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The Pseudo Core Inverses of Differences and Products of Projections in Rings with Involution

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Abstract. Let R be a unital ring with involution. The pseudo core inverses of differences and products of two projections are investigated in R. Some equivalent conditions are obtained. As applications, the pseudo core invertibility of the commutator $pq - qp$ and the anti-commutator $pq + qp$ are characterized, where p, q are projections in R.

1. Introduction

Throughout this paper, R is a unital ring with an involution $a \mapsto a^*$ satisfying $(a^*)^* = a, (a+b)^* = a^* + b^*, (ab)^* = b^*a^*$, for all $a, b \in R$. Recall that an element $a \in R$ is said to be Hermitian if $a = a^*$, an element $a \in R$ is said to be idempotent if $a = a^2$, and an element $a \in R$ is said to be a projection if a is Hermitian idempotent.

Let $a \in R$, then the Moore-Penrose inverse $a^\dagger$ of a is the unique solution, if it exists, of the system of equations

$$axa = a,\ xax = x,\ (ax)^* = ax,\ (xa)^* = xa.$$

We use $R^\dagger$ to denote the set of all Moore-Penrose invertible elements. An element $a \in R$ is said to be Drazin invertible [7] if there exists $x \in R$ such that

$$ax = xa, \quad xax = x, \quad a^k = a^{k+1}x,$$

for some positive integer k. The element x is unique if it exists and denoted by a^D. The smallest positive integer k satisfying above equations is called the Drazin index of a, denoted by $i(a)$. The set of all Drazin invertible elements in R is denoted by R^D.

In 2010, Baksalary and Trenkler [1] introduced the core inverse of a complex matrix. In 2014, Rakíc et al. [12] generalized this notion to the case of an element in a ring with involution. They proved that for $a \in R$, x

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is the core inverse of a if it satisfies the following five equations $axa = a,\ xax = x,\ (ax)^* = ax,\ ax^2 = x,\ xa^2 = a$. Such an element x is unique and denoted by $a^{\#}$ if it exists. Later Xu et al. [13] proved that these five equations can be reduced to three equations. That is to say, the core inverse $a^{\#}$ of a, if it exists, is the unique solution of equations

$$(ax)^* = ax,\ ax^2 = x,\ xa^2 = a.$$

The symbol $R^{\#}$ stands for the set of all the core invertible elements.

An element $a \in R$ is said to be pseudo core invertible [9] if there exists $x \in R$ such that

$$(ax)^* = ax, \quad ax^2 = x, \quad a^m = xa^{m+1},$$

for some positive integer m. The smallest positive integer m satisfying the above equations is called the pseudo core index of a, denoted by $I(a)$. Such x is unique if it exists and denoted by $a^{\mathbb{D}}$. The set of all pseudo core invertible elements in R is denoted by $R^{\mathbb{D}}$. If a is pseudo core invertible, then it must be Drazin invertible, and the pseudo core index coincides with the Drazin index [9]. It is obvious that if the pseudo core index is equal to 1, the pseudo core inverse of a is the core inverse of a.

Generalized inverses of differences and products of idempotents or projections in various settings attract wide attention from many authors [2, 3, 6, 8, 10, 11, 15]. For instance, Du and Deng [6] investigated the Moore-Penrose inverses of pq and $p - q$, where p, q are projections in the ring $B(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$. Li [11] studied the Moore-Penrose inverses of pq, $pq - qp$ and $pq + qp$ in C^*-algebra, where p, q are projections. Zhang et al. [15] investigated the Moore-Penrose inverses of the differences and products of projections in a ring with involution. Chen and Zhu [3] studied the Drazin inverses of the differences and products of idempotents in a ring. Li et al. [10] considered the core inverses of the differences and products of projections in a ring with involution.

Motivated by these papers, we investigate the pseudo core inverses of differences and products of two projections in a ring R with involution. Some equivalent conditions are obtained. In Section 2, we consider the pseudo core invertibility of $1 - qp$ when only p is a projection in R. In Section 3, when p, q are both projections in R, we investigate the pseudo core invertibility of $p - q$ and pq. As applications, pseudo core invertibility of the commutator $pq - qp$ and the anti-commutator $pq + qp$ are characterized.

