1}{q}$. So we have

$$h(z) = h_k(z) = \exp\sum_{m=1}^{+\infty}u_k^m\frac{(1-\rho_m{}^2)^\beta}{(1-z\rho_m e^{-i\theta})^{\beta'}},\ z\in D.$$

We estimate $h(\alpha_k)$ in the angle $\Gamma_\delta(\theta)$.

$$|h(\alpha_k)| = \exp\sum_{m=1}^{+\infty}u_k^m\Re\frac{(1-\rho_m{}^2)^\beta}{(1-\alpha_k\rho_m e^{-i\theta})^{\beta'}} = \exp\sum_{m=1}^{+\infty}u_k^m(1-\rho_m{}^2)^\beta\frac{\Re(1-\overline{\alpha_k}\rho_m e^{i\theta})^{\beta'}}{|1-\alpha_k\rho_m e^{-i\theta}|^{2\beta'}}.$$

But

$$\Re(1-\overline{\alpha_k}\rho_m e^{i\theta})^{\beta'} = \Re\left(1-r_k\rho_m e^{-i(\varphi_k-\theta)}\right)^{\beta'} = \Re\left(1-\rho_m r_k+\rho_m r_k(1-e^{-i(\varphi_k-\theta)})\right)^{\beta'} =$$
$$=\Re\left(1-\rho_m r_k+\rho_m r_k(1-e^{-i(\varphi_k-\theta)})\right)^{\beta'} == (\rho_m r_k\rho)^{\beta'}\cdot\Re\left(\frac{1-\rho_m r_k}{\rho_m r_k\rho}+e^{-i\varphi}\right)^{\beta'},$$

where $\alpha_k = r_k e^{i\varphi_k}$, $(1-e^{-i(\varphi_k-\theta)}) = \rho e^{-i\varphi}$, $|\varphi| < \frac{\pi}{2\beta'}$. Therefore we have $\Re(1-\overline{\alpha_k}\rho_m e^{i\theta})^{\beta'} \ge c_1(\rho_m r_k\rho)^{\beta'}$ by Lemma 1.3 proved in the work of F.A. Shamoyan [17].

On the other hand,

$$|1 - e^{-i(\varphi_k-\theta)}|^{\beta'} = 2^{\beta'} \sin^{\beta'}\left(\frac{\theta-\varphi_k}{2}\right),$$

whence

$$\Re \frac{1}{(1-\alpha_k\rho_m e^{-i\theta})^{\beta'}} \geq \frac{c_1(\rho_m r_k)^{\beta'} 2^{\beta'} \sin^{\beta'}\left(\frac{\theta-\varphi_k}{2}\right)}{\left((1-\rho_m r_k)^2 + 4\sin^2\left(\frac{\theta-\varphi_k}{2}\right)\rho_m r_k\right)^{\beta'}} \geq$$

$$\geq \frac{c_1(\rho_m r_k)^{\beta'} 2^{\beta'} \sin^{\beta'}\left(\frac{\theta-\varphi_k}{2}\right)}{\left((1-\rho_m r_k)^2 + 4\sin^2\left(\frac{\theta-\varphi_k}{2}\right)\right)^{\beta'}} \geq \frac{\tilde{c}_1 \cdot 2^{\beta'} \sin^{\beta'} \frac{\theta-\varphi_k}{2}}{(1-\rho_m r_k)^{2\beta'} \cdot \left(1 + \frac{4\sin^2\frac{\theta-\varphi_k}{2}}{(1-\rho_m r_k)^2}\right)^{\beta'}}.$$

Since $\{\alpha_k\}_1^\infty \subset \Gamma_\delta(\theta)$, we have

$$\frac{\left|\sin\left(\frac{\theta-\varphi_k}{2}\right)\right|}{(1-r_k)} \leq C.$$

As a result, we obtain:

$$\Re \frac{1}{(1-\alpha_k\rho_m e^{-i\theta})^{\beta'}} \geq \frac{c(\beta')}{(1-\rho_m r_k)^{\beta'}}.$$

Thus for the function $h(\alpha_k)$ in the angle $\Gamma_\delta(\theta)$ the following estimate is held:

$$|h(\alpha_k)| \geq \exp c(\beta') \sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\beta'}}.$$

Note that the series $\sum\limits_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\beta'}}$ is convergent. Indeed,

$$u_k^m \frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\beta'}} \leq u_k^m \frac{(1-{\rho_m}^2)^\beta}{(1-\rho_m)^{\beta'}} \leq u_k^m \frac{2^\beta}{(1-\rho_m)^{1/q}}.$$

By condition, $1-\rho_m = (2u_k^q)^m$. Therefore

$$u_k^m \frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\beta'}} \leq \frac{2^\beta}{2^m},$$

and the series in question is convergent in $\Gamma_\delta(\theta)$ in view of the convergence of the series $\sum\limits_{m=1}^{+\infty} \frac{1}{2^m}$.

