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Positive Solutions for Singular Semipositone Nonlinear Fractional Differential System

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Abstract.
In this paper, under suitable conditions we employ the nonlinear alternative of Leray-Schauder type and the Guo-Krasnosel'skii fixed point theorem to show the existence of positive solutions for a system of nonlinear singular Riemann-Liouville fractional differential equations with sign-changing nonlinearities, subject to integral boundary conditions. Some examples are given to illustrate our main results.

1. Introduction

In this work, we consider the following system of nonlinear boundary value problem

$$\begin{cases} D^{\alpha}u(t) + \mu_1 a(t) f(t, u(t), v(t)) - q_1(t) = 0, \ \text{in } (0,1), \ n-1 < \alpha \le n, \\ D^{\beta}v(t) + \mu_2 b(t) g(t, u(t), v(t)) - q_2(t) = 0, \ \text{in } (0,1), \ m-1 < \beta \le m, \\ u(0) = u'(0) = ... = u^{(n-2)}(0) = 0, \ u(1) = \lambda_1 \int_0^1 u(s)ds, \\ v(0) = v'(0) = ... = v^{(m-2)}(0) = 0, \ v(1) = \lambda_2 \int_0^1 v(s)ds, \end{cases} \tag{1}$$

where D^{δ} is the standard Riemann-Liouville fractional derivative of order $\delta > 0$, $n, m \in \mathbb{N}$, $n, m \ge 3$, $0 < \lambda_1 < \alpha$, $0 < \lambda_2 < \beta$, μ_1 and μ_2 are two positive parameters. The functions a, b are nonnegative continuous on $(0,1)$ and they are allowed to be singular at $t = 0$ and/or $t = 1$. The functions f, g are in $C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$. The perturbed terms $q_i : (0,1) \to [0,+\infty)$, $i = 1,2$ are measurable functions and satisfy some properties detailed below. Throughout our nonlinearities may change sign. This type of changing-sign Boundary Value Problem occurs in models for steady-state diffusion with reactions [2, 5]. Moreover, the property of the integral boundary arise in various fields of thermal conduction, semiconductor and hydrodynamic problems. In the case when the nonlinearities are nonnegative, problem (1) has been investigated in [4] and in [15]. In particular, then, our results provide an immediate generalization of [4] and[15].
On the other hand, many works have studied changing-sign boundary value problems, see, [3, 6, 10–13, 16–19] and the refrences therein.

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Recently and under coupled integral boundary, Henderson and Luca, in [7] studied the following semipositone system

$$\begin{cases} D^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, \text{ in } (0,1),\ n-1<\alpha\ \leq n, \\ D^{\beta}v(t)+\mu g(t,u(t),v(t))=0, \text{ in } (0,1),\ m-1<\beta\ \leq m, \\ u(0)=u'(0)=...=u^{(n-2)}(0)=0,\ u(1)=\int_0^1 v(s)dH(s), \\ v(0)=v'(0)=...=v^{(m-2)}(0)=0,\ v(1)=\int_0^1 u(s)dK(s), \end{cases} \tag{2}$$

where $n,m\in\mathbb{N}, n,m\geq 3$, and $\lambda,\mu>0$, D^{α} and D^{β} denote the Riemann-Liouville derivatives of orders α and β respectively, the integrals form in the boundary condition are Riemann-Stieltjes integrals, and f,g are sign-changing continuous functions. These functions may be nonsingular or singular at $t=0$ and/or $t=1$ subject to coupled integral boundary conditions. The authors presented intervals for parameters λ and μ such that the above problem (2) has at least one positive solution. However, in our work and under uncoupled integral boundary conditions, we prove existence results of positive solutions depending on real parameters μ_1 and μ_2. In the scalar case, Bourguiba and Toumi in [3] considered the following semipositone problem

$$\begin{cases} D^{\alpha}u(t)+\mu a(t)f(t,u(t))-q(t)=0, \text{ in } (0,1), \\ \\ u(0)=u'(0)=...=u^{(n-2)}(0)=0,\ u(1)=\lambda\int_0^1 u(s)ds, \end{cases} \tag{3}$$

There, the authors, gave sufficient conditions for the existence of positive solutions for problem (3) depending on the real parameter μ.
As a generalization of boundary value problem (3), we extend in this paper the result to a class of system with changing-sign nonlinearities.
The construction of this paper is displayed as follows. In the next section we recall some tools and we present properties of the Green's function. Moreover we state preliminary lemmas. Section 3 is devoted to establish existence of positive solutions for (1), respectively. However, the final Section of the paper contains examples to illustrate our results.

