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Norm Bounds for the Inverse and Error Bounds for Linear Complementarity Problems for $\{P_1, P_2\}$-Nekrasov Matrices

M. Nedović[a], Lj. Cvetković[b]

[a] *Department for Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21000 Novi Sad, Serbia.*
[b] *Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia.*

Abstract. $\{P_1, P_2\}$-Nekrasov matrices represent a generalization of Nekrasov matrices via permutations. In this paper, we obtained an error bound for linear complementarity problems for $\{P_1, P_2\}$-Nekrasov matrices. Numerical examples are given to illustrate that new error bound can give tighter results compared to already known bounds when applied to Nekrasov matrices. Also, we presented new max-norm bounds for the inverse of $\{P_1, P_2\}$-Nekrasov matrices in the block case, considering two different types of block generalizations. Numerical examples show that new norm bounds for the block case can give tighter results compared to already known bounds for the point-wise case.

1. Introduction

The paper is organized as follows.

In Section 1, preliminaries on Nekrasov and $\{P_1, P_2\}$-Nekrasov matrices are recalled. In Section 2 we recall known bounds for the norm of the inverse of $\{P_1, P_2\}$-Nekrasov matrices and we apply them to obtain new max-norm bounds for two different types of block-generalizations. In Section 3, known error bounds for linear complementarity problems are recalled and a new error bound for linear complementarity problems is given when a matrix involved is $\{P_1, P_2\}$-Nekrasov. In Section 4, new results are illustrated with numerical examples. In Section 5, final conclusions and remarks are given, together with possible directions for future research.

A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an SDD matrix if, for each $i \in N$, it holds that

$$|a_{ii}| > r_i(A) = \sum_{k \in N, k \neq i} |a_{ik}|,$$

or, in other words, $d(A) > r(A)$, where $r(A) = [r_1(A), ..., r_n(A)]^T$ is the vector of deleted row sums, and the vector of moduli of diagonal entries is denoted by $d(A) = [|a_{11}|, ..., |a_{nn}|]^T$. Matrix classes that we deal with in this paper (SDD matrices, Nekrasov matrices, $\{P_1, P_2\}$-Nekrasov matrices) are all subclasses of

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Email addresses: maja.nedovic@uns.ac.rs (M. Nedović), lila@dmi.uns.ac.rs (Lj. Cvetković)

(nonsingular) H-matrices. It is well-known that a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix if its comparison matrix $\langle A\rangle = [m_{ij}]$ defined by

$$\langle A\rangle = [m_{ij}] \in \mathbb{C}^{n,n}, \qquad m_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is an M-matrix, i.e., $\langle A\rangle^{-1} \geq 0$. According to [16], a matrix A is an H-matrix if and only if there exists a diagonal nonsingular matrix (with positive diagonal entries) W such that AW is an SDD matrix. The diagonal matrix W is called a scaling matrix for the given matrix A. For some subclasses of H-matrices we know how to construct a corresponding scaling matrix, which can be used further in obtaining eigenvalue localization, investigation of Schur complement properties, see [8, 10, 11, 31, 33], or in determining error bounds for linear complementarity problems, see [18]. As stated in [1], for any nonsingular H-matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $|A^{-1}| \leq \langle A\rangle^{-1}$.

Nekrasov matrices, see [20, 26], are defined by condition

$$|a_{ii}| > h_i(A), \quad \text{for all } \ i \in N, \tag{1}$$

or, in vector form, $d(A) > h(A)$, where the sums $h_i(A)$, $i \in N$ are defined recursively by

$$h_1(A) = r_1(A), \quad h_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \ \ i = 2, 3, \ldots n, \tag{2}$$

and $h(A) = [h_1(A), ..., h_n(A)]^T$. For a given matrix A, $A = D - L - U$ represents the standard splitting of A into its diagonal (D), strictly lower ($-L$) and strictly upper ($-U$) triangular parts.

Note that the SDD class is closed under permutations of rows and columns, but the class of Nekrasov matrices is not. This is why the following generalization of Nekrasov matrices was introduced. Given a permutation matrix, P, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called P-Nekrasov, if P^TAP is a Nekrasov matrix, i.e., if $|(P^TAP)_{ii}| > h_i(P^TAP)$, for all $i \in N$, or, in other words, $d(P^TAP) > h(P^TAP)$. The union of all P-Nekrasov classes by permutation matrices P is known as Gudkov class, see [20, 30].

Suppose that for the given matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ and two given permutation matrices P_1 and P_2,

$$d(A) > \min\left\{h^{P_1}(A), h^{P_2}(A)\right\}, \tag{3}$$

where

$$h^{P_k}(A) = P_k h(P_k^T A P_k), \ \ k = 1, 2.$$

We call such a matrix $\{P_1, P_2\}$-Nekrasov matrix, see [9].