We continue to search for a lower bound of $|h(\alpha_k)|$. To do this, we divide the internal amount into parts:

$$S = \sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\beta'}} = \sum_{(1-\rho_m)<(1-r_k)} (\ldots) + \sum_{(1-\rho_m)>(1-r_k)} (\ldots) + \sum_{(1-\rho_m)=(1-r_k)} (\ldots) =$$

$$= S_1(k) + S_2(k) + S_3(k).$$

We evaluate each of these sums separately.

$$S_2(k) = \sum_{(1-\rho_m)>(1-r_k)} u_k^m \frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\beta'}} \geq \sum_{\rho_m<r_k} u_k^m \frac{1}{(1-\rho_m^2)^{\beta'-\beta}} \geq \left(\frac{1}{2}\right)^{1/q} \sum_{\rho_m<r_k} u_k^m \frac{1}{(1-\rho_m)^{\frac{1}{q}}}.$$

By condition, $1-\rho_m=(2u_k^q)^m$. Therefore

$$S_2(k)\geq \frac{1}{2^{1/q}}\cdot\sum_{\rho_m<r_k}\left(\frac{1}{2^{1/q}}\right)^m=c(q,m_0)=\text{const}.$$

Here m_0 is a number for which $\rho_m=r_k$. Now we estimate the sum $S_3(k)$.

$$S_3(k)=\sum_{(1-\rho_m)=(1-r_k)}u_k^m\frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\beta'}}=\sum_{\rho_m=r_k}u_k^m\frac{1}{(1-r_k^2)^{1/q}}=\frac{u_k^{m_0}}{(1-r_k^2)^{1/q}}.$$

Further, we seek a lower bound for the sum $S_1(k)$.

$$\begin{aligned}S_1(k)&=\sum_{(1-\rho_m)<(1-r_k)}u_k^m\frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\beta'}}=\sum_{\rho_m>r_k}u_k^m\frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^{\frac{1}{q}}(1-r_k\rho_m)^\beta}\geq\\&\geq\frac{1}{(1-r_k^2)^{\frac{1}{q}}}\sum_{\rho_m>r_k}u_k^m\frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^\beta}\geq\\&\geq\frac{1}{(1-r_k^2)^{\beta+\frac{1}{q}}}\cdot\left(u_k^{m_0}(1-\rho_{m_0}^2)^\beta+\sum_{\rho_m>r_k,m\neq m_0^{(1)}}u_k^m\frac{(1-{\rho_m}^2)^\beta}{(1-r_k\rho_m)^\beta}\right).\end{aligned}$$

Taking into account the conditions $1-\rho_m=(2u_k^q)^m$ and $0<\beta<1/q$, we obtain:

$$S_1(k)\geq\frac{2^{m_0^{(1)}\beta}\cdot u_k^{(q\beta+1)m_0^{(1)}}}{(1-r_k^2)^{\beta+\frac{1}{q}}}\geq\frac{u_k^{2m_0^{(1)}}}{(1-r_k^2)^{\beta+\frac{1}{q}}}.$$

Here $m_0^{(1)}=\inf\limits_{\rho_m>r_k}m$. From the estimates S_1, S_2, S_3 we conclude:

$$S\geq\frac{u_k^{2m_0^{(1)}}}{(1-r_k)^{\beta+\frac{1}{q}}}+c(q,m_0)+\frac{u_k^{m_0}}{(1-r_k)^{1/q}},$$

whence

$$S(k)>\frac{u_k^{2m_0^{(1)}}}{(1-r_k)^{\frac{1}{q}}}.$$

As a result, we obtain:

$$|h(\alpha_k)|\geq\exp\frac{\mu_0(k)}{(1-r_k)^{\frac{1}{q}}},\ k=1,2,\dots,\tag{10}$$

where $\mu_0(k)=\mu(k)+\mu_1(k)\leq u_k^{2m_0^{(1)}}$, $\mu_0(k)=o(1)$, $k\to+\infty$. Lemma 1 is proved. □

For a fixed $\alpha_k\in\Gamma_\delta(\theta)$ by definition, put

$$K_\eta(\alpha_k):=\left\{z\in D:\ |z-\alpha_k|<\frac{1}{A}\exp\frac{-\eta(k)}{(1-|\alpha_k|)^{1/q}}\right\},$$

where $\eta(k)>0$, $\eta(k)=o(1)$, $k\to+\infty$, $A>1/\cos\left(\frac{\pi\delta}{2}\right)$, and $K_\eta(\alpha_k)\bigcap K_\eta(\alpha_n)=\varnothing$, $k\neq n$.

Lemma 2. *For all $t \in K_\eta(\alpha_k)$ the following estimate holds:*

$$|h(t)| \geq \exp \frac{\mu_0(k)}{(1-|\alpha_k|)^{1/q}}, \tag{11}$$

where $\mu_0(k)$ is positive infinitesimal sequence from the equation (9).

