



## New Directions in $L^B$ -valued General Fuzzy Automata: A Topological View

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**Abstract.** In the current study, we define the different L-valued operators on  $L^B$ -valued general fuzzy automata or simply  $L^B$ -valued GFA, where B is considered as a complete infinitely distributive lattice of propositions about the GFA. In particular, this study demonstrates that the L-valued successor and predecessor operators induce L-valued co-topologies while the L-valued residuated and approximation operators induce L-valued topologies on the state set of given  $L^B$ -valued GFA. Further, we show that the continuity and separation properties of such  $L^B$ -valued general fuzzy automaton can be examined in terms of these topologies. Moreover, we study and explicate the  $L^B$ -valued GFA structure space and  $L^B$ -valued GFA homotopy in more details.

### 1. Introduction

The concept of ‘fuzzy’ together with a number of some other notions in mathematics and other areas were fuzzified by Zadeh [28] in 1965. Within this real, among the first investigations was the concept of fuzzy automaton suggested by Wee [26] and Santos [17]. In their research, they dealt with the notions such as vagueness and imprecision which were often appeared in the investigations on natural languages. After a study by Bavel [7] on algebraic automata theory, the connection between topology and an automaton was initiated [20, 21]. There, it was shown that several known topological concepts and ideas can often be employed in automata theory in order to obtain certain results therein which is related, for the most part, to their connectivity and separation properties. From the application point of view, fuzzy automata propose a useful practical situation for ambiguous and confusing calculations and have highlighted their significance in solving meaningful problems with respect to learning systems, pattern recognition and data base theory [6, 12, 13]. Moreover, a study by Malik et al. [11] to a large extent contributed to the algebraic examination of fuzzy automata and fuzzy languages in which simpler notions of a fuzzy finite state machine (which is almost identical to a fuzzy automata) were established.

The concept of L-valued automata theory based on quantum logic is introduced in Ying [27]. The quantum logic can be understood as a logic whose truth value set is an orthomodular lattice, and an element of orthomodular lattice is assigned to each transition of an automaton ( Ying [27]). In Qiu [12], it

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has been shown that the concepts of L-valued source and L-valued successor operators, associated with L-valued automata, induced L-valued topologies on the state-set of an L-valued automaton. The relationship between L-valued topologies depends on the distributivity of the associated lattice. In another study, Tiwari and his coworkers introduced the  $L^M$ -valued automata theory from algebraic and topological point of view, where L is a residuated lattice and M is a completely distributive lattice. Such automata may be assumed to be more general than that existing in literature in the sense that for M as one element lattice and  $L = [0, 1]$ , it turns out to be fuzzy automaton studied in [22, 23]; while for M as one element lattice and residuated lattice L, it turns out to be fuzzy automaton studied in [1, 3, 22, 23].

Doostfatemeleh and Kremer [10] used an extension of the notion of fuzzy automata and suggested the concept of general fuzzy automata. Their key impetus in introducing the concept general fuzzy automata was the insufficiency of the current literature to deal with the applications which depend on fuzzy automata as a modeling tool, allocating membership values to active states of a fuzzy automaton. In all types of conventional automata, a zero-weight transition means no transition, while in our approach applied for general fuzzy automata; a zero-weight transition does not necessarily entail no transition. That is why we will use  $[0,1]$  as the fuzzy interval.

Moreover, the basis of some ordered algebras (e.g. BL-algebra, MV algebra, and BCK-algebra) is derived from the residuated lattice. Therefore, when we consider the residuated lattice as an ordered algebra, it bears a strong structural similarity to  $[0,1]$ . Hence, working with that is considered as both a generalization of the concept of fuzzy set and a connection between algebraic-logic and fuzzy automaton. Actually, very little has been found in literature showing a suitable solution for multi-membership problem. In essence, multi-membership is an issue which is inherent to the fuzzy-finite-state automata (FFA), and it occurs due to its fuzzy nature which is showing up under any condition. Accordingly, the problem should be resolved appropriately, using a system to fulfill its necessary requirements. The most important reasons that emphasize the necessity of multi-membership resolution can be if the multi-membership active state is final, then there will be an obvious need for a single membership value since a final state is applied to produce a crisp output for the related system. On the other hand, even if the multi-membership active state has not been final, there should be a need to allocate a single membership value to some intermediate or non-final states during the usual operation of a FFA. Moreover, the membership values of successors can be calculated for each membership value of the state but this will result in an unnecessary blow up, making it very difficult to trace the continual operation of FFA. In this respect, it seems general fuzzy automata system can appropriately resolve the multi-membership problems in an effective way.

A number of investigations have also been conducted on the development of fuzzy automata theory by Zahedi and Abolpour and some other researchers [1–4, 14–16, 18, 19, 22–25]. In the present study,  $L^B$ -valued general fuzzy automata theory is scrutinized from algebraic and topological point of view, where L stands for residuated lattice and  $\mathbf{B}$  is a set of propositions about the general fuzzy automaton, in which its underlying structure is a complete infinitely distributive lattice. In addition,  $L^B$ -valued GFA structure space and  $L^B$ -valued GFA homotopy are studied and explicated in more details.

## 2. Preliminaries

In this section, some concepts which are significantly related to  $L^B$ -valued general fuzzy automaton, complete lattice, residuated lattices and fuzzy topological spaces are introduced and explained in details.

A fuzzy set  $\mu_Q$  defined on a set Q (discrete or continuous), is a function mapping each element of Q to a unique element of the interval  $[0,1]$ .

$$\mu_Q : Q \rightarrow [0, 1].$$

Then, the fuzzy power set of Q denoted as  $\tilde{P}(Q)$ , is the set of all fuzzy subsets  $\mu_Q$ , which can be defined on the set Q.

$$\tilde{P}(Q) = \{\mu_Q | \mu_Q : Q \rightarrow [0, 1]\}$$

**Definition 2.1.** ([10]) A general fuzzy automaton (GFA) is considered as:

$$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2),$$

where (i)  $Q$  is a finite set of states,  $Q = \{q_1, q_2, \dots, q_n\}$ , (ii)  $\Sigma$  is a finite set of input symbols,  $\Sigma = \{a_1, a_2, \dots, a_m\}$ , (iii)  $\tilde{R}$  is the set of fuzzy start states,  $\tilde{R} \subseteq \tilde{P}(Q)$ , (iv)  $Z$  is a finite set of output symbols,  $Z = \{b_1, b_2, \dots, b_k\}$ , (v)  $\omega : Q \rightarrow Z$  is the output function, (vi)  $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \rightarrow [0, 1]$  is the augmented transition function. (vii) Function  $F_1 : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is called membership assignment function. Function  $F_1(\mu, \delta)$ , as it is seen, is motivated by two parameters  $\mu$  and  $\delta$ , where  $\mu$  is the membership value of a predecessor and  $\delta$  is the weight of a transition.

With this definition, the process that happens upon the transition from state  $q_i$  to  $q_j$  an input  $a_k$  is characterized by:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

It means that membership value (mv) of the state  $q_j$  at time  $t + 1$  is calculated by function  $F_1$  utilizing both the membership value of  $q_i$  at time  $t$  and the weight of the transition.

There have been many options for the function  $F_1(\mu, \delta)$ . For instance, it can be  $\max\{\mu, \delta\}$ ,  $\min\{\mu, \delta\}$ ,  $\frac{\mu + \delta}{2}$ , or any other pertinent mathematical functions.

As it can be observed in the above mentioned formulas, associated with each fuzzy transition, there exists a membership value (mv) in unit interval  $[0, 1]$ . We identify this membership value as the weight of the transition. The transition from state  $q_i$  (current state) to state  $q_j$  (next state) upon input  $a_k$  is designated as  $\delta(q_i, a_k, q_j)$ . Hereafter, we categorize this notation to refer both to a transition and its weight. Whenever  $\delta(q_i, a_k, q_j)$  is used as a value, it refers to the weight of the transition. Otherwise, it identifies the transition itself. The set of all transitions of a general fuzzy automaton  $\tilde{F}$ , is denoted as  $\Delta_{\tilde{F}}$ . However, whenever it is recognized we remove the subscript, and simply write  $\Delta$ .

(viii)  $F_2 : [0, 1]^* \rightarrow [0, 1]$ , is called multi-membership resolution function. The multi-membership resolution function determines the multi-membership active states and allocates a single membership value to them.

We let  $Q_{act}(t_i)$  be the set of all active state at time  $t_i$ ,  $\forall_i \geq 0$ . We have  $Q_{act}(t_0) = \tilde{R}$  and  $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) | \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}$ ,  $\forall_i \geq 1$ . Since  $Q_{act}(t_i)$  is a fuzzy set, to demonstrate that a state  $q$  belongs to  $Q_{act}(t_i)$  and  $T$  is a subset of  $Q_{act}(t_i)$ , we should write:  $q \in \text{Domain}(Q_{act}(t_i))$  and  $T \subseteq \text{Domain}(Q_{act}(t_i))$ ; henceforth, we simply specify them by:  $q \in Q_{act}(t_i)$  and  $T \subseteq Q_{act}(t_i)$ .

**Definition 2.2.** ([10]) The successor set of a state  $q_m$  on input symbol  $a_k$  denoted as  $Q_{succ}(q_m, a_k)$ , is the set of all states  $q_j$  which will be reached via transitions  $\delta(q_m, a_k, q_j)$ .

$$Q_{succ}(q_m, a_k) = \{q_j | \delta(q_m, a_k, q_j) \in \Delta\}$$

Similarly, we can define the predecessor set of a state set as follows:

$$Q_{pred}(q_m, a_k) = \{q_j | \delta(q_j, a_k, q_m) \in \Delta\}.$$

**Definition 2.3.** ([9]) A complete lattice is a lattice  $\mathbf{L} = (L, \leq, \wedge, \vee, 0, 1)$  in which arbitrary suprema and infima exist. A lattice  $L$  is called infinitely distributive if  $\wedge$  distributes over arbitrary joins, i.e.,  $\forall a \in L, \{b_j : j \in J\} \subseteq L$ ,

$$a \wedge \{\vee(b_j : j \in J)\} = \vee\{a \wedge b_j : j \in J\}.$$

**Definition 2.4.** ([8]) A residuated lattic is an algebra  $\mathbf{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$  such that:

- (i)  $(L, \wedge, \vee, 0, 1)$  is a lattice with the least element 0 and the greatest element 1,
- (ii)  $(L, \otimes, 1)$  is a commutative monoid with unit 1, and
- (iii)  $\otimes$  and  $\rightarrow$  form an adjoint pair, i.e., for all  $x, y, z \in L, x \otimes y \leq z \Leftrightarrow x \leq y \rightarrow z$ .

If, in addition  $(L, \wedge, \vee, 0, 1)$  is a complete lattice, then  $\mathbf{L}$  is called a complete residuated lattice.

The precomplement on  $L$  is the mapping  $\neg : L \rightarrow L$  such that  $\neg x = x \rightarrow 0, \forall x \in L$ . Some of the basic properties of complete residuated lattices, which we use, are as follows:

- (i)  $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$ ;
- (ii)  $1 \rightarrow a = a$ ;

- (iii)  $(a \rightarrow b) \otimes (b \rightarrow c) \leq a \rightarrow c$ ;
- (iv)  $a \otimes b \rightarrow c = a \rightarrow (b \rightarrow c)$ ;
- (v)  $a \otimes (\bigvee_{i \in J} b_i) = \bigvee_{i \in J} (a \otimes b_i)$ .

The concepts  $L$ -valued relations,  $L$ -valued topologies and  $L^B$ -valued general fuzzy automata have the membership values in a complete residuated lattice. For example, an  $L$ -valued set of a nonempty set  $X$  is a function from  $X$  to  $L$ , an  $L^M$ -valued set of a nonempty set  $X$  is a map from  $X$  to  $L^M$  and an  $L$ -valued relation on  $X$  is a function from  $X \times X$  to  $L$ . Throughout this study,  $L^X$  denotes the family of all  $L$ -valued subsets of  $X$  and  $\mathbf{a}$  denotes the  $a$ -valued constant  $L$ -valued subset of  $X$ .