Examples show, see [9], that it is possible that, for the set of two given permutation matrices $\{P_1, P_2\}$, the given matrix A is neither P_1-Nekrasov nor P_2-Nekrasov, but A does belong to $\{P_1, P_2\}$-Nekrasov class.

Lemma 1.1 ([9]). *Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, with $a_{ii} \neq 0$ for all $i \in N$, and given a permutation matrix, $P \in \mathbb{C}^{n,n}$, then*

$$h_i^P(A) = |a_{ii}| \left(P(|\tilde{D}| - |\tilde{L}|)^{-1} |\tilde{U}| e\right)_i,$$

where $e \in \mathbb{C}^n$ is the vector with all components equal to 1 and $(\tilde{D})$ is diagonal, $(-\tilde{L})$ strictly lower and $(-\tilde{U})$ strictly upper triangular part of the matrix P^TAP, i.e., $P^TAP = \tilde{D} - \tilde{L} - \tilde{U}$ is the standard splitting of the matrix P^TAP.

In [9] it is shown that a $\{P_1, P_2\}$-Nekrasov matrix is always a nonsingular H-matrix. In the same paper, two norm bounds for the inverse matrix of a $\{P_1, P_2\}$-Nekrasov matrix are given.

2. Max norm bounds for $\{P_1, P_2\}$-Nekrasov matrices from point-wise to the block case

2.1. Norm bounds for the point-wise case

The perturbation theory and the analysis of ill–conditioned matrices are widely applied in engineering. The condition number shows how "ill" a matrix could be. As the condition number is determined in the following way,

$$\kappa(A) = \|A\|\|A^{-1}\|,$$

as the product of a matrix norm and a norm of the inverse matrix, it is very useful to determine the upper bound for the norm of the inverse matrix. In [32], a max-norm bound is given for the inverse of strictly diagonally dominant (SDD) matrices.

Theorem 2.1 ([32]). *Given an SDD matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ the following bound applies,*

$$\|A^{-1}\|_\infty \leq \frac{1}{\min\limits_{i\in N} (|a_{ii}| - r_i(A))}.$$

This result of Varah was the basis for obtaining bounds for maximum norm of the inverse matrix for matrices belonging to different matrix classes, see [6, 7, 21, 27].

Theorem 2.2 ([22]). *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a Nekrasov matrix. Then*

$$\|A^{-1}\|_\infty \leq \max_{i\in N} \frac{z_i(A)}{|a_{ii}| - h_i(A)},$$

where $z_1(A) = 1$ and $z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1$, $i = 2, 3, \ldots n$.

In the following two theorems we recall two known norm bounds for $\{P_1, P_2\}$-Nekrasov matrices, as presented in [9].

Theorem 2.3 ([9]). *Suppose that, for a given set of permutation matrices $\{P_1, P_2\}$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is a $\{P_1, P_2\}$-Nekrasov matrix. Then,*

$$\|A^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \left(\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|} \right)}{\min\limits_{i\in N} \left(1 - \min\left\{ \frac{h_i^{P_1}(A)}{|a_{ii}|}, \frac{h_i^{P_2}(A)}{|a_{ii}|} \right\} \right)}, \tag{4}$$

where

$$z_1(A) = 1, \quad z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \quad i = 2, 3, \ldots n,$$

the corresponding vector is $z(A) = [z_1(A), ..., z_n(A)]^T$, $z^P(A) = Pz(P^T AP)$, and for the given $i \in N$ the corresponding index $k_i \in \{1, 2\}$ is chosen in such way that

$$\min\left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} = h_i^{P_{k_i}}(A).$$

Theorem 2.4 ([9]). *Suppose that, for a given set of permutation matrices $\{P_1, P_2\}$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is a $\{P_1, P_2\}$-Nekrasov matrix. Then,*

$$\|A^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \left(z_i^{P_{k_i}}(A) \right)}{\min\limits_{i\in N} \left(|a_{ii}| - \min\left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} \right)}, \tag{5}$$

where

$$z_1(A) = 1, \;\; z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \;\; i = 2, 3, \ldots n,$$

the corresponding vector is $z(A) = [z_1(A), ..., z_n(A)]^T$, $z^P(A) = Pz(P^TAP)$, *and for the given* $i \in N$ *the corresponding index* $k_i \in \{1, 2\}$ *is chosen in such way that*

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

In [34] the improved norm bound for this type of matrices is given as follows.