Proof. Without loss of generality, we assume that all interpolation nodes belong to the angle $\Gamma_\delta(\theta)$ and $\theta = 0$. For brevity denote $\beta' = \beta + \frac{1}{q}$. So we have

$$h(t) = h_k(t) = \exp \sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\beta}{(1-t\rho_m)^{\beta'}}.$$

We find a lower bound for a function $h(t)$ in the circle $K_\eta(\alpha_k)$.

$$|h(t)| = \exp \sum_{m=1}^{+\infty} u_k^m \Re \frac{(1-{\rho_m}^2)^\beta}{(1-t\rho_m)^{\beta'}} = \exp \sum_{m=1}^{+\infty} u_k^m (1-{\rho_m}^2)^\beta \frac{\Re(1-\bar{t}\rho_m)^{\beta'}}{|1-t\rho_m|^{2\beta'}}.$$

Consider the denominator.

$$\begin{aligned}
|1-t\rho_m|^{2\beta'} &= |1-\alpha_k\rho_m + \alpha_k\rho_m - t\rho_m|^{2\beta'} = \left|1-\alpha_k\rho_m + \rho_m(\alpha_k - t)\right|^{2\beta'} \leq \\
&\leq (|1-\alpha_k\rho_m| + \rho_m|\alpha_k - t|)^{2\beta'} \leq 2^{2\beta'} \cdot \left(|1-\alpha_k\rho_m|^{2\beta'} + \rho_m^{2\beta'}|\alpha_k - t|^{2\beta'}\right) = \\
&= 4^{\beta'} \cdot |1-\alpha_k\rho_m|^{2\beta'} \cdot \left(1 + \rho_m^{2\beta'}\left(\frac{|\alpha_k - t|}{|1-\alpha_k\rho_m|}\right)^{2\beta'}\right) \leq \\
&\leq 4^{\beta'} \cdot |1-\alpha_k\rho_m|^{2\beta'} \cdot \left(1 + \left(\frac{|\alpha_k - t|}{1-|\alpha_k|}\right)^{2\beta'}\right).
\end{aligned}$$

Here we have used the inequality $(a+b)^p \leq 2^p \cdot (a^p + b^p)$, valid for any positive values a, b, p. It can be argued that for sufficiently large values of A we have:

$$|\alpha_k - t| < \frac{1}{A}(1-|\alpha_k|)^{1/q} < \frac{1}{A}(1-|\alpha_k|).$$

Therefore

$$|1-t\rho_m|^{2\beta'} \leq c(\beta')|1-\alpha_k\rho_m|^{2\beta'}. \tag{12}$$

Now we consider numerator. By definition, put $t = \alpha_k + \eta e^{i\tau} = Re^{i\gamma}$. So we have

$$\begin{aligned}
&\Re(1-\bar{t}\rho_m)^{\beta'} = \Re\left(1 - R\rho_m e^{-i\gamma}\right)^{\beta'} = \\
&= \Re\left(1 - \rho_m R + \rho_m R(1-e^{-i\gamma})\right)^{\beta'} = (\rho_m R)^{\beta'} \cdot \Re\left(\frac{1-\rho_m R}{\rho_m R} + (1-e^{-i\gamma})\right)^{\beta'}.
\end{aligned}$$

The constant A is chosen so that the circle $K_\eta(\alpha_k)$ belongs to the same Stolz angle as the interpolation node α_k (in this case, $\Gamma_\delta(0)$). By Lemma 1.3 established in the work of F.A. Shamoyan [17], we obtain:

$$\Re(1-\bar{t}\rho_m)^{\beta'} \geq c_1(R\rho_m\rho_0)^{\beta'},$$

where $\rho_0 = \left|1-e^{-i\gamma}\right| = 2\sin\frac{\gamma}{2}$. But

$$R = \left|\alpha_k + \eta e^{i\tau}\right| = |\alpha_k| \cdot \left|1 + \frac{\eta}{|\alpha_k|}e^{i(\tau - \varphi_k)}\right|.$$

Therefore

$$\Re(1-\bar{t}\rho_m)^{\beta'} \geq c_1|\alpha_k|^{\beta'}(\rho_m)^{\beta'}2^{\beta'}\left(\sin\frac{\gamma}{2}\right)^{\beta'}\cdot\left|1+\frac{\eta}{|\alpha_k|}e^{i(\tau-\varphi_k)}\right|^{\beta'}=$$

$$=c_1|\alpha_k|^{\beta'}(\rho_m)^{\beta'}2^{\beta'}\left(\sin\frac{\gamma}{2}\right)^{\beta'}\cdot\left(1+2\frac{\eta}{|\alpha_k|}\cos(\tau-\varphi_k)+\frac{\eta^2}{|\alpha_k|^2}\right)^{\frac{\beta'}{2}}.$$

Obviously, we can choose the constant A in the definition of the circle $K_\eta(\alpha_k)$ so large that the following estimate holds:

$$\Re(1-\bar{t}\rho_m)^{\beta'} \geq \tilde{c}_1|\alpha_k|^{\beta'}(\rho_m)^{\beta'}2^{\beta'}\left(\sin\frac{\varphi_k}{2}\right)^{\beta'}. \tag{13}$$