Let  $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, w, F_1, F_2)$  be a general fuzzy automaton. If we fix an input  $a_k \in \Sigma$  at time  $t_i$  the proposition  $\alpha|_{a_k}$  can be computed by  $\mu^{t_i}(q_i)$  if the general fuzzy automaton  $\tilde{F}$  is in the state  $q_i$  at time  $t_i$  otherwise  $\alpha|_{a_k}$  is 0 if  $\tilde{F}$  is not in the active state  $q_i$ . Accordingly, for each state  $q_i \in Q$  we can assess the truth value of  $\alpha|_{a_k}$ , it is indicated by  $\alpha|_{a_k}(q_i)$ . As explained above  $\alpha|_{a_k}(q_i) \in [0, 1]$ . This section aims to derive the logic  $B$  which is a set of propositions about the general fuzzy automaton  $\tilde{F}$  formulated by the observer and constructing a complete infinitely distributive lattice  $\mathbf{B} = (B, \leq, \wedge, \vee, 0, 1)$ . We can establish the order  $\leq$  on  $B$  as follows:

For  $\alpha, \beta \in B, \alpha \leq \beta$  if and only if  $\alpha(q_i) \leq \beta(q_i)$  for all  $q_i \in Q$ . One can instantly check that the contradiction, i.e., the proposition with constant truth value 0, is the least element and the tautology, i.e., the proposition with constant truth value 1 is the greatest component of the  $\mathbf{B}$ . Note that any component  $i$ th of 1 is the maximum membership values of active states at time  $t_i$ , for any  $i \geq 0$ .

Let  $L = ([0, 1], \leq, \wedge, \vee, \otimes, \rightarrow)$  be a residuated lattice and  $\mathbf{B} = (B, \leq, \wedge, \vee, 0, 1)$  be a complete infinitely distributive lattice of propositions about the general fuzzy automaton  $\tilde{F}$ .

The most examined and applied structures of truth values, defined on the real unit interval  $[0,1]$  with  $x \wedge y = \min(x, y)$  and  $x \vee y = \max(x, y)$ , belong to the Lukasiewicz structure ( $x \otimes y = \max(x + y - 1, 0), x \rightarrow y = \min(1 - x + y, 1)$ ), the product structure ( $x \otimes y = x.y, x \rightarrow y = 1$  if  $x \leq y$  and  $= \frac{y}{x}$  otherwise) and the Godel structure ( $x \otimes y = \min(x, y), x \rightarrow y = 1$  if  $x \leq y$  and  $= y$  otherwise). In the present study, we use the Godel structure.

We define  $L^B$ -valued subset of  $Q \times \Sigma \times Q$ , i.e., a map  $\delta : Q \times \Sigma \times Q \rightarrow L^B$ . The range set  $L^B$  allows to interpret  $L^B$  as a map assigning each  $(q, a_k, p)$  to  $\delta(q, a_k, p) : B \rightarrow L$ . This interpretation of transition map  $\delta$  allows to represent it as the family  $\{\delta^\alpha : \alpha \in B\}$  of  $L$ -valued sets  $\delta^\alpha \in L^{Q \times \Sigma \times Q}$  of  $Q \times \Sigma \times Q$  ordered by the elements of

$$B, \text{ where the } L\text{-valued sets } \delta^\alpha \text{ are defined by } \delta^\alpha(q, a_k, p) = \delta(q, a_k, p)(\alpha) = \begin{cases} 1 & \text{if } q = p \\ \alpha(q) \vee \alpha(p) & \text{otherwise.} \end{cases}$$

**Definition 2.5.** ([5]) An  $L^B$ -valued general fuzzy automaton is a 8-tuple  $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ , where  $\tilde{\delta}$  is an  $L^B$ -valued subset of  $(Q \times L) \times \Sigma \times Q$ , i.e., a map  $\tilde{\delta} : (Q \times L) \times \Sigma \times Q \rightarrow L^B$  such that:

$$\tilde{\delta}((q_i, \mu^{t_i}(q_i)), a_k, q_j)(\alpha) = F_1(\mu^{t_i}(q_i), \delta(q_i, a_k, q_j)(\alpha)).$$

Let  $\Sigma^*$  be a monoid generated by a nonempty set  $\Sigma$ . Define a map  $\tilde{\delta}^* : (Q \times L) \times \Sigma^* \times Q \rightarrow L^B$

$$\tilde{\delta}^*((q, \mu^t(q)), \wedge, p)(\alpha) = \tilde{\delta}((q, \mu^t(q)), \wedge, p)(\alpha) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{otherwise} \end{cases}$$

such that,  $\forall q, p \in Q, \forall u \in \Sigma^*, \forall x \in \Sigma$  and  $\forall \alpha \in B$

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), ux, p)(\alpha) = \bigvee \{ \tilde{\delta}^*((q, \mu^{t_i}(q)), u, q')(\alpha) \otimes \tilde{\delta}((q', \mu^{t_j}(q')), x, p)(\alpha) \mid q' \in Q_{pred}(p, x) \}.$$

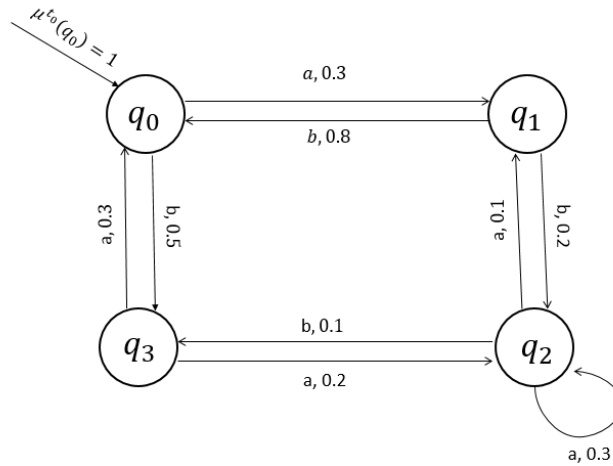
**Definition 2.6.** ([5]) Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $b \in B$ . The  $L^B$ -valued successor and the  $L^B$ -valued predecessor of  $b$  are as follows:

$$S(b)(q_m)(\alpha) = \bigvee \{ b(q_j) \otimes \tilde{\delta}((q_m, \mu^{t_i}(q_m)), u, q_j)(\alpha) \mid q_j \in Q_{succ}(q_m, u), u \in \Sigma \}, \text{ and}$$

$$P(b)(q_m)(\alpha) = \bigvee \{ b(q_j) \otimes \tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_m)(\alpha) \mid q_j \in Q_{pred}(q_m, u), u \in \Sigma \}, \text{ for } q_m \in Q_{act}(t_i), \alpha \in B.$$

**Remark 2.7.** ([5]) It can be seen that the  $L^B$ -valued successor and  $L^B$ -valued predecessor are the maps  $S, P : B \rightarrow (L^B)^Q$ . For each  $\alpha \in B$ , these can be represented as  $L$ -valued successor and predecessor operators  $S^\alpha, P^\alpha : B \rightarrow L^Q$ , where  $S^\alpha(b)(q_m) = S(b)(q_m)(\alpha)$ , and  $P^\alpha(b)(q_m) = P(b)(q_m)(\alpha)$  for  $b \in B$  and  $q_m \in Q$ .

**Example 2.8.** Consider the GFA in Figure 1, it is specified as  $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ , where  $Q = \{q_0, q_1, q_2, q_3\}$  is the set of states,  $\Sigma = \{a, b\}$  is the set of input symbols,  $\tilde{R} = \{(q_0, 1)\}$ ,  $Z = \emptyset$  and  $w$  is not applicable.



**Fig. 1** The GFA of Example 2.8

We check operation of the GFA in Example 2.8 upon input "ab<sup>2</sup>ab".

If we choose  $F_1(\mu, \delta) = \delta, F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^n (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$ , then we have:

$$\mu^{t_0}(q_0) = 1,$$

$$\mu^{t_1}(q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = \delta(q_0, a, q_1) = 0.3,$$

$$\mu^{t_2}(q_0) = F_1(\mu^{t_1}(q_1), \delta(q_1, b, q_0)) = \delta(q_1, b, q_0) = 0.8,$$

$$\mu^{t_2}(q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, b, q_2)) = \delta(q_1, b, q_2) = 0.2,$$

$$\mu^{t_3}(q_3) = F_1(\mu^{t_2}(q_0), \delta(q_0, b, q_3)) \wedge F_1(\mu^{t_2}(q_2), \delta(q_2, b, q_3))$$

$$= \delta(q_0, b, q_3) \wedge \delta(q_2, b, q_3) = 0.5 \wedge 0.1 = 0.1,$$

$$\mu^{t_4}(q_0) = F_1(\mu^{t_3}(q_3), \delta(q_3, a, q_0)) = \delta(q_3, a, q_0) = 0.3,$$

$$\mu^{t_4}(q_2) = F_1(\mu^{t_3}(q_3), \delta(q_3, a, q_2)) = \delta(q_3, a, q_2) = 0.2,$$

$$\mu^{t_5}(q_3) = F_1(\mu^{t_4}(q_0), \delta(q_0, b, q_3)) \wedge F_1(\mu^{t_4}(q_2), \delta(q_2, b, q_3))$$

$$= \delta(q_0, b, q_3) \wedge \delta(q_2, b, q_3) = 0.5 \wedge 0.1 = 0.1.$$

Table 1: Active states and their membership values (mv) at different times in Example 2.8

time	$t_0$	$t_1$	$t_2$	$t_3$	$t_4$	$t_5$
input	$\wedge$	a	b	b	a	b
$Q_{act}(t_i)$	$q_0$	$q_1$	$q_0 q_2$	$q_3$	$q_0 q_2$	$q_3$
mv	1	0.3	0.8 0.2	0.1	0.3 0.2	0.1

The set  $B = \{0, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 1\}$  of possible propositions about the general fuzzy automaton  $\tilde{F}$  is as follows:

- 0 means that the GFA is not in active states of Q,
- $-\alpha_0$  means that the GFA is in active states at time  $t_0$ ,
- $-\alpha_1$  means that the GFA is in active states at time  $t_1$ ,
- $-\alpha_2$  means that the GFA is in active states at time  $t_2$ ,
- $-\alpha_3$  means that the GFA is in active states at time  $t_3$ ,
- $-\alpha_4$  means that the GFA is in active states at time  $t_4$ ,
- $-\alpha_5$  means that the GFA is in active states at time  $t_5$ ,
- 1 means that the GFA is in at least one active state at time  $t_i$  for any  $i \geq 0$ .

We may consider  $\mathbf{B}$  with the algebra  $[0, 1]^Q$  as follows:

$$0 = (0, 0, 0, 0), \alpha_0 = (1, 0, 0, 0), \alpha_1 = (0, 0.3, 0, 0), \alpha_2 = (0.8, 0, 0.2, 0), \alpha_3 = (0, 0, 0, 0.1), \alpha_4 = (0.3, 0, 0.2, 0), \alpha_5 = (0, 0, 0, 0.1), 1 = (1, 0.3, 0.2, 0.1).$$

Here,  $\alpha(q_i)$  is the maximum membership values of active states at time  $t_i$  for any  $i \geq 0$ .