Theorem 2.5 ([34]). *Suppose that, for a given set of permutation matrices* $\{P_1, P_2\}$, *a matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *is a* $\{P_1, P_2\}$*-Nekrasov matrix. Then,*

$$\|A^{-1}\|_\infty \leq \max_{i \in N} \frac{\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|}}{1 - \min\left\{\frac{h_i^{P_1}(A)}{|a_{ii}|}, \frac{h_i^{P_2}(A)}{|a_{ii}|}\right\}}, \tag{6}$$

where

$$z_1(A) = 1, \;\; z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \;\; i = 2, 3, \ldots n,$$

the corresponding vector is $z(A) = [z_1(A), ..., z_n(A)]^T$, $z^P(A) = Pz(P^TAP)$, *and for the given* $i \in N$ *the corresponding index* $k_i \in \{1, 2\}$ *is chosen in such way that*

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

2.2. *Maximum norm bounds for the inverse of block matrices*

In this subsection, we considered the norm bounds in the block-case. New upper bounds for the block $\{P_1, P_2\}$-Nekrasov matrices of types I and II are defined.

Block generalizations have wide applications as they provide a useful tool when dealing with mathematical models given as matrices of great dimensions. Partitioning a given matrix into blocks reduces the dimension of the problem. Block H-matrices were researched in [29].

For a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and a partition $\pi = \{p_j\}_{j=0}^{l}$, $p_0 = 0 < p_1 < p_2 < ... < p_l = n$, of the index set N, one can present A in the block form as $[A_{ij}]_{l \times l}$. The given partition π defines the partition of $\mathbb{C}^n$ into a direct sum of subspaces W_i, as follows.

$$\mathbb{C}^n = W_1 \oplus W_2 \oplus ... \oplus W_l,$$

where

$$W_j = span\{e^k \mid p_{j-1} + 1 \leq k \leq p_j\}, \quad j \in L = \{1, 2, ..., l\}.$$

Here, $\{e^k\}_{k=1}^{n}$ is the standard basis of $\mathbb{C}^n$.

For rectangular blocks, $\|A_{ij}\|_\infty$ is defined as follows:

$$\|A_{ij}\|_\infty = \sup_{x \in W_j,\, x \neq 0} \frac{\|A_{ij}x\|_\infty}{\|x\|_\infty} = \sup_{\|x\|_\infty = 1} \|A_{ij}x\|_\infty.$$

Also, denote

$$(\|A_{ii}^{-1}\|_\infty)^{-1} = \inf_{x \in W_i,\, x \neq 0} \frac{\|A_{ii}x\|_\infty}{\|x\|_\infty}, \; i \in L,$$

where the last quantity is zero if A_{ii} is singular.

Now, let us recall two different ways of introducing the $l \times l$ comparison matrix for a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and a partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N.

The comparison matrix of type I is denoted by $\rangle A \langle^{\pi} = [p_{ij}]$, where

$$p_{ii} = (\|A_{ii}^{-1}\|_{\infty})^{-1}, \; p_{ij} = -\|A_{ij}\|_{\infty}, \; i, j \in L, \; i \neq j.$$

The comparison matrix of type II is denoted by $\langle A \rangle^{\pi} = [m_{ij}]$, where

$$m_{ij} = \left\{ \begin{array}{ll} 1, & i = j \textit{ and } \det A_{ii} \neq 0, \\ -\|A_{ii}^{-1} A_{ij}\|_{\infty}, & i \neq j \textit{ and } \det A_{ii} \neq 0, \\ 0, & \textit{otherwise}. \end{array} \right\}$$

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and a given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N we say that A is a block π H-matrix of type I if $\rangle A \langle^{\pi}$ is an H-matrix.

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and a given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N we say that A is a block π H-matrix of type II if $\langle A \rangle^{\pi}$ is an H-matrix.

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N and for a given set of two permutation matrices of the index set, $\{P_1, P_2\}$, we say that A is a block π $\{P_1, P_2\}$–Nekrasov matrix of type I if $\rangle A \langle^{\pi}$ is $\{P_1, P_2\}$–Nekrasov matrix.

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N and for a given set of two permutation matrices of the index set, $\{P_1, P_2\}$, we say that A is a block π $\{P_1, P_2\}$–Nekrasov matrix of type II if $\langle A \rangle^{\pi}$ is $\{P_1, P_2\}$–Nekrasov matrix.

In [5] the following results can be found.