2. The pseudo core inverse of $1 - qp$

For any two projections $p, q \in R$, Zhang et al. [15] proved that $1 - qp \in R^{\dagger}$ if and only if $1 - pqp \in R^{\dagger}$. Li et al. [10] showed that $1 - qp \in R^{\#}$ if and only if $1 - pqp \in R^{\#}$. In this section, We will show that $1 - qp \in R^{\mathbb{D}}$ if and only if $1 - pqp \in R^{\mathbb{D}}$ when only p is a projection. The following lemmas will be used in the sequel.

Lemma 2.1. [5, Theorem 3.6] *If $a, b \in R$. Then $1 - ab \in R^D$ if and only if $1 - ba \in R^D$. Furthermore, $i(1 - ab) = i(1 - ba)$.*

Lemma 2.2. [9, Theorem 4.3] *Let $a, b \in R^{\mathbb{D}}$ with $ab = ba,\ ab^* = b^*a$. Then $ab \in R^{\mathbb{D}}$ with $(ab)^{\mathbb{D}} = a^{\mathbb{D}}b^{\mathbb{D}} = b^{\mathbb{D}}a^{\mathbb{D}}$.*

Lemma 2.3. [9, Theorem 4.4] *Let $a, b \in R^{\mathbb{D}}$ with $ab = ba = 0, a^*b = 0$. Then $a + b \in R^{\mathbb{D}}$ with $(a + b)^{\mathbb{D}} = a^{\mathbb{D}} + b^{\mathbb{D}}$.*

Lemma 2.4. *Let $a, b \in R^{\mathbb{D}}$ and $p \in R$ is a projection. If $ap = pa$ and $bp = pb$, then $ap + b(1 - p) \in R^{\mathbb{D}}$ and*

$$(ap + b(1 - p))^{\mathbb{D}} = a^{\mathbb{D}}p + b^{\mathbb{D}}(1 - p).$$

Proof. Since p is a projection, we have $p \in R^{\mathbb{D}}$ and $p^{\mathbb{D}} = p$. From $ap = pa, ap^* = p^*a, bp = pb$ and $bp^* = p^*b$, we obtain $(ap)^{\mathbb{D}} = a^{\mathbb{D}}p$ and $(b(1 - p))^{\mathbb{D}} = b^{\mathbb{D}}(1 - p)$ by Lemma 2.2. Note that $apb(1 - p) = b(1 - p)ap = 0$ and $(ap)^*b(1 - p) = pa^*b(1 - p) = a^*pb(1 - p) = 0$, which gives that $(ap + b(1 - p))^{\mathbb{D}} = a^{\mathbb{D}}p + b^{\mathbb{D}}(1 - p)$ by Lemma 2.3. □

Let p be an idempotent, if $a \in R$, then a can be represented as

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p,$$

where $a_1 = pap$, $a_2 = pa(1-p)$, $a_3 = (1-p)ap$ and $a_4 = (1-p)a(1-p)$.

If p is a projection, then

$$a^* = \begin{bmatrix} a_1^* & a_3^* \\ a_2^* & a_4^* \end{bmatrix}_p.$$

Theorem 2.5. *Let $p \in R$ be a projection, $x = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}_p \in R$. The following statements are equivalent:*

(1) $x \in R^{\circledD}$, (2) $a \in (pRp)^{\circledD}$, (3) $a \in R^{\circledD}$.

In this case,

$$x^{\circledD} = \begin{bmatrix} a^{\circledD} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Proof. (1) $\Rightarrow$ (2). Suppose that $x \in R^{\circledD}$ with $I(x) = m$ and

$$x^{\circledD} = \begin{bmatrix} u & v \\ r & s \end{bmatrix}_p.$$

The equality $x^{\circledD} = x(x^{\circledD})^2$ implies that $r = 0$, $s = 0$ and $au^2 = u$. Since

$$xx^{\circledD} = \begin{bmatrix} au & av \\ 0 & 0 \end{bmatrix}_p$$

is Hermitian, we have that au is Hermitian and $av = 0$. By [9, Lemma 2.1],

$$x^{\circledD} = x^{\circledD}xx^{\circledD} = \begin{bmatrix} uau & 0 \\ 0 & 0 \end{bmatrix}_p,$$

we deduce that $v = 0$. From

$$\begin{bmatrix} ua^{m+1} & ua^m b \\ 0 & 0 \end{bmatrix}_p = x^{\circledD}x^{m+1} = x^m = \begin{bmatrix} a^m & a^{m-1}b \\ 0 & 0 \end{bmatrix}_p,$$

we get $ua^{m+1} = a^m$. Therefore, $a \in (pRp)^{\circledD}$ with $a^{\circledD} = u$.