2. Preliminaries

We present here the definitions of Riemann-Liouville fractional integral and Riemann- Liouville fractional derivative and then some auxiliary results that will be used to prove our main results. We refer the reader to [8, 14] for more details.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha>0$ for a measurable function $f:(0,+\infty)\to\mathbb{R}$ is defined as*

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,\ t>0,$$

where Γ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha>0$ for a measurable function $f:(0,+\infty)\to\mathbb{R}$ is defined as*

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t (t-s)^{n-\alpha-1}f(s)ds=(\frac{d}{dt})^n I^{n-\alpha}f(t),$$

provided that the right-hand side is pointwise defined on $\mathbb{R}^+$.
Here $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of the real number α.

Now, we recall the explicit expression of the Green's function for the linear fractional differential equation associated to (3) see ([3])

$$G_{\alpha,\lambda}(t,s)=\frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(\alpha-\lambda)((t-s)^{+})^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)},\tag{4}$$

for all $t,s\in[0,1]$ and $(t-s)^{+}=\max(t-s,0)$.

The following properties of the Green's function play an important role in this paper.

Proposition 2.3. *([3])Let $n\in\mathbb{N}$, $n\geq 3$, $n-1<\alpha\leq n$, and $\lambda\in[0,\alpha)$. Then the function $G_{\alpha,\lambda}$ defined by (4) satisfies the following properties*

i) $G_{\alpha,\lambda}$ *is nonnegative continuous function on* $[0,1]\times[0,1]$ *and* $G_{\alpha,\lambda}(t,s)>0$*, for all* $t,s\in(0,1)$.

ii) $G_{\alpha,\lambda}(t,s)\leq \eta_{\alpha,\lambda}K_{\alpha}(s)$ *for all* $t,s\in[0,1]$ *where* $K_{\alpha}(s)=\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ *and* $\eta_{\alpha,\lambda}=\dfrac{\alpha}{\alpha-\lambda}$.

iii) $G_{\alpha,\lambda}(t,s)\leq \eta_{\alpha,\lambda}t^{\alpha-1}k_{\alpha}(s)$ *for all* $t,s\in[0,1]$ *where* $k_{\alpha}(s)=\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$.

iv) $G_{\alpha,\lambda}(t,s)\geq \eta_{\alpha,\lambda}\lambda^{*}_{\alpha}t^{\alpha-1}K_{\alpha}(s)$, $\forall t,s\in[0,1]$ *where* $\lambda^{*}_{\alpha}=\dfrac{\lambda}{\alpha}$.

v) Let $\theta\in(0,\frac{1}{2})$, $s\in[0,1]$ *then* $\min\limits_{t\in[\theta,1-\theta]}G_{\alpha,\lambda}(t,s)\geq\gamma_{\alpha,\lambda}K_{\alpha}(s)$ *where* $\gamma_{\alpha,\lambda}=(\frac{\theta}{\alpha-1}+\frac{\lambda}{\alpha-\lambda})\theta^{\alpha-1}$.

Now, we state the following key Lemma.

Lemma 2.4. *([3]) Let $n\geq 3$, $n-1<\alpha\leq n$ and $0<\lambda<\alpha$. Assume that $(1-t)^{\alpha-1}q(t)\in C(0,1)\cap L(0,1)$. Then the boundary value problem*

$$\begin{cases} D^{\alpha}w(t)+q(t)=0, \ \ in \ \ (0,1),\\ \\ w(0)=w'(0)=...=w^{(n-2)}(0)=0, \ w(1)=\lambda\int_0^1 w(s)ds,\end{cases}\tag{5}$$

has a unique nonnegative solution $w(t)=\int_0^1 G_{\alpha,\lambda}(t,s)q(s)ds\in C([0,1])$ satisfying on $[0,1]$

$$w(t)\leq \eta_{\alpha,\lambda}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|q(s)|ds.\tag{6}$$

The proofs of ours results are based upon the following the nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorem.