Theorem 2.6 ([5]). *If $A = [A_{ij}]_{n \times n}$ is a block π H-matrix of type I and $\rangle A \langle^{\pi}$ is its comparison matrix of type I, then*

$$\|A^{-1}\|_{\infty} \leq \|(\rangle A \langle^{\pi})^{-1}\|_{\infty}.$$

Theorem 2.7 ([5]). *If $A = [A_{ij}]_{n \times n}$ is a block π H-matrix of type II and $\langle A \rangle^{\pi}$ is its comparison matrix of type II, then*

$$\|A^{-1}\|_{\infty} \leq \max_{i \in L} \|A_{ii}^{-1}\|_{\infty} \|(\langle A \rangle^{\pi})^{-1}\|_{\infty}.$$

Now we present two new upper bounds for the maximum norm of the inverse for block generalizations of $\{P_1, P_2\}$-Nekrasov matrices of type I and type II, respectively.

Theorem 2.8. *Suppose that, for a given set of permutation matrices $\{P_1, P_2\}$ and a given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is a block π $\{P_1, P_2\}$-Nekrasov matrix of type I. Then,*

$$\|A^{-1}\|_{\infty} \leq \max_{i \in L} \frac{z_i^{P_{k_i}}(\rangle A \langle^{\pi})}{\|A_{ii}^{-1}\|_{\infty}^{-1} - \min\left\{h_i^{P_1}(\rangle A \langle^{\pi}), h_i^{P_2}(\rangle A \langle^{\pi})\right\}} =$$

$$= \max_{i \in L} \frac{\|A_{ii}^{-1}\|_{\infty} z_i^{P_{k_i}}(\rangle A \langle^{\pi})}{1 - \min\left\{\|A_{ii}^{-1}\|_{\infty} h_i^{P_1}(\rangle A \langle^{\pi}), \|A_{ii}^{-1}\|_{\infty} h_i^{P_2}(\rangle A \langle^{\pi})\right\}},$$

where

$$z_1(A) = 1, \;\; z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \;\; i = 2, 3, \ldots n,$$

the corresponding vector is $z(A) = [z_1(A), ..., z_n(A)]^T$, $z^P(A) = Pz(P^T A P)$, and for the given $i \in N$ the corresponding index $k_i \in \{1, 2\}$ is chosen in such way that

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

Proof: From Theorem 2.6 and the fact that a block π $\{P_1, P_2\}$−Nekrasov matrix of type I is a block π H-matrix of type I, we know that

$$\|A^{-1}\|_\infty \leq \|(\rangle A\langle^{\pi})^{-1}\|_\infty.$$

From the definition of block π $\{P_1, P_2\}$−Nekrasov matrix of type I, we know that the comparison matrix $\rangle A\langle^{\pi}$ is a $\{P_1, P_2\}$−Nekrasov matrix. Therefore, we can apply the upper bound for the maximum norm of the inverse given in Theorem 2.5 to the matrix $\rangle A\langle^{\pi}$ and obtain

$$\|A^{-1}\|_\infty \leq \|(\rangle A\langle^{\pi})^{-1}\|_\infty \leq \max_{i\in L} \frac{z_i^{P_{k_i}}(\rangle A\langle^{\pi})}{\|A_{ii}^{-1}\|_\infty^{-1} - \min\left\{h_i^{P_1}(\rangle A\langle^{\pi}), h_i^{P_2}(\rangle A\langle^{\pi})\right\}} =$$

$$= \max_{i\in L} \frac{\|A_{ii}^{-1}\|_\infty z_i^{P_{k_i}}(\rangle A\langle^{\pi})}{1 - \min\left\{\|A_{ii}^{-1}\|_\infty h_i^{P_1}(\rangle A\langle^{\pi}), \|A_{ii}^{-1}\|_\infty h_i^{P_2}(\rangle A\langle^{\pi})\right\}},$$

where

$$z_1(A) = 1, \quad z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \quad i = 2, 3, \ldots n,$$

the corresponding vector is $z(A) = [z_1(A), ..., z_n(A)]^T$, $z^P(A) = Pz(P^TAP)$, and for the given $i \in N$ the corresponding index $k_i \in \{1, 2\}$ is chosen in such way that

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

□

Theorem 2.9. *Suppose that, for a given set of permutation matrices $\{P_1, P_2\}$ and a given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is a block π $\{P_1, P_2\}$-Nekrasov matrix of type II. Then,*

$$\|A^{-1}\|_\infty \leq \max_{i\in L} \|A_{ii}^{-1}\|_\infty \max_{i\in L} \frac{z_i^{P_{k_i}}(\langle A\rangle^{\pi})}{1 - \min\left\{h_i^{P_1}(\langle A\rangle^{\pi}), h_i^{P_2}(\langle A\rangle^{\pi})\right\}}, \tag{7}$$

where

$$z_1(A) = 1, \quad z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \quad i = 2, 3, \ldots n,$$

the corresponding vector is $z(A) = [z_1(A), ..., z_n(A)]^T$, $z^P(A) = Pz(P^TAP)$, and for the given $i \in N$ the corresponding index $k_i \in \{1, 2\}$ is chosen in such way that