From (13) and (12) we conclude that

$$\frac{\Re(1-\bar{t}\rho_m)^{\beta'}}{|1-t\rho_m|^{2\beta'}} \geq \frac{\tilde{c}(\beta')|\alpha_k|^{\beta'}(\rho_m)^{\beta'}2^{\beta'}\left(\sin\frac{\varphi_k}{2}\right)^{\beta'}}{|1-\alpha_k\rho_m|^{2\beta'}}.$$

Further, continuing to argue as in the proof of Lemma 1, we obtain the required estimate. Lemma 2 is proved. □

Lemma 3. *(see [2]) For any $z \in K_\eta(\alpha_k)$ the following estimate is valid:*

$$\frac{1}{2}|m_j(\alpha_k)| \leq |m_j(z)| \leq \frac{3}{2}|m_j(\alpha_k)|,$$

where

$$m_j(z)=\left(\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right), \alpha_j \in D,\ j=1,2,\dots.$$

Lemma 4. *Suppose $\{\alpha_j\}_1^\infty \subset \widetilde{\Delta}$; then there exists $\eta > 0$, such that for any $z \in K_\eta(\alpha_n)$, $n = 1,2,\dots$, the following estimate is valid*

$$|\pi_{p,n}(z,\alpha_j)| \geq \exp\frac{-\widetilde{\varepsilon}(n)}{(1-|\alpha_n|)^{\frac{1}{q}}},$$

where $p > \frac{1}{q}-1$, $\widetilde{\varepsilon}(n) > 0$, $\widetilde{\varepsilon}(n) = o(1)$, $n \to +\infty$.

Proof. The proof is almost completely repeats the arguments given in [2] (see Lemma 2.8). For understanding, we present it. We fix $n \in \mathbb{N}$ and estimate the product $\pi_{p,n}(z,\alpha_j)$ in the circle $K_\eta(\alpha_n)$. Since $K_\eta(\alpha_k)\bigcap K_\eta(\alpha_n) = \varnothing$, $k \neq n$, we have:

$$\ln\pi_{p,n}(z,\alpha_j)=\sum_{j=1,j\neq n}^{+\infty}\ln A_j(z,\alpha_j)=$$

$$=\sum_{\substack{j=1\\ j\neq n}}^{+\infty}\left[\ln\left(1-\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right)+\sum_{s=1}^{p}\frac{1}{s}\left(\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right)^s\right],$$

where the main branch of the logarithm is chosen.

We split the sum $\Sigma = \sum\limits_{\substack{j=1\\ j\neq n}}^{+\infty}\ln A_j(z,\alpha_j)$ on two parts:

$$\Sigma = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{|m_j(z)|\le\frac{1}{2}} \ln A_j(z,\alpha_j),$$

$$\Sigma_2 = \sum_{|m_j(z)|>\frac{1}{2}} \ln A_j(z,\alpha_j).$$

It is obvious that

$$\Sigma_1 = \sum_{\substack{j=1\\ j\neq n}}^{+\infty} \sum_{s=p+1}^{+\infty} \frac{1}{s}(m_j(z))^s = \sum_{\substack{j=1\\ j\neq n}}^{+\infty} (m_j(z))^{p+1} \cdot \left(\sum_{s=p+1}^{+\infty} \frac{1}{s}(m_j(z))^{s-p-1} \right).$$

Therefore we have

$$|\Sigma_1| = \left| \sum_{|m_j(z)|\le\frac{1}{2}} \ln A_j(z,\alpha_j) \right| \le \sum_{|m_j(z)|\le\frac{1}{2}} |m_j(z)|^{p+1} \sum_{s=p+1}^{+\infty} \frac{1}{s} |m_j(z)|^{s-p-1} \le$$

$$\le \sum_{|m_j(z)|\le\frac{1}{2}} |m_j(z)|^{p+1} \sum_{k=0}^{+\infty} \frac{1}{2^k} \le 2 \sum_{|m_j(z)|\le\frac{1}{2}} \frac{(1-|\alpha_j|^2)^{p+1}}{|1-\overline{\alpha_j}z|^{p+1}}.$$

By Lemma 3, we obtain:

$$|\Sigma_1| \le \frac{3^{p+1}}{2^p} \sum_{|m_j(z)|\le\frac{1}{2}} \frac{(1-|\alpha_j|^2)^{p+1}}{|1-\overline{\alpha_j}\alpha_n|^{p+1}}.$$

But the last sum is clearly less than $\sum\limits_{j=1}^{+\infty} \frac{(1-|\alpha_j|^2)^{p+1}}{|1-\overline{\alpha_j}\alpha_n|^{p+1}}$. We continue the estimate of $|\Sigma_1|$:

$$|\Sigma_1| \le 2\cdot 3^{p+1} \int\limits_0^1 \frac{(1-t)^{p+1}}{(1-t|\alpha_n|)^{p+1}} dn(t) \le 3^{p+2} \int\limits_0^1 \frac{(1-t)^p n(t)}{(1-t|\alpha_n|)^{p+1}} dt \le$$

$$\le 3^{p+2} \int\limits_0^1 \frac{(1-t)^p \varepsilon_n}{(1-t)^{\frac{1}{q}}(1-t|\alpha_n|)^{p+1}} dt,$$