(2) $\Rightarrow$ (1). Suppose that $a \in (pRp)^{\circledD}$. Then $(aa^{\circledD})^* = aa^{\circledD}$, $a(a^{\circledD})^2 = a^{\circledD}$ and $a^{\circledD}a^{m+1} = a^m$. Let $y = \begin{bmatrix} a^{\circledD} & 0 \\ 0 & 0 \end{bmatrix}_p$. It is easy to check that $xy = (xy)^*$, $xy^2 = y$ and $yx^{m+2} = x^{m+1}$. Therefore, $x \in R^{\circledD}$.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (2). From $a \in pRp$, we have $a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_p$. By $a \in R^{\circledD}$ and the equivalence of (1) and (2), we obtain $a \in (pRp)^{\circledD}$. □

Proposition 2.6. *Let $a \in R$ and $p \in R$ is a projection. Then $pa \in R^{\circledD}$ if and only if $pap \in R^{\circledD}$. In this case, $(pa)^{\circledD} = (pap)^{\circledD}$.*

Proof. Let p and a as

$$p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p \text{ and } a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p.$$

Then by Theorem 2.5, $pa = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_p \in R^{\circledD}$ if and only if $a_1 = pap \in R^{\circledD}$. Moreover, $(pa)^{\circledD} = (pap)^{\circledD}$. □

Remark 2.7. In Proposition 2.6, we cannot obtain $I(pa) = I(pap)$ in general.

For example: let $R = \mathbb{C}^{2\times 2}$ and the involution is the conjugate transpose. Let $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then p is a projection, $pa = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $pap = 0$. Thus, $pap \in R^{\textcircled{D}}$ and $pa \in R^{\textcircled{D}}$. But $1 = I(pap) \neq I(pa) = 2$.

We now present necessary and sufficient conditions for $1 - qp$ to be pseudo core invertible when only p is a projection.

Theorem 2.8. *Let $q \in R$ and $p \in R$ is a projection. The following statements are equivalent:*
(1) $1 - pqp \in R^{\textcircled{D}}$, (2) $p - pqp \in R^{\textcircled{D}}$, (3) $p - pq \in R^{\textcircled{D}}$, (4) $1 - qp \in R^{\textcircled{D}}$.
In this case,

$$(1 - pqp)^{\textcircled{D}} = (p - pqp)^{\textcircled{D}}p + 1 - p,$$

$$(1 - qp)^{\textcircled{D}} = (1 + qp - pqp)(p - pqp)^{\textcircled{D}} + 1 - p,$$

and

$$(p - pq)^{\textcircled{D}} = (p - pqp)^{\textcircled{D}} = (1 - pqp)^{\textcircled{D}}p = p(1 - pqp)^{\textcircled{D}} = p(1 - qp)^{\textcircled{D}}.$$

Proof. (1)⇒(2). Let $a = 1 - pqp, b = p$, then $a, b \in R^{\textcircled{D}}$. Since $ab = p - pqp = ba$ and $ab^* = b^*a$, we have $ab = p - pqp \in R^{\textcircled{D}}$ and $(p - pqp)^{\textcircled{D}} = (1 - pqp)^{\textcircled{D}}p = p(1 - pqp)^{\textcircled{D}}$ by Lemma 2.2.

(2)⇒(1). Let $a = p - pqp, b = 1$, then $a, b \in R^{\textcircled{D}}$. Note that $ap = pa$ and $bp = pb$. We have $ap + b(1 - p) = 1 - pqp \in R^{\textcircled{D}}$ and $(1 - pqp)^{\textcircled{D}} = (p - pqp)^{\textcircled{D}}p + 1 - p$ by Lemma 2.4.

(2)⇔(3). By Proposition 2.6, and in this case we have $(p - pqp)^{\textcircled{D}} = (p - pq)^{\textcircled{D}}$.