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

Proof: In a similar manner as previous, the proof follows from Theorem 2.7, the fact that every block π $\{P_1, P_2\}$-Nekrasov matrix of type II is also a block π H-matrix of type II, the definition of block π $\{P_1, P_2\}$-Nekrasov matrix of type II and Theorem 2.5. □

Remark 2.10. *Upper bounds given in Theorem 2.8 and in Theorem 2.9 are always tighter than corresponding bounds given in [15].*

3. Linear complementarity problem

The linear complementarity problem (LCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, Ax + q \geq 0, (Ax + q)^T x = 0,$$

or to show that such vector doesn't exist. Here, $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and $q \in \mathbb{R}^n$. Many mathematical problems can be described in LCP form. It is well known that LCP(A, q) has a unique solution for any $q \in \mathbb{R}^n$ if and only if the matrix A is a $P-$matrix, a real square matrix with all its principal minors positive, see [4]. An $H-$matrix with positive diagonals is a $P-$matrix. In defining an upper error bound for LCP(A, q) where A is a $P-$matrix, the following fact can be a starting point, see [2]

$$\|x - x^*\|_\infty \leq \max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \|r(x)\|_\infty.$$

Here, x^* is a solution of the LCP(A, q), $r(x) = min(x, Ax + q)$, $D = diag(d_i), 0 \leq d_i \leq 1$, and the min operator denotes the componentwise minimum of two vectors. Obviously, the upper bound for the maximum norm of the inverse matrix of $I - D + DA$ plays an important role in determining LCP error bound.

If A is a certain structure matrix, more results on LCP(A, q) can be found in [3, 12–14, 17, 18, 23–25, 28, 35].

The following theorem gives a way of determining error bound for LCP when a matrix involved is a Nekrasov matrix with positive diagonal entries, by constructing a corresponding scaling matrix.

Theorem 3.1 ([18]). *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, be a Nekrasov matrix with $a_{ii} > 0$ for $i \in N$ such that for each $i = 1, 2, ..., n-1$, $a_{ij} \neq 0$ for some $j > i$. Let $W = diag(w_1, ..., w_n)$ with $w_i = \frac{h_i(A)}{a_{ii}}$ for $i = 1, 2, ..., n-1$ and $w_n = \frac{h_n(A)}{a_{nn}} + \varepsilon, \varepsilon \in (0, 1 - \frac{h_n(A)}{a_{nn}})$. Then*

$$\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq \left\{ \frac{\max_{i \in N} w_i}{\min_{i \in N} s_i}, \frac{\max_{i \in N} w_i}{\min_{i \in N} w_i} \right\},$$

where for each $i = 1, 2, ..., n-1, s_i = \sum_{j=i+1}^{n} |a_{ii}|(1 - w_j)$ and $s_n = \varepsilon a_{nn}$.

In [23] it is stated that this bound is not always effective and a new bound is proposed, based on the following results.

Lemma 3.2 ([23]). *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a matrix with $a_{ii} > 0$ for $i \in N$ and let $B = I - D + DA = [b_{ij}]$, where $D = diag(d_i)$ with $0 \leq d_i \leq 1$. Then*

$$z_i(B) \leq \eta_i(A),$$

and

$$\frac{z_i(B)}{b_{ii}} \leq \frac{\eta_i(A)}{\min(a_{ii}, 1)},$$

where

$$z_1(B) = \eta_1(A) = 1,\ z_i(B) = \sum_{j=1}^{i-1} \frac{|b_{ij}|}{|b_{jj}|} z_j(B) + 1,$$

and

$$\eta_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{\min(a_{jj}, 1)} \eta_j(A) + 1,\ i = 2, 3, ..., n.$$

Theorem 3.3 ([23]). *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, be a Nekrasov matrix with $a_{ii} > 0$ for $i \in N$. Then*

$$\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq \max_{i \in N} \frac{\eta_i(A)}{\min\{a_{ii} - h_i(A), 1\}}.$$

As said before, linear complementarity problems are studied by many researchers as these problems have wide applications. In many recent papers, error bounds for LCP are given, when a matrix involved is a matrix of a special structure. Here, we give error bound for LCP for $\{P_1, P_2\}$-Nekrasov matrices, which is useful in two different ways. First, our bound can work for some matrices that do not satisfy the Nekrasov condition. In other words, this bound can work in cases when bounds given for Nekrasov matrices ([18, 23]) cannot be applied. Second, even if a matrix is a Nekrasov matrix, our bound, in some cases, gives a tighter estimation.