where $\varepsilon_n > 0$, $\varepsilon_n = o(1)$, $n \to +\infty$. In the last inequality we used the condition (4). Further, we have:

$$|\Sigma_1| \le \frac{3^{p+2}\varepsilon_n}{(1-|\alpha_n|)^{1/q-\delta}} \int\limits_0^1 \frac{(1-t)^{p-1/q}}{(1-t|\alpha_n|)^{p+1+\delta-1/q}} dt \le$$

$$\le \frac{3^{p+2}\varepsilon_n}{(1-|\alpha_n|)^{1/q-\delta}} \cdot \frac{1}{(1-|\alpha_n|)^{\delta}} = \frac{3^{p+2}\varepsilon_n}{(1-|\alpha_n|)^{1/q}}.$$

Thus we obtain:

$$|\Sigma_1| \le \frac{\varepsilon_n^{(1)}}{(1-|\alpha_n|)^{\frac{1}{q}}},$$

where $\varepsilon_n^{(1)} > 0$, $\varepsilon_n^{(1)} = o(1)$, $n \to +\infty$.

Now we estimate Σ_2. First note that if $|m_j(z)| > \frac{1}{2}$, then $|m_j(\alpha_n)| > \frac{1}{2}$ in the circle $K_\eta(\alpha_n)$ for sufficiently large η. We find a lower bound for $\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right|$:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| = \left[\ln\frac{|z-\alpha_j|}{|\alpha_n-\alpha_j|} + \ln\left|\frac{1-\overline{\alpha}_j\alpha_n}{1-\overline{\alpha}_j z}\right|\right] + \\ + \sum_{s=1}^{p}\frac{1}{s}\Re\left(\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right)^s - \sum_{s=1}^{p}\frac{1}{s}\Re\left(\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j\alpha_n}\right)^s.$$

We choose η so large that the following inequality holds:

$$\frac{1}{2} \le \left|\frac{z-\alpha_j}{\alpha_n-\alpha_j}\right| = \left|1+\frac{z-\alpha_n}{\alpha_n-\alpha_j}\right| \le \frac{3}{2}.$$

Taking into account Lemma 3, we obtain:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -\ln 2 + |m_j(z)|^{p+1}\times\left[\sum_{s=1}^{p}\frac{1}{s}\Re(m_j(z))^s\cdot\frac{1}{|m_j(z)|^{p+1}} - \sum_{s=1}^{p}\frac{1}{s}\Re(m_j(\alpha_n))^s\cdot\frac{1}{|m_j(z)|^{p+1}}\right].$$

Since $|\Re w| \le |w|$, $w \in \mathbb{C}$, we have:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -\ln 2 - |m_j(z)|^{p+1}\times\left[\sum_{s=1}^{p}\frac{1}{s}\frac{1}{|m_j(z)|^{p+1-s}} + \sum_{s=1}^{p}\frac{1}{s}\frac{1}{|m_j(\alpha_n)|^{p+1-s}}\right].$$

Using the inequality $|m_j(z)| > \frac{1}{2}$ and $|m_j(\alpha_n)| > \frac{1}{2}$ in the circle $K_\eta(\alpha_n)$, we obtain:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -\ln 2 - |m_j(z)|^{p+1}\times\sum_{s=1}^{p}\frac{1}{s}2\cdot 2^{p+1-s} \ge -\ln 2 - 2^{p+2}|m_j(z)|^{p+1}\times\sum_{s=1}^{p}\frac{1}{s\cdot 2^s},$$

whence we conclude:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -|m_j(z)|^{p+1}\cdot\left(\frac{\ln 2}{|m_j(z)|^{p+1}} + 2^{p+2} - 4\right) \ge -|m_j(z)|^{p+1}\cdot\left(2^{p+1}\ln 2 + 2^{p+2} - 4\right),$$

i.e.

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -c_2(p)|m_j(z)|^{p+1}.$$

Further, from (5) we have

$$\ln\left|A_j(z,\alpha_j)\right| \ge -c_2(p)\left|\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right|^{p+1} - \frac{\mu(n)}{(1-|\alpha_n|)^{1/q}}.$$

Integrating the estimates for amounts Σ_1 and Σ_2, we conclude:

$$\sum_{\substack{j=1\\ j\neq n}}^{+\infty}\ln|A_j(z,\alpha_j)| = \sum_{|m_j(z)|\le\frac{1}{2}}\ln|A_j(z,\alpha_j)| + \sum_{|m_j(z)|>\frac{1}{2}}\ln|A_j(z,\alpha_j)| \ge \\ \ge -\frac{\varepsilon_n^{(1)}}{(1-|\alpha_n|)^{\frac{1}{q}}} - \frac{\varepsilon_n^{(2)}}{(1-|\alpha_n|)^{\frac{1}{q}}} - \frac{\mu(n)}{(1-|\alpha_n|)^{\frac{1}{q}}}.$$