(4)⇒(2). Suppose that $1 - qp \in R^{\textcircled{D}}$ with $I(1 - qp) = m$ and $x = (1 - qp)^{\textcircled{D}}$. Next we prove that px is the pseudo core inverse of $p - pqp$.

$$\begin{aligned} & p(1 - qp)x - p(1 - qp)xp = p(1 - qp)x(1 - p) \\ & = p(1 - qp)x(1 - qp)^{m+1}(1 - p) = p(1 - qp)^{m+1}(1 - p) = p(1 - p) = 0. \end{aligned}$$

Hence $p(1 - qp)x = p(1 - qp)xp$ is Hermitian. Moreover,

$$px(1 - p) = px(1 - qp)^{m+1}(1 - p) = p(1 - qp)^m(1 - p) = p(1 - p) = 0.$$

And,

$$\begin{aligned} & p(1 - qp)^{m+1} = p(1 - qp)(1 - qp)^m = (p - pqp)(1 - qp)^m \\ & = (p - pqp)p(1 - qp)^m = (p - pqp)^2(1 - qp)^{m-1} = \cdots = (p - pqp)^{m+1}. \end{aligned}$$

Then we conclude that

(i) $(p - pqp)px = p(1 - qp)x$ is Hermitian.

(ii) $(p - pqp)(px)^2 = (p - pqp)pxpx = p(1 - qp)xpx = p(1 - qp)x^2 = px$.

(iii) $\begin{aligned}[t] & px(p - pqp)^{m+1} = pxp(1 - qp)^{m+1} = px(p - 1 + 1)(1 - qp)^{m+1} \\ & = px(1 - qp)^{m+1} - px(1 - p)(1 - qp)^{m+1} \\ & = px(1 - qp)^{m+1} = p(1 - qp)^m = (p - pqp)^m. \end{aligned}$

Therefore, $p - pqp \in R^{\textcircled{D}}$ and $(p - pqp)^{\textcircled{D}} = p(1 - qp)^{\textcircled{D}}$.

(2)⇒(4). Let $p - pqp \in R^{\textcircled{D}}$ with $I(p - pqp) = m$ and

$$x = (1 + b)(p - pqp)^{\textcircled{D}} + 1 - p$$

where $b = qp - pqp$. Next we prove that x is the pseudo core inverse of $1 - qp$.

Note that $(p - pqp)^{\textcircled{D}} \in pRp$, $pb = 0$, and

$$\begin{aligned}(1 - qp)(1 + b) &= (1 - qp)(1 + qp - pqp)\\ &= 1 + qp - pqp - qp - qpqp + qpqp\\ &= 1 - pqp.\end{aligned}$$

Then

$$\begin{aligned}(1 - qp)x &= (1 - qp)(1 + b)(p - pqp)^{\textcircled{D}} + (1 - qp)(1 - p)\\ &= (1 - pqp)(p - pqp)^{\textcircled{D}} + 1 - p\\ &= (1 - pqp)p(p - pqp)^{\textcircled{D}} + 1 - p\\ &= (p - pqp)(p - pqp)^{\textcircled{D}} + 1 - p,\end{aligned}$$

thus $(1 - qp)x$ is Hermitian. Since

$$\begin{aligned}&[(p - pqp)(p - pqp)^{\textcircled{D}} - p]x\\ &= [(p - pqp)(p - pqp)^{\textcircled{D}} - p](1 + b)(p - pqp)^{\textcircled{D}} + [(p - pqp)(p - pqp)^{\textcircled{D}} - p](1 - p)\\ &= [(p - pqp)(p - pqp)^{\textcircled{D}} - p](p - pqp)^{\textcircled{D}}\\ &= (p - pqp)(p - pqp)^{\textcircled{D}}(p - pqp)^{\textcircled{D}} - p(p - pqp)^{\textcircled{D}}\\ &= 0,\end{aligned}$$

we have

$$\begin{aligned}(1 - qp)x^2 &= [(p - pqp)(p - pqp)^{\textcircled{D}} + 1 - p]x\\ &= x + [(p - pqp)(p - pqp)^{\textcircled{D}} - p]x\\ &= x.\end{aligned}$$

Since $bp = b$, $(1 - p)(1 - qp) = 1 - p - b$ and $p(1 - qp)^{m+1} = (p - pqp)^{m+1}$,
it gives that

$$\begin{aligned}x(1 - qp)^{m+1} &= (1 + b)(p - pqp)^{\textcircled{D}}(1 - qp)^{m+1} + (1 - p)(1 - qp)^{m+1}\\ &= (1 + b)(p - pqp)^{\textcircled{D}}p(1 - qp)^{m+1} + (1 - p - b)(1 - qp)^{m}\\ &= (1 + b)(p - pqp)^{\textcircled{D}}(p - pqp)^{m+1} + (1 - p - b)(1 - qp)^{m}\\ &= (1 + b)(p - pqp)^{m} + (1 - p - b)(1 - qp)^{m}\\ &= [(1 + b)p + 1 - p - b](1 - qp)^{m}\\ &= (1 - qp)^{m}.\end{aligned}$$