First, let us consider the shifted matrix $B = I - D + DA$, where $D = diag(d_i)$ with $0 \le d_i \le 1$, and prove that the matrix B inherits $\{P_1, P_2\}$-Nekrasov property from the matrix A.

Theorem 3.4. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $n \ge 2$, $a_{ii} > 0$, $B = I - D + DA$, where $D = diag(d_i)$ with $0 \le d_i \le 1$. If for the set of two given permutations of the index set, $\{P_1, P_2\}$, A is a $\{P_1, P_2\}$-Nekrasov matrix, then B is also a $\{P_1, P_2\}$-Nekrasov matrix.*

Proof: Let $\tilde{A} = P^T AP = [\tilde{a}_{ij}]$, $\tilde{B} = P^T BP = [\tilde{b}_{ij}]$ and $\tilde{D} = P^T DP = diag[\tilde{d}]$, where P is a fixed permutation of the index set N. Then,

$$\tilde{b_{ii}} = 1 - \tilde{d}_i + \tilde{d}_i\tilde{a_{ii}},$$

$$\tilde{b_{ij}} = \tilde{d}_i\tilde{a_{ij}}, \ i \ne j.$$

It is easy to see that

$$|\tilde{b_{ii}}| = |1 - \tilde{d}_i + \tilde{d}_i\tilde{a_{ii}}| \ge \tilde{d}_i|\tilde{a_{ii}}|.$$

Also,

$$h_1(\tilde{B}) = r_1(\tilde{B}) = \tilde{d}_1 h_1(\tilde{A}).$$

Assume that

$$h_j(\tilde{B}) \le \tilde{d}_j h_j(\tilde{A}), \quad j = 1, 2, ..., i-1.$$

By induction we obtain

$$h_i(\tilde{B}) = \sum_{j=1}^{i-1} |\tilde{b_{ij}}| \frac{h_j(\tilde{B})}{|\tilde{b_{jj}}|} + \sum_{j=i+1}^{n} |\tilde{b_{ij}}| = \sum_{j=1}^{i-1} |\tilde{d}_i\tilde{a_{ij}}| \frac{h_j(\tilde{B})}{|1 - \tilde{d}_j + \tilde{d}_j\tilde{a_{jj}}|} + \sum_{j=i+1}^{n} \tilde{d}_i|\tilde{a_{ij}}| =$$

$$= \tilde{d}_i(\sum_{j=1}^{i-1} |\tilde{a_{ij}}| \frac{h_j(\tilde{B})}{|1 - \tilde{d}_j + \tilde{d}_j\tilde{a_{jj}}|} + \sum_{j=i+1}^{n} |\tilde{a_{ij}}|) \le \tilde{d}_i(\sum_{j=1}^{i-1} |\tilde{a_{ij}}| \frac{h_j(\tilde{A})}{|\tilde{a_{jj}}|} + \sum_{j=i+1}^{n} |\tilde{a_{ij}}|) = \tilde{d}_i h_i(\tilde{A}).$$

Therefore,

$$h_i(\tilde{B}) \le \tilde{d}_i h_i(\tilde{A}), \quad i = 1, 2, ..., n.$$

From these considerations, we conclude that, if $|\tilde{a_{ii}}| > h_i(\tilde{A})$, then

$$|\tilde{b_{ii}}| \ge \tilde{d}_i|\tilde{a_{ii}}| > \tilde{d}_i h_i(\tilde{A}) \ge h_i(\tilde{B}),$$

for $d_i > 0$, and also, if $d_i = 0$,

$$|\tilde{b_{ii}}| = 1 > 0 = h_i(\tilde{B}).$$

In other words, if A is a $\{P_1, P_2\}$-Nekrasov matrix, then, B is also a $\{P_1, P_2\}$-Nekrasov matrix. Notice also that

$$\frac{h_i(\tilde{B})}{|\tilde{b_{ii}}|} \le \frac{h_i(\tilde{A})}{|\tilde{a_{ii}}|}.$$

□

Now, applying the previous result, Lemma 3.2 and the norm bound from Theorem 2.5, we define a new error bound for LCP with $\{P_1, P_2\}$-Nekrasov matrix.