By the definition of  $L^B$ -valued general fuzzy automaton we have:

$$\delta(q_0, a, q_1)(\alpha_1) = \alpha_1(q_0) \vee \alpha_1(q_1) = 0 \vee 0.3 = 0.3,$$

$$\delta(q_2, a, q_2)(\alpha_1) = 1,$$

$$\delta(q_2, b, q_3)(\alpha_3) = \alpha_3(q_2) \vee \alpha_3(q_3) = 0 \vee 0.1 = 0.1,$$

$$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), ab, q_2)(\alpha_1) = \tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1)(\alpha_1) \otimes \tilde{\delta}((q_1, \mu^{t_1}(q_1)), b, q_2)(\alpha_1)$$

$$= F_1((\mu^{t_0}(q_0), \delta(q_0, a, q_1)(\alpha_1)) \otimes F_1((\mu^{t_1}(q_1), \delta(q_1, b, q_2)(\alpha_1)))$$

$$= \delta(q_0, a, q_1)(\alpha_1) \otimes \delta(q_1, b, q_2)(\alpha_1)$$

$$= [\alpha_1(q_0) \vee \alpha_1(q_1)] \otimes [\alpha_1(q_1) \vee \alpha_1(q_2)]$$

$$= [0 \vee 0.3] \otimes [0.3 \vee 0] = 0.3 \otimes 0.3 = 0.3,$$

$$\tilde{\delta}^*((q_1, \mu^{t_1}(q_1)), b^2a, q_0)(\alpha_2) = \tilde{\delta}^*((q_1, \mu^{t_1}(q_1)), b^2, q_3)(\alpha_2) \otimes \tilde{\delta}((q_3, \mu^{t_3}(q_3)), a, q_0)$$

$$(\alpha_2) = [(\tilde{\delta}((q_1, \mu^{t_1}(q_1)), b, q_0)(\alpha_2) \otimes \tilde{\delta}((q_0, \mu^{t_2}(q_0)), b, q_3)(\alpha_2))$$

$$\vee (\tilde{\delta}((q_1, \mu^{t_1}(q_1)), b, q_2)(\alpha_2) \otimes \tilde{\delta}((q_2, \mu^{t_2}(q_2)), b, q_3)(\alpha_2))] \otimes \tilde{\delta}((q_3, \mu^{t_3}(q_3)), a, q_0)$$

$$(\alpha_2) = [(F_1(\mu^{t_1}(q_1), \delta(q_1, b, q_0)(\alpha_2)) \otimes F_1(\mu^{t_2}(q_0), \delta(q_0, b, q_3)(\alpha_2))) \vee (F_1(\mu^{t_1}(q_1), \delta(q_1, b, q_2)(\alpha_2)) \otimes F_1(\mu^{t_2}(q_2), \delta(q_2, b,$$

$$q_3)(\alpha_2))] \otimes F_1(\mu^{t_3}(q_3), \delta(q_3, a, q_0)(\alpha_2))$$

$$= [(\delta(q_1, b, q_0)(\alpha_2) \otimes \delta(q_0, b, q_3)(\alpha_2)) \vee (\delta(q_1, b, q_2)(\alpha_2) \otimes \delta(q_2, b, q_3)(\alpha_2))] \otimes \delta(q_3, a, q_0)(\alpha_2)$$

$$= [((\alpha_2(q_1) \vee \alpha_2(q_0)) \otimes ((\alpha_2(q_0) \vee \alpha_2(q_3)) \vee ((\alpha_2(q_1) \vee \alpha_2(q_2)) \otimes ((\alpha_2(q_2) \vee \alpha_2(q_3))))))] \otimes (\alpha_2(q_3) \vee \alpha_2(q_0))$$

$$= [((0 \vee 0.8) \otimes (0.8 \vee 0)) \vee ((0 \vee 0.2) \otimes (0.2 \vee 0))] \otimes (0 \vee 0.8)$$

$$= [(0.8 \otimes 0.8) \vee (0.2 \otimes 0.2)] \otimes 0.8 = 0.8 \otimes 0.8 = 0.8.$$

**Proposition 2.9.** ([5]) Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $S, P : B \rightarrow (L^B)^Q$  be the induced  $L^B$ -valued successor and predecessor operators, respectively. Then for all  $\lambda, \gamma, (\lambda_j : j \in J) \in B$  and  $\alpha \in B$ ,

(i)  $S(\mathbf{a}) = \mathbf{a}$  and  $P(\mathbf{a}) = \mathbf{a}, \forall \mathbf{a} \in L;$

(ii) if  $\lambda \leq \gamma$  then  $P(\lambda) \leq P(\gamma)$  and  $S(\lambda) \leq S(\gamma);$

(iii)  $\lambda \leq P(\lambda)$  and  $\lambda \leq S(\lambda);$

(iv)  $P(\cup \{\lambda_j : j \in J\}) = \cup \{P(\lambda_j : j \in J)\}$  and  $S(\cup \{\lambda_j : j \in J\}) = \cup \{S(\lambda_j : j \in J)\};$

(v)  $P(P(\lambda)) = P(\lambda)$  and  $S(S(\lambda)) = S(\lambda)$ ;

(vi)  $P(\cap \{\lambda_j : j \in J\}) \leq \cap \{P(\lambda_j : j \in J)\}$  and  $S(\cap \{\lambda_j : j \in J\}) \leq \cap \{S(\lambda_j : j \in J)\}$ ;

(vii)  $\alpha \leq \beta \Rightarrow P(\lambda)(q_m)(\alpha) \geq P(\lambda)(q_m)(\beta)$  and  $S(\lambda)(q_m)(\alpha) \geq S(\lambda)(q_m)(\beta)$ ,  
 $\forall q_m \in Q_{act}(t_i), \beta \in B$ .

**Definition 2.10.** ([5]) For an  $L^B$ -valued general fuzzy automaton  $\tilde{F}$ ,  $\alpha \in B$  is called an  $L^B$ -valued subproposition of  $\tilde{F}$  if  $P(\alpha) \leq \alpha$ . Further, this  $L^B$ -valued subproposition is called separated if  $P(\alpha^c) \leq \alpha^c$ .

**Proposition 2.11.** ([5]) For an  $L^B$ -valued subproposition  $\alpha$  of an  $L^B$ -valued GFA  $\tilde{F}$ , the following conditions are equivalent:

- (i)  $\alpha$  is separated;
- (ii)  $P(\alpha^c) = \alpha^c$ .

**Definition 2.12.** ([5]) Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . The  $L^B$ -valued residuated operator of  $\alpha$  is given by

$$R(\alpha)(q_m)(b) = \wedge \{ \tilde{\delta}((p, \mu^{t_i}(p)), u, q_m)(b) \rightarrow \alpha(p) \mid p \in Q_{pred}(q_m, u), u \in \Sigma \}$$

for  $q_m \in Q_{act}(t_i)$  and  $b \in B$ .

**Remark 2.13.** ([5]) It can be seen that  $L^B$ -valued residuated operator is a map  $R : B \rightarrow (L^B)^Q$ , which induces a family of  $L$ -valued residuated operators  $\{R^b : B \rightarrow L^Q \mid b \in B\}$ , where

$$R^b(\alpha)(q_m) = R(\alpha)(q_m)(b), \forall \alpha \in B, q_m \in Q_{act}(t_i).$$

**Proposition 2.14.** ([5]) Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $R : B \rightarrow (L^B)^Q$  be the induced  $L^B$ -valued residuated operator. Then for all  $\lambda, \gamma, (\lambda_j : j \in J) \in B$ ,

- (i)  $R(\mathbf{a}) = (\mathbf{a}), \forall \mathbf{a} \in L$ ;
- (ii) if  $\lambda \leq \gamma$  then  $R(\lambda) \leq R(\gamma)$ ;
- (iii)  $R(\lambda) \leq \lambda$ ;
- (iv)  $R(\cap \{\lambda_j \mid j \in J\}) = \cap \{R(\lambda_j \mid j \in J)\}$ ;
- (v)  $R(R(\lambda)) = R(\lambda)$ ;
- (vi)  $\cup \{R(\lambda_j \mid j \in J)\} \leq R(\cup \{\lambda_j \mid j \in J\})$ ;
- (vii)  $\alpha \leq \beta \Rightarrow R(\lambda)(q_m)(\alpha) \leq R(\lambda)(q_m)(\beta), \forall q_m \in Q_{act}(t_i), \alpha, \beta \in B$ .

**Proposition 2.15.** ([5]) Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . The  $L$ -valued residuated operator viewed as function  $R^\alpha : B \rightarrow L^Q$ , which sends each  $b \in B$  to  $R^\alpha(b)$ , is a Kuratowski  $L$ -valued interior operator on  $Q$ .

**Definition 2.16.** ([5]) Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton. Then  $\alpha \in B$  is called:

- (i) dynamic if  $S(\alpha) \leq P(\alpha)$  ;
- (ii) dynamically closed if there exists  $\beta \leq \alpha$  such that  $S(\beta) \leq P(\beta)$  and  $P(\beta) = \alpha$ ;
- (iii) primary if  $\alpha$  is a minimal dynamically closed subproposition of  $\tilde{F}$ .

**Definition 2.17.** ([5]) Let  $\mathbf{A} = (A, \leq, \wedge, \vee, 0, 1)$  be a complete infinitely distributive lattice of propositions about the general fuzzy automaton  $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$  and  $\mathbf{B} = (B, \leq, \wedge, \vee, 0, 1)$  be a complete infinitely distributive lattice of propositions about the general fuzzy automaton  $\tilde{F}' = (Q', \Sigma, \tilde{R}', Z, w', \tilde{\delta}', F_1, F_2)$ . A homomorphism from an  $L^A$ -valued GFA  $\tilde{F}$  to an  $L^B$ -valued GFA  $\tilde{F}'$  is a pair  $(f, g)$  of maps, where  $f : Q \rightarrow Q'$  and  $g : A \rightarrow B$  are functions such that:

$$(i) \tilde{\delta}'((f(q), \mu^t(f(q))), u, f(p))(g(\alpha)) \geq \tilde{\delta}((q, \mu^t(q)), u, p)(\alpha),$$

$$(ii) w(q) = z \Leftrightarrow w(f(q)) = z,$$

$$(iii) f(q_0) = q'_0,$$

for all  $\alpha \in A$  and  $p, q \in Q$ .

The pair  $(f, g)$  is called a strong homomorphism if for all  $\alpha \in A$

$$\tilde{\delta}'((f(q), \mu^t(f(q))), u, f(p))(g(\alpha)) = \vee \{ \tilde{\delta}((q, \mu^t(q)), u, p')(\alpha) \mid p' \in Q, f(p) = f(p') \}.$$

### 3. Main Results

In what follows the L-valued operators and their associated L-valued topologies are discussed. Moreover,  $L^B$ -valued GFA structure space and  $L^B$ -valued GFA homotopy are investigated. Further, the relationships between  $L^B$ -valued general fuzzy automaton and associated L-valued subpropositions are studied and explicated in more details.

#### 3.1. L-valued Operators and their Associated L-valued Topologies

In this subsection, some concepts as L-valued topologies/co-topologies induced by the  $L^B$ -valued successor/predecessor/residuated/approximation operator for fixed element of  $B$  are suggested and examined. In particular, it is shown that the L-valued successor and predecessor operators induce L-valued co-topologies and the L-valued residuated/approximation operator induces L-valued topology on the state set of given  $L^B$ -valued general fuzzy automaton. Further, it is also demonstrated that the induced L-valued topologies and co-topologies can be applied to characterize the algebraic concepts of an  $L^B$ -valued general fuzzy automaton.

**Proposition 3.1.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ .

(a) L-valued successor and the L-valued predecessor viewed as functions  $S^\alpha : B \rightarrow L^Q$  and  $P^\alpha : B \rightarrow L^Q$ , turns out to be Alexandroff Kuratowski L-valued closure operators on  $Q$ .

(b) These two operators induce two L-valued co-topologies on  $Q$  which we shall respectively denote as  $T^\alpha(Q)$  and  $T'^\alpha(Q)$ , where  $T^\alpha(Q) = \{b \in B \mid S^\alpha(b) = b\}$  and  $T'^\alpha(Q) = \{b \in B \mid P^\alpha(b) = b\}$ .

(c) The L-valued co-topologies  $T^\alpha(Q)$  and  $T'^\alpha(Q)$  are dual with in the sense that  $b \in B$  is  $T^\alpha(Q)$ -closed if and only if  $b^c$  is  $T'^\alpha(Q)$ -closed.

*Proof.* It follows from Proposition 2.9 and Definition 2.6.  $\square$

**Proposition 3.2.** The L-valued topology on  $Q$  given by the Alexandroff Kuratowski L-valued interior operator  $R^\alpha$  is precisely  $T^\alpha(Q)$ .

*Proof.* To prove this proposition, we need to show that  $R^\alpha(b) = b$  iff  $S^\alpha(b) = b$  for all  $b \in B$ . First, let  $S^\alpha(b) = b$ .

Then

$$\begin{aligned} S^\alpha(b) = b &\Rightarrow S(b)(q_i)(\alpha) = b(q_i) \\ &\Rightarrow \vee \{ b(q_j) \otimes \tilde{\delta}((q_i, \mu^{t_i}(q_i)), u, q_j)(\alpha) \mid q_j \in Q_{succ}(q_i, u), u \in \Sigma \} = b(q_i) \\ &\Rightarrow b(q_j) \otimes \tilde{\delta}((q_i, \mu^{t_i}(q_i)), u, q_j)(\alpha) \leq b(q_i), q_j \in Q_{succ}(q_i, u), u \in \Sigma \\ &\Rightarrow b(q_j) \leq \tilde{\delta}((q_i, \mu^{t_i}(q_i)), u, q_j)(\alpha) \rightarrow b(q_i), q_i \in Q_{pred}(q_j, u), u \in \Sigma \\ &\Rightarrow b(q_j) \leq \wedge \{ \tilde{\delta}((q_i, \mu^{t_i}(q_i)), u, q_j)(\alpha) \rightarrow b(q_i) \mid q_i \in Q_{pred}(q_j, u), u \in \Sigma \} \\ &\Rightarrow b(q_j) \leq R^\alpha(b)(q_j). \end{aligned}$$

Also, as  $R^\alpha(b)(q_j) \leq b(q_j)$ , we have  $R^\alpha(b) = b$ .