As a result, we obtain:

$$\ln|\pi_{p,n}(z,\alpha_n)| = \sum_{\substack{j=1\\ j\neq n}}^{+\infty} |\ln A_j(z,\alpha_j)| \geq \frac{-\widetilde{\varepsilon}(n)}{(1-|\alpha_n|)^{\frac{1}{q}}},$$

where $\widetilde{\varepsilon}(n) > 0$, $\widetilde{\varepsilon}(n) = o(1)$, $n \to +\infty$. $\square$

To state and prove further results, we introduce additional notation. First we remark that the function $\pi_{p,k}(z,\alpha_k) \cdot \left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha_k}z}\right)^{p+p_k+1}$ is analytic in D and does not vanish in certain neighborhood of the point $z = \alpha_k$ for all $p > \frac{1}{q} - 1$. For any $k \in \mathbb{N}$ we consider a function

$$\tau_k(z) = \left\{\pi_{p,k}(z,\alpha_k) \cdot \left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha_k}z}\right)^{p+p_k+1} \cdot h(z)\right\}^{-1},$$

where $h(z)$ is defined above (see (8)).

It can be argued that in any sufficiently small ε-neighborhood of the point α_k the following expansion is valid:

$$\tau_k(z) = \sum_{\nu=0}^{\infty} a_\nu(\alpha_k)(z-\alpha_k)^\nu,\ |z-\alpha_k| < \varepsilon,$$

where $a_\nu(\alpha_k) = \frac{1}{\nu!}\frac{d^\nu}{dz^\nu}\left[\left\{\pi_{p,k}(z,\alpha_k) \cdot \left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha_k}z}\right)^{p+p_k+1} \cdot h(z)\right\}^{-1}\right]_{z=\alpha_k}$.

Lemma 5. *If $\{\alpha_k\}_1^\infty \in \widetilde{\Delta}$, then the following estimates are valid:*

$$|a_\nu(\alpha_k)| \leq a(\nu),\ 0 \leq \nu \leq p + p_k,\ k = 1, 2, \ldots,$$

where $a(\nu)$ depends only on the ν.

This lemma is proved in the same way as in [20]; we give the proof for completeness.

Proof. So we have

$$a_\nu(\alpha_k) = \frac{1}{\nu!}\frac{d^\nu}{dz^\nu}\left[\left\{\left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha_k}z}\right)^{p+p_k+1} \pi_{p,k}(z,\alpha_k)h(z)\right\}^{-1}\right]_{z=\alpha_k},$$

where $\nu = \overline{0, p_k}$, $k = 1, 2, \ldots$. We use the Leibniz formula:

$$a_\nu(\alpha_k) = \sum_{j=0}^{\nu} C_\nu^j \frac{1}{(1-|\alpha_k|^2)^{p+p_k+1}} \left((1-\overline{\alpha_k}z)^{p+p_k+1}\right)^{(\nu-j)} \left(\frac{1}{\pi_{p,k}(z,\alpha_k)h(z)}\right)^{(j)}\Bigg|_{z=\alpha_k}.$$

We find an upper bound of $|a_\nu(\alpha_k)|$ for all $\nu = \overline{0, p+p_k}$, $k = 1, 2, \ldots$:

$$|a_\nu(\alpha_k)| \leq \sum_{j=0}^{\nu} C_\nu^j \frac{|\alpha_k^{\nu-j}|(p+p_k+1)\cdot\ldots\cdot(p+p_k+1-\nu+j+1)}{(1-|\alpha_k|^2)^{p+p_k+1}} \times$$

$$\times(1-|\alpha_k|^2)^{p+p_k+1-\nu+j+1}\left|\left(\frac{1}{\pi_{p,k}(z,\alpha_k)h(z)}\right)^{(j)}\right|_{z=\alpha_k} \leq$$

$$\leq \sum_{j=0}^{\nu} C_\nu^j \cdot c(\nu)(1-|\alpha_k|^2)^{-\nu+j} \cdot \left|\left(\frac{1}{\pi_{p,k}(z,\alpha_k)h(z)}\right)^{(j)}\right|_{z=\alpha_k}.$$

Now we use Cauchy's integral formula in a disc $K_\eta(\alpha_k)$:

$$\left|\left(\frac{1}{\pi_{p,k}(z,\alpha_k)h(z)}\right)^{(j)}\right|_{z=\alpha_k} = \left|\frac{j!}{2\pi i}\int\limits_{\partial K_\eta(\alpha_k)} \frac{\pi_{p,k}^{-1}(t,\alpha_k)h^{-1}(t)}{(t-\alpha_k)^{j+1}}dt\right| \le \frac{j!}{2\pi}\int\limits_{\partial K_\eta(\alpha_k)} \frac{|\pi_{p,k}^{-1}(t,\alpha_k)|\cdot|h^{-1}(t)|}{|t-\alpha_k|^{j+1}}dt,\ k\ge 1.$$