Therefore, $1 - qp \in R^{\textcircled{D}}$ and $(1 - qp)^{\textcircled{D}} = (1 + qp - pqp)(p - pqp)^{\textcircled{D}} + 1 - p$. □

Substitute $1 - p$ for p in Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let $q \in R$ and $p \in R$ is a projection. The following statements are equivalent:*
(1) $1 - (1 - p)q(1 - p) \in R^{\textcircled{D}}$, (2) $(1 - p)(1 - q)(1 - p) \in R^{\textcircled{D}}$,
(3) $(1 - p)(1 - q) \in R^{\textcircled{D}}$, (4) $1 - q + qp \in R^{\textcircled{D}}$.

Remark 2.10. From the proof of Theorem 2.8, we obtain that $I(1 - qp) = I(p - pqp)$. Since the pseudo core index coincides with the Drazin index and by Lemma 2.1, we have that $I(1 - pqp) = I(1 - qp) = I(p - pqp)$. But $I(p - pqp) \neq I(p - pq)$ in general by Remark 2.7, where p is a projection.

From Theorem 2.8 and Remark 2.10, we have following result immediately.

Corollary 2.11. *Let $q \in R$ and $p \in R$ is a projection. The following statements are equivalent:*
(1) $1 - pqp \in R^{\textcircled{\#}}$, (2) $p - pqp \in R^{\textcircled{\#}}$, (3) $1 - qp \in R^{\textcircled{\#}}$.

Zhang et al. [15] proved that $1 - pqp \in R^{\dagger}$ if and only if $1 - qpq \in R^{\dagger}$ when p, q are projections. But we will show that it does not hold for the cases of pseudo core inverses when only p is a projection.

Remark 2.12. Let $q \in R$ and $p \in R$ is a projection. Then

(1) $1 - pqp \in R^{\textcircled{D}} \nRightarrow 1 - qpq \in R^{\textcircled{D}}$ or $1 - pq \in R^{\textcircled{D}}$ in general.

(2) $1 - pqp \in R^{\textcircled{D}} \nRightarrow q - qpq \in R^{\textcircled{D}}$ or $q - qp \in R^{\textcircled{D}}$ in general.

(3) $1 - pqp \in R^{\textcircled{D}} \nRightarrow p - qp \in R^{\textcircled{D}}$ in general.

(4) $1 - pqp \in R^{\textcircled{D}} \nRightarrow q - pq \in R^{\textcircled{D}}$ in general.

For example: let $R = \mathbb{C}^{2\times 2}$ and the involution is the transpose.

(1) Let $p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $q = \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix}$. Then we obtain $1 - pqp = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R^{\textcircled{D}}$, but $1 - pq = 1 - qpq = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} \notin R^{\textcircled{D}}$.

(2) Let $p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $q = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}$. Then we have $1 - pqp = 1 \in R^{\textcircled{D}}$, but $q - qp = q - qpq = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} \notin R^{\textcircled{D}}$.

(3) Let $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix}$. Then we obtain $1 - pqp = 1 \in R^{\textcircled{D}}$, but $p - qp = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} \notin R^{\textcircled{D}}$.

(4) Let $p = 0$, $q = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}$. Then we have $1 - pqp = 1 \in R^{\textcircled{D}}$, but $q - pq = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} \notin R^{\textcircled{D}}$.

3. The pseudo core inverse of $p - q$ and pq

In this section, we consider pseudo core inverses of $p - q$ and pq in a ring R when p and q are projections. As applications, pseudo core invertibility of the commutator $pq - qp$ and the anti-commutator $pq + qp$ are characterized. First let us look at the following lemmas.

Lemma 3.1. *Let $a \in R$.*

(1) *If $a^* = a$, then $a \in R^D$ if and only if $a \in R^{\textcircled{D}}$. In this case, $a^{\textcircled{D}} = a^D$.*

(2) *If $a^* = -a$, then $a \in R^D$ if and only if $a \in R^{\textcircled{D}}$. In this case, $a^{\textcircled{D}} = a^D$.*

Proof. (1). If $a^* = a$ and $a \in R^D$, then $(a^D)^* = (a^*)^D$, and $(aa^D)^* = (a^D)^* a^* = (a^*)^D a^* = a^D a = aa^D$, $a(a^D)^2 = a^D$, $a^D a^{m+1} = a^m$. Thus, $a \in R^{\textcircled{D}}$ and $a^{\textcircled{D}} = a^D$. If $a \in R^{\textcircled{D}}$, then $a \in R^D$ by [9, Theorem 2.3].