Conversely, let  $R^\alpha(b) = b$ . Then

$$\begin{aligned} R^\alpha(b) = b &\Rightarrow R(b)(q_j)(\alpha) = b(q_j) \\ &\Rightarrow \wedge \{ \tilde{\delta}((q_i, \mu^{t_i}(q_i)), u, q_j)(\alpha) \rightarrow b(q_i) \mid q_i \in Q_{pred}(q_j, u), u \in \Sigma \} = b(q_j) \end{aligned}$$



$$\begin{aligned} &\Rightarrow b(q_j) \leq \tilde{\delta}((q_i, \mu^{t_i}(q_i)), u, q_j)(\alpha) \rightarrow b(q_i), q_i \in Q_{pred}(q_j, u), u \in \Sigma \\ &\Rightarrow b(q_j) \otimes \tilde{\delta}((q_i, \mu^{t_i}(q_i)), u, q_j)(\alpha) \leq b(q_i), q_j \in Q_{succ}(q_i, u), u \in \Sigma \\ &\Rightarrow \vee \{b(q_j) \otimes \tilde{\delta}((q_i, \mu^{t_i}(q_i)), u, q_j)(\alpha) | q_j \in Q_{succ}(q_i, u), u \in \Sigma\} \leq b(q_i) \\ &\Rightarrow S^\alpha(b)(q_i) \leq b(q_i). \end{aligned}$$

Also, as  $b(q_i) \leq S^\alpha(b)(q_i)$ , whereby  $S^\alpha(b) = b$ . Hence,  $R^\alpha(b) = b$  iff  $S^\alpha(b) = b$ .  $\square$

**Proposition 3.3.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $b \in B$ . Then for fixed  $\alpha \in B$ ,  $b$  is an  $L^B$ -valued subproposition of  $B$  if and only if  $b$  is  $T^\alpha$ -closed.

*Proof.* The proof is straightforward from definition of subproposition and Proposition 3.1.  $\square$

**Proposition 3.4.** For fixed  $\alpha \in B$  and an  $L^B$ -valued subproposition  $b$  of an  $L^B$ -valued general fuzzy automaton  $\tilde{F}$ ,  $b$  is separated iff  $b$  is  $T^\alpha$ -closed.

*Proof.* It follows from Propositions 2.11 and 3.1.  $\square$

Before stating the next part, we introduce the following concept of an  $L$ -valued regular closed subset in an  $L$ -valued topology.

**Definition 3.5.** A closed  $L$ -valued subset in an  $L$ -valued topological space is called regular closed if it is equal to the closure of its interior.

The following is towards the  $L$ -valued topological characterization of the concept of dynamically closed  $L$ -valued subset in an  $L^B$ -valued general fuzzy automaton.

**Proposition 3.6.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton. Then a closed  $L$ -valued subset  $b \in B$  is dynamically closed if and only if  $b$  is regular closed.

*Proof.* Let  $b \in B$  be dynamically closed. Then there exists  $a \in B$ ,  $a \leq b$  such that for all  $\alpha \in B$ ,  $S^\alpha(a) \leq P^\alpha(a) = b$ . Now,  $S^\alpha(a)(q_j) \leq b(q_j) \Rightarrow S(a)(q_j)(\alpha) \leq b(q_j)$

$$\begin{aligned} &\Rightarrow \vee \{a(q_i) \otimes \tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_i)(\alpha) | q_i \in Q_{succ}(q_j, u), u \in \Sigma\} \leq b(q_j) \\ &\Rightarrow a(q_i) \otimes \tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_i)(\alpha) \leq b(q_j), q_i \in Q_{succ}(q_j, u), u \in \Sigma \\ &\Rightarrow a(q_i) \leq \tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_i)(\alpha) \rightarrow b(q_j), q_j \in Q_{pred}(q_i, u), u \in \Sigma \\ &\Rightarrow a(q_i) \leq \wedge \{\tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_i)(\alpha) \rightarrow b(q_j) | q_j \in Q_{pred}(q_i, u), u \in \Sigma\} \\ &\Rightarrow a(q_i) \leq R(b)(q_i)(\alpha) \\ &\Rightarrow a(q_i) \leq R^\alpha(b)(q_i) \\ &\Rightarrow P^\alpha(a)(q_i) \leq P^\alpha(R^\alpha(b))(q_i) \\ &\Rightarrow b(q_i) \leq P^\alpha(R^\alpha(b))(q_i). \end{aligned}$$

Again,  $R^\alpha(b) \leq b \Rightarrow P^\alpha(R^\alpha(b)) \leq P^\alpha(b) = b$ . Therefore  $P^\alpha(R^\alpha(b)) = b$ .

Conversely, let  $P^\alpha(R^\alpha(b)) = b$  and  $a = R^\alpha(b)$ . Then  $a \leq b$ . Now,

$$\begin{aligned} S^\alpha(R^\alpha(b))(q_j) &= \vee \{R^\alpha(b)(q_i) \otimes \tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_i)(\alpha) | q_i \in Q_{succ}(q_j, u), u \in \Sigma\} \\ &= \vee \{(\wedge \{\tilde{\delta}((q_k, \mu^{t_k}(q_k)), v, q_i)(\alpha) \rightarrow b(q_k) | q_k \in Q_{pred}(q_i, v), v \in \Sigma\}) \\ &\quad \otimes \tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_i)(\alpha) | q_i \in Q_{succ}(q_j, u), u \in \Sigma\} \\ &= \vee \{\tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_i)(\alpha) \otimes (\wedge \{\tilde{\delta}((q_k, \mu^{t_k}(q_k)), v, q_i)(\alpha) \rightarrow b(q_k) | \\ &\quad q_k \in Q_{pred}(q_i, v), v \in \Sigma\}) | q_i \in Q_{succ}(q_j, u), u \in \Sigma\} \\ &\leq \vee \{\tilde{\delta}((q_j, \mu^{t_j}(q_j)), u, q_i)(\alpha) \otimes (\tilde{\delta}((q_k, \mu^{t_k}(q_k)), v, q_i)(\alpha) \rightarrow \\ &\quad b(q_k)) | q_j \in Q_{pred}(q_i, u), u \in \Sigma\} \leq b(q_j) = P^\alpha(R^\alpha(b))(q_j) \end{aligned}$$

i. e.,  $S^\alpha(a) \leq P^\alpha(a) = b$ . Hence proved.  $\square$

**Remark 3.7.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$  be a primary. Then  $\alpha$  is a minimal regular closed subproposition of  $\tilde{F}$ .

**Definition 3.8.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . Consider an  $L$ -valued relation  $E$  on  $Q$  given by if  $p \in Q_{pred}(q, u) \Rightarrow E(p, q) = P^\alpha(\mathbf{1})(q) = P(\mathbf{1})(q)(\alpha) = \bigvee \{ \mathbf{1}(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) | p \in Q_{pred}(q, u), u \in \Sigma \}$  for all  $q \in Q$ .

We can show that  $L$ -valued relation  $E$  on  $Q$  is  $L$ -valued reflexive and  $L$ -valued transitive. To show  $E$  is  $L$ -valued reflexive, we need to demonstrated that  $E(p, p) = 1$ . Let  $p \in Q_{pred}(p, u), E(p, p) = P^\alpha(\mathbf{1})(p) = P(\mathbf{1})(p)(\alpha) = \bigvee \{ \mathbf{1}(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, p)(\alpha) | p \in Q_{pred}(p, u), u \in \Sigma \} = \tilde{\delta}((p, \mu^t(p)), \wedge, p)(\alpha) = 1$ . Thus,  $E$  is  $L$ -valued reflexive. To show  $E$  is  $L$ -valued transitive, we need to display that  $E(p, r) \geq E(p, q) \otimes E(q, r)$  where,  $p \in Q_{pred}(q, u), q \in Q_{pred}(r, v)$ , i.e.,  $P^\alpha(\mathbf{1})(r) \geq P^\alpha(\mathbf{1})(q) \otimes P^\alpha(\mathbf{1})(r)$ . Now,  $P^\alpha(\mathbf{1})(q) \otimes P^\alpha(\mathbf{1})(r) = [\bigvee \{ \mathbf{1}(q') \otimes \tilde{\delta}((q', \mu^{t_i}(q')), u, q)(\alpha) | q' \in Q_{pred}(q, u), u \in \Sigma \}] \otimes [\bigvee \{ \mathbf{1}(r') \otimes \tilde{\delta}((r', \mu^{t_j}(r')), v, r)(\alpha) | r' \in Q_{pred}(r, v), v \in \Sigma \}] = \bigvee \{ (\mathbf{1}(q') \otimes \tilde{\delta}((q', \mu^{t_i}(q')), u, q)(\alpha)) \otimes (\bigvee \{ \mathbf{1}(r') \otimes \tilde{\delta}((r', \mu^{t_j}(r')), v, r)(\alpha) | r' \in Q_{pred}(r, v), v \in \Sigma \}) | q' \in Q_{pred}(q, u), u \in \Sigma \} = \bigvee \{ (\tilde{\delta}((q', \mu^{t_i}(q')), u, q)(\alpha) \otimes (\bigvee \{ \tilde{\delta}((r', \mu^{t_j}(r')), v, r)(\alpha) | r' \in Q_{pred}(r, v), v \in \Sigma \})) | q' \in Q_{pred}(q, u), u \in \Sigma \} = \bigvee \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \otimes \tilde{\delta}((q, \mu^{t_i}(q)), v, r)(\alpha) | p \in Q_{pred}(q, u), q \in Q_{pred}(v, r), u, v \in \Sigma \} = \bigvee \{ \tilde{\delta}^*((p, \mu^t(p)), uv, r)(\alpha) | p \in Q_{pred}(q, u), q \in Q_{pred}(v, r), u, v \in \Sigma \} = \bigvee \{ \mathbf{1}(p) \otimes \tilde{\delta}^*((p, \mu^t(p)), uv, r)(\alpha) | p \in Q_{pred}(q, u), q \in Q_{pred}(v, r), u, v \in \Sigma \} = P^\alpha(\mathbf{1})(r) = E(p, r)$ . Thus,  $E(p, r) = E(p, q) \otimes E(q, r)$ . Hence,  $E$  is  $L$ -valued transitive.

**Definition 3.9.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . The  $L$ -valued approximation operator on  $Q$  is as follows:

$$D^\alpha(q) = D(\alpha)(q) = \bigvee \{ E(p, q) \otimes \alpha(p) | p \in Q_{pred}(q, u), u \in \Sigma \} \text{ for all } q \in Q.$$

**Proposition 3.10.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . An  $L$ -valued relation  $E$  on a non-empty set  $Q$  is  $L$ -valued reflexive and  $L$ -valued transitive if and only if  $L$ -valued approximation operator  $D$  is a Kuratowski saturated  $L$ -valued closure operator on  $Q$ .

*Proof.* Let  $E$  be an  $L$ -valued reflexive and  $L$ -valued transitive relation on  $Q$ . To show that  $D$  is an  $L$ -valued closure operator on  $Q$ , we need to verify the following conditions ( $\forall \lambda, \lambda_j \in B, j \in J$ ):

i)  $D(\mathbf{a}) = \mathbf{a}, \forall a \in L$ ;

ii)  $\lambda \leq D(\lambda)$ ;

iii)  $D(\cup \{ \lambda_j | j \in J \}) = \cup \{ D(\lambda_j) | j \in J \}$ ;

iv)  $D(D(\lambda)) = D(\lambda)$ .