Applying estimates from Lemmas 2 and 4, we have:

$$\left|\left(\frac{1}{\pi_{p,k}(z,\alpha_k)h(z)}\right)^{(j)}\right|_{z=\alpha_k} \le \frac{j!}{2\pi}\exp\frac{-\mu_0(k)+\widetilde{\varepsilon}(k)}{(1-|\alpha_k|)^{1/q}}\int\limits_{\partial K_\eta(\alpha_k)}\frac{1}{|t-\alpha_k|^{j+1}}dt =$$
$$= \frac{j!}{A^j}\exp\frac{-\mu_0(k)+\widetilde{\varepsilon}(k)+j\eta(k)}{(1-|\alpha_k|)^{1/q}},\ k\ge 1.$$

We continue the estimate of $|a_\nu(\alpha_k)|$:

$$|a_\nu(\alpha_k)| \le c(\nu)\exp\left\{\ln(1-|\alpha_k|^2)^{-\nu+j}\right\}\cdot\exp\left\{\frac{-\mu_0(k)+\widetilde{\varepsilon}(k)+\nu\eta(k)}{(1-|\alpha_k|)^{1/q}}\right\} =$$
$$= c(\nu)\exp\left\{\frac{-\mu_0(k)+\widetilde{\varepsilon}(k)+\nu\eta(k)}{(1-|\alpha_k|)^{1/q}} + o\left((1-|\alpha_k|)^{-1/q}\right)\right\}.$$

Choosing the sequences $\eta(k)$, $\mu_1(k)$ so that $-\mu_0(k)+\widetilde{\varepsilon}(k)+\nu\eta(k)<0$, we obtain the required estimate. Lemma 5 is proved. □

Consider the polynomials

$$q_k(z) = \sum_{\nu=0}^{p_k-s_k} a_\nu(\alpha_k)(z-\alpha_k)^\nu,\ k=1,2,\ldots.$$

Now we define a system of analytic functions in D:

$$\widetilde{\Omega}_k(z) = \frac{(z-\alpha_k)^{s_k-1}q_k(z)}{(s_k-1)!\tau_k(z)}. \tag{14}$$

It obvious that these functions can be written in the form:

$$\widetilde{\Omega}_k(z) = h(z)\frac{\pi_{p,k}(z,\alpha_k)}{(s_k-1)!}\cdot\left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha}_k z}\right)^{p+p_k+1}\cdot\sum_{\nu=0}^{p_k-s_k} a_\nu(\alpha_k)(z-\alpha_k)^{\nu+s_k-1},$$

$k=1,2,\ldots$, where $p>\frac{1}{q}-1$.

We note that the method of constructing such system of functions was first proposed by M. Djrbashian in [5].

The following assertion holds:

Lemma 6. *Functions of the system* (14) *have the following interpolating properties:*

$$\widetilde{\Omega}_k^{(r)}(\alpha_k) = \begin{cases}1,\ r=s_k-1;\\ 0,\ r\ne s_k-1,\ 0\le r\le p_k-1.\end{cases}$$

Indeed, this follows immediately from the expansion:

$$\widetilde{\Omega}_k(z) = \frac{(z-\alpha_k)^{s_k-1}}{(s_k-1)!} - \left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha_k}z}\right)^{p+p_k+1}\cdot\frac{\pi_{p,k}(z,\alpha_k)h(z)}{(s_k-1)!}\times\sum_{\nu=p_k-s_k+1}^{+\infty} a_\nu(\alpha_k)(z-\alpha_k)^{\nu+s_k-1} =$$
$$= \frac{(z-\alpha_k)^{s_k-1}}{(s_k-1)!} - \lambda(z),$$

and $\lambda(z)$ has zero of multiplicity p_k in the point $z=\alpha_k$.

3. Proof of main result

We prove Theorem 2.

Proof. Suppose that the interpolation nodes satisfy the following conditions: $\{\alpha_k\}_1^\infty \subset \Gamma_\delta(\theta)$ for a certain $0 < \delta < 1$ and $\{\alpha_k\}_1^\infty \in \widetilde{\Delta}$. For any sequence $\{w_k\} \in \tilde{l}^q$ we construct interpolation function $f(z)$ in the following way:

$$f(z) = \sum_{k=1}^{+\infty} w_k \widetilde{\Omega}_k(z) \frac{h(z)}{h(\alpha_k)}, \; z \in D, \tag{15}$$

where $h = h_k(z)$ defined by (8).

Applying Lemma 6, we get: $f^{(s_k-1)}(\alpha_k) = w_k, k = 1, 2, \ldots$.