(2). If $a^* = -a$ and $a \in R^D$, then $(a^D)^* = (a^*)^D$, $(-a)^D = -a^D$ and $(aa^D)^* = (a^D)^* a^* = (a^*)^D a^* = (-a)^D(-a) = -a^D(-a) = aa^D$, $a(a^D)^2 = a^D$, $a^D a^{m+1} = a^m$. Thus, $a \in R^{\textcircled{D}}$ and $a^{\textcircled{D}} = a^D$. If $a \in R^{\textcircled{D}}$, then $a \in R^D$ by [9, Theorem 2.3]. $\square$

Lemma 3.2. [3, Proposition 3.1] *Let $p, q \in R$ be two idempotents. The following statements are equivalent:*
(1) $1 - qp \in R^D$, (2) $1 - pqp \in R^D$, (3) $p - pqp \in R^D$,
(4) $1 - pq \in R^D$, (5) $1 - qpq \in R^D$, (6) $q - qpq \in R^D$.

From Remark 2.12, we have $1 - pqp \in R^{\textcircled{D}}$ does not imply $1 - qpq \in R^{\textcircled{D}}$ in general when only p is a projection. Next we will show that they are equivalent when p, q are projections.

Theorem 3.3. *Let $p, q \in R$ be two projections. The following statements are equivalent:*
(1) $1 - qp \in R^{\circledD}$, (2) $1 - pqp \in R^{\circledD}$, (3) $p - pqp \in R^{\circledD}$,
(4) $1 - pq \in R^{\circledD}$, (5) $1 - qpq \in R^{\circledD}$, (6) $q - qpq \in R^{\circledD}$.
In this case, the pseudo core index of (1)–(6) are the same.

Proof. (1)⇔(2)⇔(3). By Theorem 2.8 we have (1)⇔(2)⇔(3). From Remark 2.10, we obtain that the pseudo core index of (1)–(3) are the same.

(4)⇔(5)⇔(6). Exchange p and q in (1)⇔(2)⇔(3). We also obtain that the pseudo core index of (4)–(6) are the same.

(2)⇔(5). Since both (2) and (5) are Hermitian, by Lemma 3.1, (2) is equivalent to $1 - pqp \in R^D$ and (5) is equivalent to $1 - qpq \in R^D$, therefore they are equivalent by Lemma 3.2. Since the pseudo core index coincides with the Drazin index and by Lemma 2.1, we obtain that (2) and (5) have the same pseudo core index. □

From Theorem 2.8 and Theorem 3.3, we have following corollary.

Corollary 3.4. [10, Theorem 3.4] *Let $p, q \in R$ be two projections. The following statements are equivalent:*
(1) $1 - qp \in R^{\#\!\!\!\bigcirc}$, (2) $1 - pqp \in R^{\#\!\!\!\bigcirc}$, (3) $p - pqp \in R^{\#\!\!\!\bigcirc}$,
(4) $1 - pq \in R^{\#\!\!\!\bigcirc}$, (5) $1 - qpq \in R^{\#\!\!\!\bigcirc}$, (6) $q - qpq \in R^{\#\!\!\!\bigcirc}$.
In this case,

$$(p - pqp)^{\#\!\!\!\bigcirc} = p(1 - qp)^{\#\!\!\!\bigcirc}$$

and

$$(1 - qp)^{\#\!\!\!\bigcirc} = (1 + qp - pqp)(p - pqp)^{\#\!\!\!\bigcirc} + 1 - p.$$

Li et al. [10] showed that $p - q$ is core invertible if and only if $p(1 - q)$ and $(1 - p)q$ are core invertible. Zhang et al. [15] proved that $p - q$ is Moore–Penrose invertible if and only if $p(1 - q)$ is Moore–Penrose invertible if and only if $(1 - p)q$ is Moore-Penrose invertible in a $*$-reducing ring R, where p, q are projections. We will show the case of pseudo core invertibility in any ring R with involution.