Theorem 3.5. *Suppose that, for a given set of permutation matrices* $\{P_1, P_2\}$*, a matrix* $A = [a_{ij}] \in \mathbb{R}^{n,n},\ n \geq 2,\ a_{ii} > 0,\ i = 1, ..., n,$ *is a* $\{P_1, P_2\}$*-Nekrasov matrix,* D *is a diagonal matrix* $D = diag[d],\ d = [d_1, ..., d_n]^T,\ 0 \leq d_i \leq 1,\ i = 1, ..., n.$ *Then,*

$$\|(I - D + DA)^{-1}\|_\infty \leq \max_{i \in N} \frac{\frac{\max(\eta_i^{P_1}(A), \eta_i^{P_2}(A))}{\min(a_{ii}, 1)}}{1 - \min\left\{\frac{h_i^{P_1}(A)}{a_{ii}}, \frac{h_i^{P_2}(A)}{a_{ii}}\right\}},$$

where $\eta^P(A) = P\eta(P^T AP)$.

Proof: Consider the matrix $B = I - D + DA,$ where, again, D is a diagonal matrix $D = diag[d],\ d = [d_1, ..., d_n]^T,\ 0 \leq d_i \leq 1,\ i = 1, ..., n.$ If the given matrix A is a $\{P_1, P_2\}$-Nekrasov matrix, then, B is also a $\{P_1, P_2\}$-Nekrasov matrix for the same set of permutation matrices $\{P_1, P_2\}$. Therefore, we can apply the upper bound from Theorem 2.5 to the norm of the inverse matrix of the matrix B, as follows.

$$\|B^{-1}\|_\infty \leq \max_{i \in N} \frac{\frac{z_i^{P_{k_i}}(B)}{|b_{ii}|}}{1 - \min\left\{\frac{h_i^{P_1}(B)}{|b_{ii}|}, \frac{h_i^{P_2}(B)}{|b_{ii}|}\right\}} \leq \max_{i \in N} \frac{\frac{\eta_i^{P_{k_i}}(A)}{\min(a_{ii}, 1)}}{1 - \min\left\{\frac{h_i^{P_1}(A)}{|a_{ii}|}, \frac{h_i^{P_2}(A)}{|a_{ii}|}\right\}},$$

because

$$\frac{h_i^{P_1}(B)}{|b_{ii}|} \leq \frac{h_i^{P_1}(A)}{|a_{ii}|}.$$

Here,

$$z_1(B) = 1,\ \ z_i(B) = \sum_{j=1}^{i-1} |b_{ij}| \frac{z_j(B)}{|b_{jj}|} + 1,\ \ i = 2, 3, \ldots n,$$

the corresponding vector is $z(B) = [z_1(B), ..., z_n(B)]^T$, $z^P(B) = Pz(P^T BP)$, and for the given $i \in N$ the corresponding index $k_i \in \{1, 2\}$ is chosen in such way that

$$\min\left\{h_i^{P_1}(B), h_i^{P_2}(B)\right\} = h_i^{P_{k_i}}(B).$$

As for $i \in N$ it holds that $\eta_i^{P_{k_i}}(A) \leq \max(\eta_i^{P_1}(A), \eta_i^{P_2}(A)),$ this proves the statement.
□

4. Numerical examples

Example 1. Consider the following matrix:

$$A_1 = \begin{pmatrix} 1 & \frac{-2}{5} & \frac{-2}{5} & 0 \\ \frac{-1}{2} & 1 & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-2}{5} & \frac{-2}{5} & 1 & 0 \\ \frac{-1}{5} & \frac{-2}{5} & \frac{-2}{5} & 1 \end{pmatrix}$$

Matrix A_1 is a Nekrasov matrix. This matrix was considered in [23]. Also, A_1 does satisfy the $\{P_1, P_2\}$-Nekrasov condition, where P_1 is the identical permutation of order 4 and P_2 is the following permutation matrix

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now, for the matrix A_1, we calculate error bounds for linear complementarity problem. The matrix is a Nekrasov matrix, but as $a_{34} = 0$ we cannot apply the bound from Theorem 3.1 (obtained in [18]). However, we can apply the error bound given in Theorem 3.3 (obtained in [23]) and our new error bound given in Theorem 3.5. The results are presented in the following table.

Error Bound LCP	Theorem 3.1 [18]	Theorem 3.3 [23]	Theorem 3.5
A_1	-	15	12.5

In other words, from Theorem 3.5 we have

$$\max_{d\in[0,1]^n} \|(I - D + DA_1)^{-1}\|_\infty \leq 12.5.$$

The bound is obtained using only the entries of the original matrix A_1. Obviously, although the matrix A_1 is a Nekrasov matrix, in this case via permutations we can obtain a tighter bound.