Proof of (i) is obviously satisfied and (ii) follows using the reflexivity of  $E$ . (iii) is satisfied, since for

$$q \in Q \text{ and } \lambda_j \in B, j \in J, D(\cup \{ \lambda_j | j \in J \})(q) = \bigvee \{ E(p, q) \otimes (\bigvee \{ \lambda_j(p) | j \in J \}) | p \in Q_{pred}(q, u), u \in \Sigma \}$$

$$= \bigvee \{ \bigvee \{ E(p, q) \otimes \lambda_j(p) | p \in Q_{pred}(q, u), u \in \Sigma \} | j \in J \}$$

$$= \cup \{ D(\lambda_j | j \in J) \}(q). \text{ Finally, (iv) is also satisfied by using the transitivity of } E \text{ as: } \forall q \in Q$$

$$D(D(\alpha))(q) = \bigvee \{ E(p, q) \otimes D(\alpha)(p) | p \in Q_{pred}(q, u), u \in \Sigma \}$$

$$= \bigvee \{ E(p, q) \otimes (\bigvee \{ E(r, p) \otimes \alpha(r) | r \in Q_{pred}(p, v), v \in \Sigma \}) | p \in Q_{pred}(q, u), u \in \Sigma \}$$

$$= \bigvee \{ (\bigvee \{ E(p, q) \otimes E(r, p) | p \in Q_{pred}(q, u) \}) \otimes \alpha(r) | r \in Q_{pred}(p, v), u, v \in \Sigma \}$$

$$\leq \bigvee \{ E(r, q) \otimes \alpha(r) | r \in Q_{pred}(q, w), w \in \Sigma^* \} = D(\alpha)(q).$$

Conversely, let  $D$  be an  $L$ -valued closure operator on  $Q$  and  $q \in Q$ . Then  $1 = \mathbf{1}(q) \leq D(\mathbf{1})(q)$ . Thus,  $D(\mathbf{1})(q) = 1$ , hence  $\bigvee \{E(p, q) \otimes \mathbf{1}(p) \mid p \in Q_{\text{pred}}(q, u), u \in \Sigma\} = 1 = E(p, p)$ . Hence,  $E$  is an  $L$ -valued reflexive. Next, let  $q \in Q, r \in Q_{\text{pred}}(q, u)$  for  $u \in \Sigma$ . Then  $D(D(\mathbf{1}))(q) \leq D(\mathbf{1})(q)$ , i.e.,

$$\begin{aligned} & \bigvee \{E(p, q) \otimes D(\mathbf{1})(p) \mid p \in Q_{\text{pred}}(q, u), u \in \Sigma\} \leq D(\mathbf{1})(q) \\ \Rightarrow & \bigvee \{E(p, q) \otimes (\bigvee \{E(r, p) \otimes \mathbf{1}(r) \mid r \in Q_{\text{pred}}(p, v), v \in \Sigma\}) \mid p \in Q_{\text{pred}}(q, u), u \in \Sigma\} \leq \bigvee \{E(q', q) \otimes \mathbf{1}(q') \mid q' \in Q_{\text{pred}}(q, w), w \in \Sigma\}, \text{ or that } \bigvee \{E(r, p) \otimes E(p, q) \mid p \in Q_{\text{pred}}(q, u), u \in \Sigma\} \leq \bigvee \{E(q', q) \otimes \mathbf{1}(q') \mid q' \in Q_{\text{pred}}(q, w), w \in \Sigma\}. \end{aligned}$$

Thus,  $\bigvee \{E(r, p) \otimes E(p, q) \mid p \in Q_{\text{pred}}(q, u), r \in Q_{\text{pred}}(p, v)\} \leq E(r, q)$ . Hence,  $E$  is an  $L$ -valued transitive also.  $\square$

**Proposition 3.11.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton. Then  $D$  is a Kuratowski saturated  $L$ -valued closure operator on  $Q$ .

*Proof.* It follows from Proposition 3.10.  $\square$

Thus,  $D$  induces an saturated  $L$ -valued topology on  $Q$ , say  $\tau(Q)$ .

**Remark 3.12.** (i) Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . Similar to above, if we define another  $L$ -valued relation  $E^*$  on  $Q$ , given by if  $p \in Q_{\text{succ}}(q, u) \Rightarrow E^*(p, q) = S^\alpha(\mathbf{1})(q) = S(\mathbf{1})(q)(\alpha) = \bigvee \{\mathbf{1}(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \mid p \in Q_{\text{succ}}(q, u), u \in \Sigma\}$  for all  $q \in Q$ . Then  $E^*(p, q) = E(q, p)$  is also an  $L$ -valued reflexive and transitive relation on  $Q$ , and hence it will induce another  $L$ -valued approximation operator, say  $D^*$ , on  $Q$ , which will induce an  $L$ -valued topology on  $Q$ , say  $\tau^*(Q)$ .

(ii) It can be seen that  $D(\lambda) = P(\lambda)$  and  $D^*(\lambda) = S(\lambda)$  for all  $\lambda \in B$ .

**Definition 3.13.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ .  $b \in B$  is called an  $L$ -valued successor subproposition of  $\tilde{F}$  if

$$b(p) \leq \bigwedge \{\tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q) \mid q \in Q_{\text{succ}}(p, u), u \in \Sigma\}.$$

**Proposition 3.14.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ .  $b \in B$  is an  $L$ -valued successor subproposition of  $\tilde{F}$  if and only if  $P^\alpha(b) = b$  (i.e.,  $b$  is  $L$ -valued  $T'^\alpha(Q)$ -closed).

*Proof.* Let  $P^\alpha(b) = b$ . Then

$$\begin{aligned} P^\alpha(b) = b & \Rightarrow P^\alpha(b)(q) = b(q) \\ \Rightarrow & \bigvee \{b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \mid p \in Q_{\text{pred}}(q, u), u \in \Sigma\} = b(q) \\ \Rightarrow & b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq b(q), p \in Q_{\text{pred}}(q, u), u \in \Sigma \\ \Rightarrow & b(p) \leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q), q \in Q_{\text{succ}}(p, u), u \in \Sigma \\ \Rightarrow & b(p) \leq \bigwedge \{\tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q) \mid q \in Q_{\text{succ}}(p, u), u \in \Sigma\}. \end{aligned}$$

Then  $b$  is an  $L$ -valued successor subproposition of  $\tilde{F}$ .

Conversely, let  $b$  be an  $L$ -valued successor subproposition of  $\tilde{F}$ . Then

$$\begin{aligned} b(p) & \leq \bigwedge \{\tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q) \mid q \in Q_{\text{succ}}(p, u), u \in \Sigma\} \\ \Rightarrow & b(p) \leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q), q \in Q_{\text{succ}}(p, u), u \in \Sigma \\ \Rightarrow & b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq b(q), p \in Q_{\text{pred}}(q, u), u \in \Sigma \\ \Rightarrow & \bigvee \{b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \mid p \in Q_{\text{pred}}(q, u), u \in \Sigma\} \leq b(q) \\ \Rightarrow & P^\alpha(b)(q) \leq b(q). \end{aligned}$$

Also, as  $b(q) \leq P^\alpha(b)(q)$ , whereby  $P^\alpha(b) = b$ . Hence,  $P^\alpha(b) = b$  iff  $b$  be an  $L$ -valued successor subproposition of  $\tilde{F}$ .  $\square$

**Proposition 3.15.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ .  $b \in B$  is an  $L$ -valued separated successor subproposition of  $\tilde{F}$  if and only if  $b$  is  $T'^\alpha(Q)$ -clopen.

*Proof.* Follows from Propositions 2.11 and 3.14.  $\square$

**Definition 3.16.** An  $L^B$ -valued general fuzzy automaton  $\tilde{F}$  is called reversible if

$$\tilde{\delta}((p, \mu^{t_i}(p)), u, q)(\alpha) \leq \vee\{\tilde{\delta}((q, \mu^{t_j}(q)), v, p)(\alpha) | v \in \Sigma\}, \forall p, q \in Q, u \in \Sigma \text{ and } \alpha \in B.$$

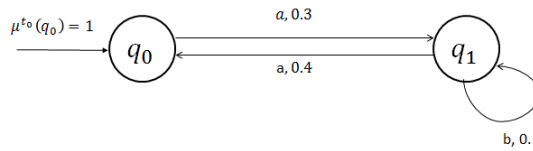
**Proposition 3.17.** An  $L^B$ -valued general fuzzy automaton  $\tilde{F}$  is reversible if and only if  $L$ -valued topology  $\tau(Q)$  is  $R_0$ .

*Proof.* Let  $\tilde{F}$  be reversible. Then  $\forall p, q \in Q, u \in \Sigma$  and  $\alpha \in B, \tilde{\delta}((p, \mu^{t_i}(p)), u, q)(\alpha) \leq \vee\{\tilde{\delta}((q, \mu^{t_j}(q)), v, p)(\alpha) | v \in \Sigma\}$ .

We show that  $L$ -valued topology  $\tau(Q)$  is  $R_0$ , for which it suffices to show that  $P^\alpha(\mathbf{1})(q) \leq P^\alpha(\mathbf{1})(p), \forall p, q \in Q$  and  $\alpha \in B$ . As  $P^\alpha(\mathbf{1})(q) = \vee\{\tilde{\delta}((p, \mu^{t_i}(p)), u, q)(\alpha) | p \in Q_{pred}(q, u), u \in \Sigma\}$  and  $P^\alpha(\mathbf{1})(p) = \vee\{\tilde{\delta}((q, \mu^{t_j}(q)), v, p)(\alpha) | q \in Q_{pred}(p, v), v \in \Sigma\}$ , from the reversibility of  $\tilde{F}, P^\alpha(\mathbf{1})(q) \leq P^\alpha(\mathbf{1})(p), \forall p, q \in Q$ . Therefore, the  $L$ -valued topology  $\tau(Q)$  is  $R_0$ . Conversely, let  $\tau(Q)$  be an  $R_0$   $L$ -valued topology on  $Q$ . Then  $\forall p, q \in Q, P^\alpha(\mathbf{1})(q) \leq P^\alpha(\mathbf{1})(p)$ , i.e.,  $\tilde{\delta}((p, \mu^{t_i}(p)), u, q)(\alpha) \leq \tilde{\delta}((q, \mu^{t_j}(q)), v, p)(\alpha), \forall u, v \in \Sigma, \forall p, q \in Q$  and  $\alpha \in B$ .

Hence,  $\tilde{F}$  is reversible.  $\square$

**Example 3.18.** Consider the GFA in Figure 2, it is specified as  $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ , where  $Q = \{q_0, q_1\}$  is the set of states,  $\Sigma = \{a, b\}$  is the set of input symbols,  $\tilde{R} = \{(q_0, 1)\}, Z = \emptyset$  and  $w$  is not applicable.



**Fig. 2** The GFA of Example 3.18

We check operation of the GFA in Example 3.18 upon input "aba".

If we choose  $F_1(\mu, \delta) = \delta, F_2(q) = \mu^{t+1}(q_m) = \wedge_{i=1}^n (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$ , then we have:

$$\mu^{t_0}(q_0) = 1,$$

$$\mu^{t_1}(q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = \delta(q_0, a, q_1) = 0.3,$$

$$\mu^{t_2}(q_1) = F_1(\mu^{t_1}(q_1), \delta(q_1, b, q_1)) = \delta(q_1, b, q_1) = 0.1,$$

$$\mu^{t_2}(q_0) = F_1(\mu^{t_2}(q_1), \delta(q_1, a, q_0)) = \delta(q_1, a, q_0) = 0.4.$$

Table 2: Active states and their membership values (mv) at different times in Example 3.18

time	$t_0$	$t_1$	$t_2$	$t_3$
input	$\wedge$	a	b	a
$Q_{act}(t_i)$	$q_0$	$q_1$	$q_1$	$q_0$
mv	1	0.3	0.1	0.4

The set  $B = \{0, \alpha_0, \alpha_1, \alpha_2, \alpha_3, 1\}$  of possible propositions about the general fuzzy automaton  $\tilde{F}$  is as follows: -0 means that the GFA is not in active states of  $Q$ ,

- $\alpha_0$  means that the GFA is in active states at time  $t_0$ ,
- $\alpha_1$  means that the GFA is in active states at time  $t_1$ ,
- $\alpha_2$  means that the GFA is in active states at time  $t_2$ ,
- $\alpha_3$  means that the GFA is in active states at time  $t_3$ ,
- 1 means that the GFA is in at least one active state at time  $t_i$  for any  $i \geq 0$ .