Theorem 3.5. *Let $p, q \in R$ be two projections. The following statements are equivalent:*

(1) $p - q \in R^{\circledD}$;

(2) $p(1 - q) \in R^{\circledD}$;

(3) $(1 - p)q \in R^{\circledD}$.

Proof. (1)⇔(2). From [3], $p - q \in R^D$ if and only if $p(1 - q)p \in R^D$. Since $p - q$ and $p(1 - q)p$ are Hermitian, we have that $p - q \in R^{\circledD}$ if and only if $p(1 - q)p \in R^{\circledD}$ if and only if $p(1 - q) \in R^{\circledD}$ by Lemma 3.1 and Proposition 2.6.

(1)⇔(3). In (1)⇔(2), replacing p, q by $1 - p, 1 - q$, respectively. □

In [10, Theorem 2.7], Li et al. proved that $(p - q)^2 \in R^{\#\!\!\!\bigcirc}$ if and only if $p(1 - q)p \in R^{\#\!\!\!\bigcirc}$ and $(1 - p)q(1 - p) \in R^{\#\!\!\!\bigcirc}$. From Theorem 3.5, Proposition 2.6 and [9, Theorem 2.5], we have that $p(1 - q)p \in R^{\circledD}$ if and only if $(1 - p)q(1 - p) \in R^{\circledD}$ if and only if $(p - q)^k \in R^{\#\!\!\!\bigcirc}$ for some positive integer k.

Remark 3.6. In Theorem 3.5, the pseudo core index of (1) and (2) are not equal in general. Let $R = \mathbb{C}^{3\times 3}$ and the involution is the transpose. Let $p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 1 & -1 & -i \\ -1 & 1 & i \\ -i & i & -1 \end{bmatrix}$. Then p, q are projections. By calculations, we have $p - q = \begin{bmatrix} 0 & 1 & i \\ 1 & -1 & -i \\ i & -i & 1 \end{bmatrix} \in R^{\circledD}$, $p - pq = \begin{bmatrix} 0 & 1 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in R^{\circledD}$. Thus $I(p - q) = 3 \neq 2 = I(p - pq)$.

Substitute $1 - p$ for p in Theorem 3.5, we have the following corollary.

Corollary 3.7. *Let $p, q \in R$ be two projections. The following statements are equivalent:*

(1) $pq \in R^{\textcircled{D}}$;

(2) $1 - p - q \in R^{\textcircled{D}}$;

(3) $(1 - p)(1 - q) \in R^{\textcircled{D}}$.

In 2014, Chen and Zhu [3] proved that $p - q \in R^D$ if and only if $1 - pq \in R^D$ if and only if $p + q - pq \in R^D$. In 2017, Wang and Chen [14] showed that $p - q \in R^D$ if and only if $p(p - q)^n p \in R^D$, where p, q are idempotents. Next we consider the pseudo core inverses case where p, q are two projections.

Theorem 3.8. *Let $p, q \in R$ be two projections, $n \geq 1$. The following statements are equivalent:*

(1) $p - q \in R^{\textcircled{D}}$;

(2) $1 - pq \in R^{\textcircled{D}}$;

(3) $p + q - pq \in R^{\textcircled{D}}$;

(4) $p(p - q)^n p \in R^{\textcircled{D}}$.

Proof. (1)⇔(2). From the proof of (1)⇔(2) in Theorem 3.5, we have that $p - q \in R^{\textcircled{D}}$ if and only if $p(1 - q)p \in R^{\textcircled{D}}$. Thus $p - q \in R^{\textcircled{D}}$ if and only if $1 - pq \in R^{\textcircled{D}}$ by Theorem 3.3.

(1)⇔(3). In (1)⇔(2), replacing p, q by $1 - p, 1 - q$, respectively.

(1)⇔(4). Since $p - q \in R^D$ if and only if $p(p - q)^n p \in R^D$ by [14], $p - q$ and $p(p - q)^n p$ are Hermitian, we obtain that (1)⇔(4) by Lemma 3.1. □

Remark 3.9. In Theorem 3.8, the pseudo core index of (1) and (2) are not equal in general. Indeed, by Remark 3.6 we have that $p - q \in R^{\textcircled{D}}$ and $I(p - q) = 3$. By calculations, we have $1 - pq = \begin{bmatrix} 0 & 1 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in R^{\textcircled{D}}$, thus $I(1 - pq) = 1 \neq 3 = I(p - q)$.