Example 2. Consider the following matrix:

$$A_2 = \begin{pmatrix} -4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -5 & 4 & 1 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 2 & -0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 12 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & -0.4 & 14 \end{pmatrix}$$

The matrix A_2 is not a Nekrasov matrix, but it is a $\{P_1, P_2\}$-Nekrasov matrix if P_1 is the identical permutation of order 10 and P_2 is counteridentical permutation of order 10. Also, for the partition $\pi = \{0, 5, 10\}$ this matrix is a block $\pi - \{P_1, P_2\}-$ Nekrasov matrix of type I and also a block $\pi - \{P_1, P_2\}-$ Nekrasov matrix of type II. For this matrix, the block-case gives better results for the max-norm bound of the inverse, as the following table shows.

Norm Bound	Theorem 2.5	Theorem 2.8 (Block I)	Theorem 2.9 (Block II)
A_2	5.03448	1.42236	1.614

The exact value of the norm of the inverse is $\|A_2^{-1}\|_\infty = 1.07622$.

Example 3. Consider the following matrix, A_3.

$$\begin{pmatrix}
0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 \\
2 & 0 & -2 & 1 & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 \\
1 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-3} \\
-1 & 1 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10^{-3} & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-3} \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 10^{-3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-3} & 10^{-3} & 2 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1.5 & 0 & 0 & 0 \\
10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 10^{-3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & -0.5 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0.5 & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 2 & 0 & 2 \\
0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}$$

In [19] it has been pointed out that self-interactions can be removed from the community matrix, modelling a continuous-time, dynamical ecological system composed of several populations, resting at a feasible equilibrium point. It means that the community matrix has a block form, while all its diagonal entries are equal to zero. Motivated by this, we consider the 21 by 21 matrix A_3. It is a hollow matrix, meaning that all the diagonal entries are equal to zero, therefore A_3 does not belong to the class of H-matrices (in the point-wise sense). But one can see that, for the partition π into 3 by 3 blocks, this matrix does belong to $\pi - \{P_1, P_2\}-$ Nekrasov matrices of type II. Permutations are chosen to be the identical and counter-identical permutation of order 7. For this matrix neither of the point-wise bounds can be applied, nor the matrix belongs to block-Nekrasov type. Bound obtained according to Theorem 2.9 is equal to 10.5111, while the exact value of the norm of the inverse is equal to 2.86631.

This example, involving a hollow matrix, is just an illustration showing that in some practical applications matrices of this type occur, and their hollow, but block-organized structure follows from the properties of the observed (ecological) system. For all these matrices, well-known results on bounding the norm of the inverse, developed for many different subclasses of H-matrices (point-wise), do not work. Therefore, we use the block approach.

5. Conclusions and remarks

In this paper we presented an error bound for LCP when the matrix involved is a $\{P_1, P_2\}-$Nekrasov matrix. The benefit of this error bound is twofold. First, for matrices that do not belong to the class of Nekrasov matrices, but do belong to $\{P_1, P_2\}-$Nekrasov class for some choice of permutations, our bounds can be applied, while the well-known error bounds for LCP with Nekrasov matrices cannot be applied. Second, if the given matrix is a Nekrasov matrix, our bound in some cases can work better than the bound in [23]. It is worth mentioning here that the bound given in [25] can also work better than that of [23] in some cases. This bound given in [25] is obtained from the result of [18] by optimization of the parameter involved.

In this paper, we also presented upper bounds for the norm of the inverse matrix, in the block case - when the matrix involved belongs to block $\{P_1, P_2\}-$Nekrasov class. We considered two types of block generalizations. The benefit of these results is the following - if the given matrix is large, but it has a certain pattern in the structure, the block approach gives a way to collect information on the original matrix by analyzing another (comparison) matrix of a smaller dimension. Moreover, it should be pointed out, that there are matrices that are not H-matrices in the point-wise sense (for instance, when there is a zero diagonal entry) but they do belong to block $\{P_1, P_2\}-$Nekrasov class, or some other subclass of block H-matrices. For such matrices, the well-known norm bounds given for many different subclasses of H-matrices (point-wise) cannot be applied. In these cases, we use the block-approach only.

In the block case it would be an interesting question to investigate the following two problems. First, if the given matrix A is in the specific subclass of block H-matrices, does it guarantee that the matrix $I-D+DA$ is in the same subclass, for any choice of diagonal matrix D with diagonal entries in $[0, 1]$? This question asks for further research on the subject. If the answer is yes, for some matrix class, then the results obtained for bounding the norm of the inverse matrix in the block case could be used for obtaining error bound for LCP. Second, we should have in mind that a matrix could be a block H-matrix, but not an H-matrix in the point-wise sense. In this paper, our bound for LCP is based on the starting assumption that the given matrix is an H-matrix in the point-wise sense with positive diagonal entries, which guarantees that it is a P-matrix. In the block-case, it demands further investigation.

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