By the Definition 3.16  $\tilde{F}$  is reversible and  $L$ -valued topology  $\tau$  on  $Q$  is  $R_0$ . For example we have:  $\tilde{\delta}((q_1, \mu^{t_2}(q_1)), a, q_0)(\alpha_2) = F_1(\mu^{t_2}(q_1), \delta(q_1, a, q_0)(\alpha_2)) = \delta(q_1, a, q_0)(\alpha_2) = \alpha_2(q_0) \vee \alpha_2(q_1) = 0 \vee 0.1 = 0.1$  and  $\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1)(\alpha_2) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)(\alpha_2)) = \delta(q_0, a, q_1)(\alpha_2) = \alpha_2(q_0) \vee \alpha_2(q_1) = 0 \vee 0.1 = 0.1$ , then  $\tilde{\delta}((q_1, \mu^{t_2}(q_1)), a, q_0)(\alpha_2) \leq \tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1)(\alpha_2)$ . Also,  $P^{\alpha_2}(\mathbf{1})(q_0) \leq P^{\alpha_2}(\mathbf{1})(q_1)$  Since  $P^{\alpha_2}(\mathbf{1})(q_0) = \mathbf{1}(q_1) \otimes \tilde{\delta}((q_1, \mu^{t_2}(q_1)), a, q_0)(\alpha_2) = 0.3 \otimes 0.1 = 0.1$ ,  $P^{\alpha_2}(\mathbf{1})(q_1) = \{\mathbf{1}(q_0) \otimes \tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1)(\alpha_2)\} \vee \{\mathbf{1}(q_1) \otimes \tilde{\delta}((q_1, \mu^{t_1}(q_1)), b, q_1)(\alpha_2)\} = 0.1 \otimes 0.1 = 0.1$ .

**Proposition 3.19.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ .  $\tilde{F}$  is reversible if and only if for any  $L$ -valued successor subproposition  $b \in B$  of  $\tilde{F}$ ,  $P^\alpha(b) = b$ .

*Proof.* Let  $\tilde{F}$  be reversible. Then  $\tau(Q)$  is  $R_0$   $L$ -valued topology. Thus, for any  $L$ -valued closed subset  $b$  on  $Q$ ,  $P^\alpha(b) \leq b$ . Hence,  $P^\alpha(b) = b$ . Conversely, let  $b \in B$  be an  $L$ -valued successor subproposition of  $\tilde{F}$  such that  $P^\alpha(b) = b$ . Then as  $b$  is  $\tau(Q)$   $L$ -valued closed and contains the closure of its each point, the  $L$ -valued topology  $\tau(Q)$  is  $R_0$ . Thus,  $\tilde{F}$  is reversible.  $\square$

**Corollary 3.20.** An  $L^B$ -valued general fuzzy automaton  $\tilde{F}$  is reversible if and only if each  $L$ -valued successor subproposition of  $\tilde{F}$  is separated.

*Proof.* It follows from Propositions 3.14 and 3.19.  $\square$

### 3.2. $L^B$ -valued GFA Structure Space and $L^B$ -valued GFA Homotopy

Let  $\tau$  be an  $L$ -valued topology on  $Q$  induced by Kuratowski saturated  $L$ -valued closure operator  $D$  in previous subsection. Then  $\tau$  is said to be the  $L^B$ -valued GFA structure generated by  $L$ -valued topology associated with an  $L^B$ -valued GFA and the ordered pair  $(Q, \tau)$  is said to be an  $L^B$ -valued GFA structure space. Moreover, members of  $\tau$  are said to be an  $L^B$ -valued GFA open subpropositions and their complements are said to be an  $L^B$ -valued GFA closed subpropositions.

**Definition 3.21.** Let  $\tilde{F}$  be an  $L^B$ -valued GFA and let  $(Q, \tau)$  be an  $L^B$ -valued GFA structure space. Let  $p, q \in Q, u \in \Sigma$  and  $\alpha \in B$ . Then the proposition point  $q_\alpha$  is defined as:

$$q_\alpha(p) = \begin{cases} \alpha(q), & p \in Q_{pred}(q, u) \\ 0, & \text{otherwise} \end{cases}.$$

**Definition 3.22.** Let  $(Q, \tau)$  and  $(Q', \tau')$  be any two  $L^B$ -valued GFA structure space. A function  $f : (Q, \tau) \rightarrow (Q', \tau')$  is said to be:

- (i)  $L^B$ -valued GFA continuous if  $f^{-1}(\lambda)$  is an  $L^B$ -valued GFA open (resp.  $L^B$ -valued GFA closed) subproposition in  $(Q, \tau)$  for every  $L^B$ -valued open (resp.  $L^B$ -valued closed) subproposition  $\lambda$  in  $(Q', \tau')$ .
- (ii)  $L^B$ -valued GFA homomorphism if  $f$  is bijective and  $f, f^{-1}$  are  $L^B$ -valued GFA continuous functions.

**Definition 3.23.** Let  $(Q, \tau)$  be an  $L^B$ -valued GFA structure space and let  $A \subseteq Q$ . Then the  $L^B$ -valued GFA characteristic function  $\chi_A$  of  $A$  is defined as:

$$\chi_A(q) = \begin{cases} 1, & q \in A \\ 0, & \text{otherwise} \end{cases}.$$

**Definition 3.24.** Let  $(Q, \tau)$  be an  $L^B$ -valued GFA structure space. If  $A \subseteq Q$  and  $\chi_A$  is the  $L^B$ -valued GFA characteristic function of  $A$ , then the collection  $\tau/A = \{\lambda/A = \lambda \wedge \chi_A : \lambda \in \tau\}$  is an  $L^B$ -valued GFA structure on  $A$  which is called the  $L^B$ -valued GFA subspace structure and the pair  $(A, \tau/A)$  is called  $L^B$ -valued GFA structure subspace of  $(Q, \tau)$ .

**Definition 3.25.** Let  $(Q, \tau)$  be a structure space where  $Q$  is the non-empty set of states of an  $L^B$ -valued GFA  $\tilde{F}$ . The collection,  $w(\tau) = \{\chi_P : P \in \tau\}$  such that  $D(\chi_P) = \chi_P$  is an  $L^B$ -valued structure on  $Q$  introduced by  $\tau$ . Then  $(Q, w(\tau))$  is called the  $L^B$ -valued GFA structure space introduced by  $(Q, \tau)$ .

**Note 3.26.** Let  $\zeta$  be an Euclidean topology on  $I$ , where  $I = [0, 1]$  and  $(I, w(\zeta))$  be an  $L^B$ -valued GFA structure space introduced by the Euclidean space  $(I, \zeta)$ .

**Definition 3.27.** Let  $(Q, \tau)$  and  $(Q', \tau')$  be any two  $L^B$ -valued GFA structure spaces and  $(I, w(\zeta))$  be an  $L^B$ -valued GFA structure introduced by the Euclidean space  $(I, \zeta)$ . Let  $f, g : (Q, \tau) \rightarrow (Q', \tau')$  be any two  $L^B$ -valued GFA continuous functions. If there exists an  $L^B$ -valued GFA continuous function  $H : (Q, \tau) \times (I, w(\zeta)) \rightarrow (Q', \tau')$  such that  $H(q_\alpha, 0) = f(q_\alpha)$  and  $H(q_\alpha, 1) = g(q_\alpha)$ , for each proposition point  $q_\alpha$  of  $Q$ , then  $f$  is said to be an  $L^B$ -valued GFA homotopic to  $g$ . Moreover, the function  $H$  is said to be an  $L^B$ -valued GFA homotopy between  $f$  and  $g$ , denoted as  $f \simeq g$ .

**Proposition 3.28.** Let  $(Q, \tau)$  and  $(Q', \tau')$  be any two  $L^B$ -valued GFA structure spaces. Let  $U$  and  $V$  be subsets of  $Q$ . Let  $\lambda = \frac{\lambda}{U} \vee \frac{\lambda}{V}$ , where  $\frac{\lambda}{U}$  and  $\frac{\lambda}{V}$  are the  $L^B$ -valued GFA open subpropositions in  $(Q, \tau)$ . Let  $f : (U, \frac{\tau}{U}) \rightarrow (Q', \tau')$  and  $h : (V, \frac{\tau}{V}) \rightarrow (Q', \tau')$  be any two  $L^B$ -valued GFA continuous functions. If  $\frac{f}{U \cap V} = \frac{h}{U \cap V}$ , then  $g : (Q, \tau) \rightarrow (Q', \tau')$  which is defined by,

$$g(q) = \begin{cases} f(q), & \text{for } q \in U \\ h(q), & \text{for } q \in V \end{cases}$$

is an  $L^B$ -valued GFA continuous function.

*Proof.* Let  $\alpha$  be an  $L^B$ -valued GFA open subproposition in  $(Q', \tau')$ . Let  $\frac{\lambda}{U}$  and  $\frac{\lambda}{V}$  be the  $L^B$ -valued GFA open subpropositions in  $(Q, \tau)$ . Then  $\frac{\lambda}{U} \vee \frac{\lambda}{V}$  is an  $L^B$ -valued GFA open subproposition in  $(Q, \tau)$ . Now,

$$g^{-1}(\alpha) = g^{-1}(\alpha) \wedge \lambda = g^{-1}(\alpha) \wedge \left(\frac{\lambda}{U} \vee \frac{\lambda}{V}\right) = (g^{-1}(\alpha) \wedge \frac{\lambda}{U}) \vee (g^{-1}(\alpha) \wedge \frac{\lambda}{V}) = f^{-1}(\alpha) \vee h^{-1}(\alpha).$$

Since  $f$  and  $h$  are  $L^B$ -valued GFA continuous functions,  $f^{-1}(\alpha)$  and  $h^{-1}(\alpha)$  are the  $L^B$ -valued GFA open subpropositions in  $(U, \frac{\lambda}{U})$  and  $(V, \frac{\lambda}{V})$ , respectively. Then  $g^{-1}(\alpha)$  is an  $L^B$ -valued GFA open subproposition in  $(Q, \tau)$ . Hence,  $g$  is an  $L^B$ -valued GFA continuous function.  $\square$

**Proposition 3.29.** The homotopy relation  $\simeq$  is an equivalence relation.

*Proof.* It is clear.  $\square$

**Note 3.30.** The equivalence class of the  $L^B$ -valued GFA function  $f$  under the equivalence relation  $\simeq$  is denoted by  $[f]$ .

**Definition 3.31.** Let  $(Q, \tau)$  and  $(Q', \tau')$  be any two  $L^B$ -valued GFA structure spaces. Let  $f : (Q, \tau) \rightarrow (Q', \tau')$  and  $g : (Q', \tau') \rightarrow (Q, \tau)$  be  $L^B$ -valued GFA continuous function such that  $f \circ g = id_{Q'}$  and  $g \circ f = id_Q$ , where  $id_Q$  and  $id_{Q'}$  are the identity functions of  $(Q, \tau)$  and  $(Q', \tau')$ , respectively. Then  $(Q, \tau)$  and  $(Q', \tau')$  are said to  $L^B$ -valued GFA homotopic equivalent spaces and  $f$  is called an  $L^B$ -valued GFA homotopy equivalence.

**Definition 3.32.** Let  $(Q, \tau)$  and  $(Q', \tau')$  be any two  $L^B$ -valued GFA structure spaces. An  $L^B$ -valued GFA function  $f : (Q, \tau) \rightarrow (Q', \tau')$  is said to be a constant function if for each proposition point  $q_\delta$  in  $Q$   $f(q_\delta) = r_\lambda$  where  $r_\lambda$  is a fixed proposition point in  $Q'$ .

**Definition 3.33.** An  $L^B$ -valued GFA structure  $(Q, \tau)$  is said to be  $L^B$ -valued GFA contractible space if the identity function  $id_Q : (Q, \tau) \rightarrow (Q, \tau)$  is  $L^B$ -valued GFA homotopic to a constant function  $h : (Q, \tau) \rightarrow (Q, \tau)$ .

**Definition 3.34.** Let  $(Q, \tau)$  and  $(Q', \tau')$  be any two  $L^B$ -valued GFA structure spaces. Let  $f, g : (Q, \tau) \rightarrow (Q', \tau')$  be any two  $L^B$ -valued GFA continuous functions and  $f \simeq g$ . If  $g$  is a constant function, then  $f$  is called an  $L^B$ -valued GFA null-homotopic function.

**Proposition 3.35.** An  $L^B$ -valued GFA structure space  $(Q, \tau)$  is  $L^B$ -valued GFA contractible if for any arbitrary  $L^B$ -valued GFA structure space  $(Q', \tau')$  every  $L^B$ -valued GFA function  $f : (Q, \tau) \rightarrow (Q', \tau')$  is  $L^B$ -valued GFA null-homotopic.