From Theorem 3.3, Theorem 3.5 and Theorem 3.8, we have following corollary.

Corollary 3.10. Statements (1)-(6) of Theorem 3.3 are equivalent to each of the following statements:
(7) $p - pq \in R^{\textcircled{D}}$, (8) $p - qp \in R^{\textcircled{D}}$, (9) $q - qp \in R^{\textcircled{D}}$, (10) $q - pq \in R^{\textcircled{D}}$.

Remark 3.11. The pseudo core index of Theorem 3.3(1) does not equal to the index of Corollary 3.10(7) in general. Indeed, from Remark 3.6, we have that $p - pq \in R^{\textcircled{D}}$ and $I(p - pq) = 2$. By calculations, we have $1 - qp = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ i & 0 & 1 \end{bmatrix} \in R^{\textcircled{D}}$. Then $I(1 - qp) = 1 \neq 2 = I(p - pq)$.

Replacing p and q by $\overline{p} = 1 - p$ and $\overline{q} = 1 - q$ in Theorem 3.3 and Corollary 3.10, respectively, we obtain the following result.

Corollary 3.12. Let $p, q \in R$ be two projections. The following statements are equivalent:
(11) $p + q - qp \in R^{\textcircled{D}}$, (12) $p + \overline{p}q\overline{p} \in R^{\textcircled{D}}$ (13) $\overline{p}q\overline{p} \in R^{\textcircled{D}}$,
(14) $p + q - pq \in R^{\textcircled{D}}$, (15) $q + \overline{q}p\overline{q} \in R^{\textcircled{D}}$, (16) $\overline{q}p\overline{q} \in R^{\textcircled{D}}$
(17) $q - pq \in R^{\textcircled{D}}$, (18) $q - qp \in R^{\textcircled{D}}$, (19) $p - qp \in R^{\textcircled{D}}$,
(20) $p - pq \in R^{\textcircled{D}}$.

Chen and Zhu [3] proved that $pq - qp \in R^D$ (resp., $pq + qp \in R^D$) if and only if $pq \in R^D$ and $p - q \in R^D$ (resp., $p + q \in R^D$), where p, q are idempotents. We will give relevant results of pseudo core invertibility of commutator $pq - qp$ (resp., anti-commutator $pq + qp$) in a ring with involution, where p, q are projections.

Theorem 3.13. *Let $p, q \in R$ be two projections. The following statements are equivalent:*

(1) $pq - qp \in R^{\textcircled{D}}$;

(2) $pq \in R^{\textcircled{D}}$ *and* $p - q \in R^{\textcircled{D}}$.

Proof. (1)⇒(2). Since $pq - qp \in R^{\textcircled{D}}$, we have $pq - qp \in R^D$. Thus, $pq \in R^D$ and $p - q \in R^D$ by [3, Theorem 3.6]. Since $(p - q)^* = p - q$, we obtain $p - q \in R^{\textcircled{D}}$ by Lemma 3.1. From [4], $pq \in R^D$ implies that $pqp \in R^D$. Since $(pqp)^* = pqp$, we get $pqp \in R^{\textcircled{D}}$ by Lemma 3.1. Therefore, $pq \in R^{\textcircled{D}}$ by Proposition 2.6.

(2)⇒(1). Since $pq \in R^{\textcircled{D}}$ and $p - q \in R^{\textcircled{D}}$, we have $pq \in R^D$ and $p - q \in R^D$, thus, $pq - qp \in R^D$ by [3, Theorem 3.6]. Since $(pq - qp)^* = -(pq - qp)$, we obtain $pq - qp \in R^{\textcircled{D}}$ by Lemma 3.1. □

Similarly to the proof in Theorem 3.13, we have following result.

Theorem 3.14. *Let $p, q \in R$ be two projections. The following statements are equivalent:*

(1) $pq + qp \in R^{\textcircled{D}}$;

(2) $pq \in R^{\textcircled{D}}$ *and* $p + q \in R^{\textcircled{D}}$.

From Corollary 3.7 and Theorem 3.14, we have following corollary, which generalizes the result of core inverses in [10] to the cases of pseudo core inverses.

Corollary 3.15. *Let $p, q \in R$ be two projections. The following statements are equivalent:*

(1) $pq + qp \in R^{\textcircled{D}}$;

(2) $p + q - 1 \in R^{\textcircled{D}}$ *and* $p + q \in R^{\textcircled{D}}$.

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