Now we prove that the function $f(z)$ belongs to the class Π_q. Since $\{\alpha_k\}_1^\infty \subset \Gamma_\delta(\theta)$ and all members of the sequence $\{w_k\}_1^\infty$ satisfy condition (6), then by Lemma 1 we have for all $k = 1, 2, \ldots$:

$$|f(z)| \le \sum_{\alpha_k \in \Gamma_\delta(\theta)} \left| \frac{w_k}{h(\alpha_k)} \right| \cdot |\widetilde{\Omega}_k(z)| \cdot |h(z)| \le$$

$$\le c_0 \sum_{\alpha_k \in \Gamma_\delta(\theta)} \exp \frac{\mu_1(k) - \mu_0(k)}{(1 - |\alpha_k|)^{\frac{1}{q}}} \cdot |\widetilde{\Omega}_k(z)| \cdot |h(z)| \le$$

$$\le c_0 \sum_{\alpha_k \in \Gamma_\delta(\theta)} |\widetilde{\Omega}_k(z)| \cdot |h(z)|.$$

Here we took into account that

$$\exp \frac{\mu_1(k) - \mu_0(k)}{(1 - |\alpha_k|)^{\frac{1}{q}}} \le 1.$$

Thus we have:

$$|f(z)| \le c_0 \sum_{k=1}^{+\infty} |\widetilde{\Omega}_k(z)| \cdot |h(z)|.$$

We obtain an upper estimate on $\widetilde{\Omega}_k(z)$ for all $k = 1, 2, \ldots$. Recall that

$$|\widetilde{\Omega}_k(z)| = |h(z)| \frac{|\pi_{p,k}(z, \alpha_k)|}{(s_k - 1)!} \cdot \left(\frac{1 - |\alpha_k|^2}{|1 - \overline{\alpha}_k z|} \right)^{p+p_k+1} \times \left| \sum_{\nu=0}^{p_k - s_k} a_\nu(\alpha_k)(z - \alpha_k)^{\nu+s_k-1} \right|.$$

We fix a number $k = k_0$. Taking into account the well-known estimate of M. M. Djrbashian's infinite product (see, for example, [17]), we get:

$$\ln^+ |\pi_{p,k}(z, \alpha_j)| \le c_p \sum_{j=1}^{+\infty} \left(\frac{1 - |\alpha_j|^2}{|1 - \overline{\alpha}_j z|} \right)^{p+1}.$$

Therefore for all $k \ge k_0$ the following estimate holds:

$$|f(z)| \le c_0 \sum_{k=1}^{+\infty} |\widetilde{\Omega}_k(z)| \cdot |h_{k_0}(z)| \le$$

$$\le c_0 |h_{k_0}(z)|^2 \exp \left\{ c_p \sum_{j=1}^{+\infty} \left(\frac{1 - |\alpha_j|^2}{|1 - \overline{\alpha}_j z|} \right)^{p+1} \right\} \times \left[\sum_{k=1}^{+\infty} \left(\frac{1 - |\alpha_k|^2}{|1 - \overline{\alpha}_k z|} \right)^{p+p_k+1} \cdot \left| \sum_{\nu=0}^{p_k - s_k} a_\nu(\alpha_k)(z - \alpha_k)^{\nu+s_k-1} \right| \right].$$

Notice that

$$\left(\frac{1-|\alpha_k|^2}{|1-\overline{\alpha_k}z|}\right)^{p+p_k+1}|z-\alpha_k|^{s_k-1+\nu}\leq\left(\frac{1-|\alpha_k|^2}{|1-\overline{\alpha_k}z|}\right)^{p+1}.$$

Therefore we have

$$|f(z)|\leq C_0|h_{k_0}(z)|^2\exp\left\{c_p\sum_{j=1}^{+\infty}\left(\frac{1-|\alpha_j|}{|1-\overline{\alpha}_jz|}\right)^{p+1}\right\}\cdot\sum_{k=1}^{+\infty}\left(\frac{1-|\alpha_k|^2}{|1-\overline{\alpha_k}z|}\right)^{p+1},$$

whence we conclude:

$$|f(z)|\leq C_0|h_{k_0}(z)|^2\left[\exp\left\{c_p\sum_{j=1}^{+\infty}\left(\frac{1-|\alpha_j|}{|1-\overline{\alpha}_jz|}\right)^{p+1}\right\}\right]^2. \tag{16}$$

As stated by F. A. Shamoyan in the recent paper [18] (see the proof of sufficiency in Theorem 3), if condition (4) is met, then Djrbashian's product π_p with zeros $\{\alpha_k\}_1^\infty$ located in the Stolz angles belongs to the class Π_q for all $p>\frac{1}{q}-1$. Therefore the majorant in the inequality (16) belongs to the class Π_q and hence the function f belongs to the class Π_q for all $0<q<1$.

Theorem 2 is proved. □

Remark 3. *Notice that we managed to avoid the Blaschke condition* $\sum_{k=1}^{+\infty}(1-|\alpha_k|)<+\infty$ *and replace it with* (4) *in Theorem 2; at the same time we replace the Carleson condition with the condition of weak separation* (5).

The converse statement in the class Π_q $(q>1)$ *was established in [8] for the case of simple nodes. The necessity of the condition "interpolation nodes contain in the Stolz angles" for the case* $q=1$ *is proved by Naftalevic in [9]. The question of the validity of converse Theorem 2 is still open. Apparently, the proof will be based on the factorization of functions from the class* Π_q. *Shamoyan's result introduced in Theorem 3 and Remark 1 give hope that sufficient conditions for interpolation are also necessary. Here we can trace some parallel with the results for the Nevanlinna-type classes (see [2]).*

The author thanks for Professor F. A. Shamoyan for carefully reading of the manuscript and helpful comments.

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