*Proof.* Assume that for any arbitrary  $L^B$ -valued GFA structure space  $(Q', \tau')$  the function  $f : (Q, \tau) \rightarrow (Q', \tau')$  is  $L^B$ -valued GFA null-homotopic. Then the identity function  $id_Q : (Q, \tau) \rightarrow (Q, \tau)$  is  $L^B$ -valued GFA null-homotopic; and, hence  $(Q, \tau)$  is  $L^B$ -valued GFA contractible. Now, assume that  $(Q, \tau)$  is an  $L^B$ -valued GFA contractible space. Then there exists a constant function  $h : (Q, \tau) \rightarrow (Q, \tau)$  by  $h(q_\delta) = q'_\delta$  and  $L^B$ -valued GFA homotopy  $F : (Q, \tau) \times (I, w(\zeta)) \rightarrow (Q, \tau)$  such that  $F(q_\delta, 0) = id_Q(q_\delta); F(q_\delta, 1) = h(q_\delta) = q'_\delta$ .

If there is an  $L^B$ -valued GFA  $f : (Q, \tau) \rightarrow (Q', \tau')$ , then  $H : (Q, \tau) \times (I, w(\zeta)) \rightarrow (Q', \tau')$  is defined by  $H(q_\delta, s) = f(F(q_\delta, s))$ , where  $s \in I$  has the following properties:

$$H(q_\delta, 0) = f(F(q_\delta, 0)) = f(id_Q(q_\delta)) = f(q_\delta) = (f(q))_\delta, H(q_\delta, 1) = f(F(q_\delta, 1)) = f(h(q_\delta)) = q'_\delta.$$

Hence,  $H$  is an  $L^B$ -valued GFA homotopy from  $f$  to a constant map with value  $f(q')$ . Thus,  $f$  is  $L^B$ -valued GFA null-homotopic.  $\square$

**Proposition 3.36.** Let  $(Q, \tau)$ ,  $(Q', \sigma)$  and  $(Q'', \rho)$  be any three  $L^B$ -valued GFA structure spaces. let  $f, g : (Q, \tau) \rightarrow (Q', \sigma)$  be  $L^B$ -valued GFA continuous functions such that  $f \simeq g$ . If  $h : (Q', \sigma) \rightarrow (Q'', \rho)$  is an  $L^B$ -valued continuous function, then  $hof, hog : (Q, \tau) \rightarrow (Q'', \rho)$  are  $L^B$ -valued GFA continuous and  $hof \simeq hog$ .

*Proof.* Let  $(I, w(\zeta))$  be an  $L^B$ -valued GFA structure space introduced by the Euclidean space  $(I, \zeta)$ . Since  $h, f$  and  $g$  are  $L^B$ -valued GFA functions,  $hof$  and  $hog$  are  $L^B$ -valued GFA continuous. Since  $f \simeq g$  by definition of  $L^B$ -valued GFA homotopy there is an  $L^B$ -valued continuous function  $G : (Q, \tau) \times (I, w(\zeta)) \rightarrow (Q', \sigma)$  such that  $G(q_\delta, 0) = f(q_\delta)$ ,  $G(q_\delta, 1) = g(q_\delta)$ , for each proposition poin  $q_\delta$  of  $Q$ . Now,  $H : (Q, \tau) \times (I, w(\zeta)) \rightarrow (Q'', \rho)$  is given by  $H(q_\delta, t) = h(G(q_\delta, t))$ , where  $t \in I$ . Since  $h$  and  $g$  are  $L^B$ -valued GFA continuous functions,  $H = hog$  is an  $L^B$ -valued GFA continuous function. Moreover,  $H$  satisfies the following conditions:

$$H(q_\delta, 0) = h(G(q_\delta, 0)) = h(f(q_\delta)) = (hof)(q_\delta);$$

$$H(q_\delta, 1) = h(G(q_\delta, 1)) = h(g(q_\delta)) = (hog)(q_\delta).$$

Hence,  $hof \simeq hog$ .  $\square$

**Proposition 3.37.** Let  $(Q, \tau)$ ,  $(Q', \sigma)$  and  $(Q'', \rho)$  be any three  $L^B$ -valued GFA structure spaces. Suppose that  $f_1, f_2 : (Q, \tau) \rightarrow (Q', \sigma)$  are  $L^B$ -valued GFA homotopic functions and that  $g_1, g_2 : (Q', \sigma) \rightarrow (Q'', \rho)$  are  $L^B$ -valued GFA homotopic functions, then  $g_1 \circ f_1 \simeq g_2 \circ f_2$ .

*Proof.* Let  $H : (Q, \tau) \times (I, w(\zeta)) \rightarrow (Q', \sigma)$  be an  $L^B$ -valued GFA homotopy  $f_1$  to  $f_2$  and let  $G : (Q', \sigma) \times (I, w(\zeta)) \rightarrow (Q'', \rho)$  be an  $L^B$ -valued GFA homotopy from  $g_1$  to  $g_2$ . Now, let us define a function  $F : (Q, \tau) \times (I, w(\zeta)) \rightarrow (Q'', \rho)$  by  $F(q_\delta, s) = G(H(q_\delta, s), s)$ . Since  $F$  is a composition of two  $L^B$ -valued GFA continuous functions,  $G$  and  $H$ ,  $F$  is also an  $L^B$ -valued GFA continuous function. It is seen that

$$F(q_\delta, 0) = G(H(q_\delta, 0), 0) = g_2(f_2(q_\delta));$$

$$F(q_\delta, 1) = G(H(q_\delta, 1), 1) = g_1(f_1(q_\delta)).$$

Hence,  $F$  is an  $L^B$ -valued GFA homotopy from  $g_1 \circ f_1$  to  $g_2 \circ f_2$ . Therefore  $g_1 \circ f_1 \simeq g_2 \circ f_2$ .  $\square$

**Definition 3.38.** Let  $(Q, \tau)$  and  $(Q', \sigma)$  be any two  $L^B$ -valued GFA structure spaces. Let  $f_1 : (Q, \tau) \rightarrow (Q', \sigma)$  be an  $L^B$ -valued GFA continuous function. If there exists an  $L^B$ -valued GFA continuous function  $f_2 : (Q', \sigma) \rightarrow (Q, \tau)$  which satisfies the following conditions:

(i)  $f_1 \circ f_2 \simeq id_{Q'}$  and (ii)  $f_2 \circ f_1 \simeq id_Q$ , where  $id_Q$  and  $id_{Q'}$  are the identity functions of  $Q$  and  $Q'$ , respectively, then the function  $f_1$  is called an  $L^B$ -valued GFA homotopy equivalence and the  $L^B$ -valued GFA structure spaces  $(Q, \tau)$  and  $(Q', \sigma)$  are called  $L^B$ -valued GFA homotopic equivalent spaces.

**Proposition 3.39.** *Let  $(Q, \tau)$  and  $(Q', \sigma)$  be any two  $L^B$ -valued GFA structure spaces. Every  $L^B$ -valued GFA function that is  $L^B$ -valued GFA homotopic to an  $L^B$ -valued GFA homotopy equivalence is an  $L^B$ -valued GFA homotopy equivalence.*

*Proof.* Let  $f_1 : (Q, \tau) \rightarrow (Q', \sigma)$  be an  $L^B$ -valued GFA homotopy equivalence and let  $g : (Q, \tau) \rightarrow (Q', \sigma)$  be  $L^B$ -valued GFA homotopic to  $f_1$ . Then by definition of  $L^B$ -valued GFA homotopy equivalence, there exists an  $L^B$ -valued GFA continuous function  $f_2 : (Q', \sigma) \rightarrow (Q, \tau)$  such that  $f_2 \circ f_1 \simeq id_Q$  and  $f_1 \circ f_2 \simeq id_{Q'}$ , respectively. Since  $f_1, g : (Q, \tau) \rightarrow (Q', \sigma)$  are  $L^B$ -valued GFA homotopic and by Proposition 3.37 it is seen that  $f_2 \circ f_1 \simeq f_2 \circ g \simeq id_Q$ ;  $f_1 \circ f_2 \simeq g \circ f_2 \simeq id_{Q'}$ . Therefore  $g$  is an  $L^B$ -valued GFA homotopy equivalence.  $\square$

**Definition 3.40.** Let  $(Q, \tau)$  and  $(Q', \sigma)$  be any two  $L^B$ -valued GFA structure spaces. If the bijective function  $f : (Q, \tau) \rightarrow (Q', \sigma)$  and its inverse function are  $L^B$ -valued GFA continuous functions, then the function  $f$  is said to be an  $L^B$ -valued GFA homeomorphism. Moreover,  $(Q', \sigma)$  are said to be an  $L^B$ -valued GFA homeomorphic spaces.

**Proposition 3.41.** *Every  $L^B$ -valued GFA homeomorphic spaces are  $L^B$ -valued GFA homotopy equivalent spaces*

*Proof.* Let  $(Q, \tau)$  and  $(Q', \sigma)$  be any two  $L^B$ -valued GFA structure spaces. Since  $f_1 : (Q, \tau) \rightarrow (Q', \sigma)$  and  $f_2 : (Q', \sigma) \rightarrow (Q, \tau)$  are  $L^B$ -valued GFA homeomorphisms,  $f_1 \circ f_2 \simeq id_{Q'}$  and  $f_2 \circ f_1 \simeq id_Q$ , where  $id_Q$  and  $id_{Q'}$  are the identity functions of  $Q$  and  $Q'$ , respectively. Hence, by Proposition 3.29,  $(Q, \tau)$  and  $(Q', \sigma)$  are  $L^B$ -valued GFA homotopy equivalent spaces.  $\square$

### 3.3. Relationships Between $L^B$ -valued GFA and Associated $L$ -valued Subpropositions

In this subsection, we study the concepts associated with  $L$ -valued successor (predecessor) subpropositions of an  $L^B$ -valued general fuzzy automaton  $\tilde{F}$  with the membership values in a complete residuated lattice. Finally, the continuity properties of such  $L^B$ -valued general fuzzy automaton are discussed in terms of the topologies.

**Definition 3.42.** A reverse  $L^B$ -valued general fuzzy automaton of an  $L^B$ -valued GFA  $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$  is an  $L^B$ -valued GFA  $\tilde{\tilde{F}} = (Q, \Sigma, \tilde{R}, Z, \tilde{\tilde{\delta}}, \omega, F_1, F_2)$  where  $\tilde{\tilde{\delta}} : (Q \times L) \times \Sigma \times Q \rightarrow L^B$  is a map such that

$$\tilde{\tilde{\delta}}((p, \mu^{t_i}(p)), u, q)(\alpha) = \vee \{ \tilde{\delta}((q, \mu^{t_i}(q)), v, p)(\alpha) | v \in \Sigma \}.$$

**Definition 3.43.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . Then  $b \in B$  is called:

i) an  $L$ -valued predecessor subproposition of  $\tilde{F}$  if

$$b(q) \leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(p) | p \in Q_{pred}(q, u), u \in \Sigma \};$$

ii) a double  $L$ -valued subproposition of  $\tilde{F}$  if it is both  $L$ -valued successor and predecessor subproposition of  $\tilde{F}$ .

**Proposition 3.44.** *Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . Then the following are equivalent:*

i)  $b \in B$  is an  $L$ -valued successor subproposition of  $\tilde{F}$ .

ii)  $b \in B$  is  $\tau(Q)$ -open.

iii)  $b \in B$  is a solution to an  $L$ -valued relational equation  $\chi \otimes E = \chi$ , where  $\chi \in B$  is an unknown.



*Proof.* (i)  $\rightarrow$  (ii) Let  $b$  be an  $L$ -valued successor subproposition of  $\tilde{F}$ . Then

$$\begin{aligned} b(p) &\leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q) \mid q \in Q_{succ}(p, u), u \in \Sigma \} \\ \Rightarrow b(p) &\leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ \Rightarrow b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) &\leq b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ \Rightarrow b(p) \otimes (\vee \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \mid q \in Q_{succ}(p, u), u \in \Sigma \}) &\leq b(q) \\ \Rightarrow b(p) \otimes (\vee \{ \mathbf{1}(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \mid p \in Q_{pred}(q, u), u \in \Sigma \}) &\leq b(q) \\ \Rightarrow b(p) \otimes (\vee \{ E(p, q) \mid p \in Q_{pred}(q, u), u \in \Sigma \}) &\leq b(q) \\ \Rightarrow \vee \{ b(p) \otimes E(p, q) \mid p \in Q_{pred}(q, u), u \in \Sigma \} &\leq b(q) \\ \Rightarrow D(b)(q) &\leq b(q). \end{aligned}$$

Also, as  $b(q) \leq D(b)(q)$ , whereby  $D(b) = b$ . Then  $b$  is  $\tau(Q)$ -open.

(ii)  $\rightarrow$  (iii) Let  $b$  be  $\tau(Q)$ -open. Then  $b \otimes E \leq b$ . Also, from the reflexivity of  $E$ ,  $b \leq b \otimes E$ . Thus  $b \otimes E = b$ , or that  $b$  is a solution of  $L$ -valued relation equation  $\chi \otimes E = \chi$ .

(iii)  $\rightarrow$  (i) is trivial.  $\square$

**Proposition 3.45.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . Then the following are equivalent:

i)  $b \in B$  is an  $L$ -valued predecessor subproposition of  $\tilde{F}$ .

ii)  $b \in B$  is  $\tau^*(Q)$ -open.

iii)  $b \in B$  is a solution to an  $L$ -valued relation equation  $\chi \otimes E = \chi$ , where  $\chi \in B$  is an unknown.

*Proof.* It is similar to the above mentioned lines.  $\square$

**Proposition 3.46.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . Then;

i) if  $b \in B$  is an  $L$ -valued predecessor subproposition, then  $\neg b$  is an  $L$ -valued successor subproposition.

ii) if  $b \in B$  is an  $L$ -valued successor subproposition, then  $\neg b$  is an  $L$ -valued predecessor subproposition.

*Proof.* (i) Let  $b$  be an  $L$ -valued predecessor subproposition of  $\tilde{F}$ . Then

$$\begin{aligned} b(q) &\leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(p) \mid p \in Q_{pred}(q, u), u \in \Sigma \} \\ \Rightarrow b(q) &\leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(p), p \in Q_{pred}(q, u), u \in \Sigma \\ \Rightarrow b(q) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) &\leq b(p), p \in Q_{pred}(q, u), u \in \Sigma \\ \Rightarrow \neg(b(q) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha)) &\geq \neg b(p), p \in Q_{pred}(q, u), u \in \Sigma \\ \Rightarrow (b(q) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha)) \rightarrow 0 &\geq \neg b(p), p \in Q_{pred}(q, u), u \in \Sigma \\ \Rightarrow (\tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \otimes b(q)) \rightarrow 0 &\geq \neg b(p), p \in Q_{pred}(q, u), u \in \Sigma \\ \Rightarrow \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow (b(q) \rightarrow 0) &\geq \neg b(p), p \in Q_{pred}(q, u), u \in \Sigma \\ \Rightarrow \neg b(p) \leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow \neg b(q), &q \in Q_{succ}(p, u), u \in \Sigma \\ \Rightarrow \neg b(p) \leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow \neg b(q) \mid q \in Q_{succ}(p, u), &u \in \Sigma \}. \end{aligned}$$

Then  $\neg b$  is an  $L$ -valued successor subproposition of  $\tilde{F}$ .

(ii) The proof is similar to what has been explained before.  $\square$

**Proposition 3.47.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ . Then  $b \in B$  is an  $L$ -valued successor subproposition of  $\tilde{F}$  if and only if  $b : (Q, E) \rightarrow (L, \rightarrow)$  is an preserving map.

*Proof.* Let  $b$  be an  $L$ -valued successor subproposition of  $\tilde{F}$ . Then

$$\begin{aligned} b(p) &\leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q) \mid q \in Q_{succ}(p, u), u \in \Sigma \} \\ \Rightarrow b(p) &\leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q), q \in Q_{succ}(p, u), u \in \Sigma \end{aligned}$$

$$\begin{aligned} &\Rightarrow b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ &\Rightarrow \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \otimes b(p) \leq b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ &\Rightarrow \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq b(p) \rightarrow b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ &\Rightarrow \mathbf{1}(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq b(p) \rightarrow b(q), p \in Q_{pred}(q, u), u \in \Sigma \\ &\Rightarrow \vee \{ \mathbf{1}(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) | p \in Q_{pred}(q, u), u \in \Sigma \} \leq b(p) \rightarrow b(q) \\ &\Rightarrow E(p, q) \leq b(p) \rightarrow b(q). \end{aligned}$$

Thus  $b : (Q, E) \rightarrow (L, \rightarrow)$  preserves order. Converse the following similarly.  $\square$

**Proposition 3.48.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton and  $\alpha \in B$ .  $b \in B$  is an  $L$ -valued predecessor subproposition of  $\tilde{F}$  if and only if  $b : (Q, E^*) \rightarrow (L, \rightarrow)$  is an order preserving map.

*Proof.* Similar to that of Proposition 3.47.  $\square$

**Proposition 3.49.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton,  $b \in B$  be an  $L$ -valued successor (predecessor) subproposition of  $\tilde{F}$  and  $\alpha \in B$ . Then for each  $a \in L$ ,  $b \rightarrow a$  is an  $L$ -valued predecessor (successor) subproposition of  $\tilde{F}$ .

*Proof.* Let  $b$  be an  $L$ -valued successor subproposition of  $\tilde{F}$ . Then

$$\begin{aligned} &b(p) \leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q) | q \in Q_{succ}(p, u), u \in \Sigma \} \\ &\Rightarrow b(p) \leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ &\Rightarrow b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq b(q), q \in Q_{succ}(p, u), u \in \Sigma. \end{aligned}$$

To show that  $b \rightarrow a$  is an  $L$ -valued predecessor subproposition of  $\tilde{F}$  it is enough to show that  $(b(q) \rightarrow a) \leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow (b(p) \rightarrow a), p \in Q_{pred}(q, u), u \in \Sigma$ , or that  $(b(q) \rightarrow a) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \otimes b(p) \leq a, p \in Q_{pred}(q, u), u \in \Sigma$ . Now,  $(b(q) \rightarrow a) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \otimes b(p) \leq (b(q) \rightarrow a) \otimes b(q) \leq a$ . Thus  $b \rightarrow a$  is an  $L$ -valued predecessor subproposition of  $\tilde{F}$ . Similarly, it can be prove that if  $b$  is an  $L$ -valued predecessor subproposition of  $\tilde{F}$ , then for each  $a \in L, b \rightarrow a$  is an  $L$ -valued successor subproposition.  $\square$

**Proposition 3.50.** Let  $\tilde{F}$  be an  $L^B$ -valued general fuzzy automaton,  $b \in B$  be an  $L$ -valued successor (predecessor) subproposition of  $\tilde{F}$  and  $\alpha \in B$ . Then for each  $a \in L, a \otimes b$  is an  $L$ -valued successor (predecessor) subproposition of  $\tilde{F}$ .

*Proof.* Let  $b$  be an  $L$ -valued successor subproposition of  $\tilde{F}$  and  $a \in L$ . Then

$$\begin{aligned} &b(p) \leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q) | q \in Q_{succ}(p, u), u \in \Sigma \} \\ &\Rightarrow b(p) \leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ &\Rightarrow b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ &\Rightarrow a \otimes b(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq a \otimes b(q), q \in Q_{succ}(p, u), u \in \Sigma \\ &\Rightarrow (a \otimes b(p)) \leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow (a \otimes b(q)), q \in Q_{succ}(p, u), u \in \Sigma \\ &\Rightarrow (a \otimes b(p)) \leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow (a \otimes b(q)) | q \in Q_{succ}(p, u), u \in \Sigma \}. \end{aligned}$$

Thus  $a \otimes b$  is an  $L$ -valued successor subproposition of  $\tilde{F}$ . The proof for the case of  $L$ -valued predecessor subproposition of  $\tilde{F}$  follows similarly.  $\square$

**Definition 3.51.** Let  $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$  and  $\tilde{F}' = (Q', \Sigma, \tilde{R}', Z, \tilde{\delta}', \omega', F_1, F_2)$  be two  $L^A$ -valued and  $L^B$ -valued general fuzzy automata, where  $A$  and  $B$  are regarded as a complete infinitely distributive lattice of propositions about the general fuzzy automata  $\tilde{F}$  and  $\tilde{F}'$ , respectively. Also, let  $E$  and  $H$  be  $L$ -valued relation on  $Q$  and  $Q'$ . A map  $f : (Q, E) \rightarrow (Q', H)$  between  $L$ -valued general fuzzy automata is called order preserving if  $E(p, q) \leq H(f(p), f(q)), \forall p, q \in Q$ .

**Proposition 3.52.** Let  $\tilde{F}$  and  $\tilde{F}'$  be two  $L^A$ -valued and  $L^B$ -valued general fuzzy automata and  $f : (Q, E) \rightarrow (Q', H)$  be order preserving. Then the inverse image of  $L$ -valued successor (predecessor) subproposition of  $\tilde{F}'$  is an  $L$ -valued successor (predecessor) subproposition of  $\tilde{F}$ .

*Proof.* Let  $p \in Q_{pred}(q, u), \alpha \in A$  and  $\lambda \in B$  be an  $L$ -valued successor subproposition of  $\tilde{F}'$ . To show that  $f^{-1}(\lambda)$  is an  $L$ -valued successor subproposition of  $\tilde{F}$  it is enough to show that

$$\begin{aligned}
 & f^{-1}(\lambda)(p) \leq \wedge \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow f^{-1}(\lambda)(q) \mid q \in Q_{succ}(p, u), u \in \Sigma \} \\
 & \Rightarrow f^{-1}(\lambda)(p) \leq \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \rightarrow f^{-1}(\lambda)(q), q \in Q_{succ}(p, u), u \in \Sigma \\
 & \Rightarrow f^{-1}(\lambda)(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \leq f^{-1}(\lambda)(q), q \in Q_{succ}(p, u), u \in \Sigma \\
 & \Rightarrow f^{-1}(\lambda)(p) \otimes (\vee \{ \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \mid q \in Q_{succ}(p, u), u \in \Sigma \}) \leq f^{-1}(\lambda)(q) \\
 & \Rightarrow f^{-1}(\lambda)(p) \otimes (\vee \{ \mathbf{1}(p) \otimes \tilde{\delta}((p, \mu^t(p)), u, q)(\alpha) \mid p \in Q_{pred}(q, u), u \in \Sigma \}) \leq f^{-1}(\lambda)(q) \\
 & \Rightarrow f^{-1}(\lambda)(p) \otimes E(p, q) \leq f^{-1}(\lambda)(q). \\
 & \text{Now, } f^{-1}(\lambda)(p) \otimes E(p, q) \leq \lambda(f(p)) \otimes E(p, q) \leq \lambda(f(p)) \otimes H(f(p), f(q)) \leq \lambda(f(q)) \leq f^{-1}(\lambda)(q).
 \end{aligned}$$

Thus  $f^{-1}(\lambda)$  is an  $L$ -valued subproposition of  $\tilde{F}$ .  $\square$

**Proposition 3.53.** *Let  $\tilde{F}$  and  $\tilde{F}'$  be two  $L^A$ -valued and  $L^B$ -valued general fuzzy automata. If  $f : \tilde{F} \rightarrow \tilde{F}'$  is homomorphism, then  $f : (Q, \tau(Q)) \rightarrow (Q', \tau'(Q'))$  is  $L$ -valued continuous.*

*Proof.* It follows from Propositions 3.44, 3.45 and 3.52.  $\square$

#### 4. Conclusion

This study was an attempt to establish the relationships among  $L$ -valued successor/predecessor /residuated/ approximation operator,  $L$ -valued topology/co-topology, and  $L^B$ -valued general fuzzy automata. It also showed that the continuity and separation properties of such  $L^B$ -valued general fuzzy automaton could be examined in terms of those topologies. Moreover in this study,  $L^B$ -valued GFA structure space and  $L^B$ -valued GFA homotopy were scrutinized in more details. The discussions presented here offer hope for some new insights in quantum computation, specifically, in the family of  $L$ -valued topology/co-topology. Possibly, much more investigations can be done than what has been suggested in this study. For example, it may be possible to introduce an  $L$ -valued topology on the state-set of product of two  $L^B$ -valued general fuzzy automata. Also, the decompositions of an  $L^B$ -valued general fuzzy automaton can be proposed and it will be interesting to set that up to which extent these concepts depend on the distributivity of associated lattice.

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