Lemma 2.5. *([1]) Let X be a Banach space with $\Omega\subset X$ closed and convex. Assume U is a relatively open subset of Ω with $0\in U$, and let $S:\overline{U}\to\Omega$ be a completely continuous operator. Then either*

(i) S has a fixed point in $\overline{U}$, or

(ii) there exist $u\in\partial U$ and $\rho\in(0,1)$ such that $u=\rho Su$.

Lemma 2.6. *([9]) Let P be the cone of a real Banach space E and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\overline{\Omega}_1\subset\Omega_2$. Suppose that $T:P\cap(\overline{\Omega}_2\setminus\Omega_1)\longrightarrow P$ is completely continuous operator such that either*
(i) $\|Tx\|\geq\|x\|$, $x\in P\cap\partial\Omega_1$ and $\|Tx\|\leq\|x\|$, $x\in P\cap\partial\Omega_2$, or
(ii) $\|Tx\|\leq\|x\|$, $x\in P\cap\partial\Omega_1$ and $\|Tx\|\geq\|x\|$, $x\in P\cap\partial\Omega_2$.
holds. Then the operator T has at least one fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Now, let $E = C([0,1]) \times C([0,1])$, endowed with the norm
$\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Then E is a Banach space.
Let $\theta \in [0, \frac{1}{2})$ and set $J_\theta = [\theta, 1-\theta]$. In the sequel we need the following notations. For a measurable function $h : (0,1) \to [0,+\infty)$ and $\alpha > 0$, we note

$$\sigma_\alpha^\theta(h) = \int_\theta^{1-\theta} K_\alpha(t)h(t)dt$$

and

$$\sigma_\alpha(h) = \int_0^1 k_\alpha(t)h(t)dt.$$

where K_α and k_α are defined in Proposition 2.3.

$$G_\alpha = G_{\alpha,\lambda_1} \text{and } G_\beta = G_{\beta,\lambda_2},$$

where G_{α,λ_1}, G_{β,λ_2} are given by (4). Also we denote

$$\gamma_\alpha = \gamma_{\alpha,\lambda_1},\ \gamma_\beta = \gamma_{\beta,\lambda_2}. \tag{7}$$

$$\eta_\alpha = \eta_{\alpha,\lambda_1},\ \eta_\beta = \eta_{\beta,\lambda_2}. \tag{8}$$

where $\gamma_{\alpha,\lambda_1}$, γ_{β,λ_2}, η_{α,λ_1} and η_{β,λ_2} are defined in Proposition 2.3.

Now, we define the cone Ω in E by

$$\Omega = \{(u,v) \in E :\ u(t) \geq 0,\ v(t) \geq 0, u(t) \geq \lambda_\alpha^* t^{\alpha-1} \|u\|, v(t) \geq \lambda_\beta^* t^{\beta-1} \|v\|, t \in [0,1]\},$$

and for $r > 0$, let

$$\Omega_r = \{(u,v) \in \Omega : \|(u,v)\| < r\}.$$

In the rest of the paper, we adopt the following hypotheses:

(H$_1$) $a, b \in C((0,1), [0+\infty))$, $a(t), b(t) \neq 0$ on any subinterval of $(0,1)$ and $0 < \sigma_\alpha^0(a), \sigma_\beta^0(b) < \infty$.

(H$_2$) $q_1, q_2 : (0,1) \to [0,+\infty)$ with $0 < \sigma_\alpha(q_1), \sigma_\beta(q_2) < \infty$.

(H$_3$) $f, g \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$.

(H$_4$) There exist $t_1, t_2 \in (0,1)$ such that $f(t_1,u,v) > 0$ and $g(t_2,u,v) > 0$ for each $u, v \in (0,+\infty)$.

The purpose of this paper is to investigate the existence of positive solutions of problem (1). By positive solution we mean a pair of functions $(u,v) \in C([0,1]) \times C([0,1])$ satisfying problem (1) with $u(t), v(t) \geq 0$ for all $t \in [0,1]$ and $u(t) > 0$ or $v(t) > 0$ for all $t \in (0,1]$.
We consider now the intermediary system of nonlinear fractional differential equations

$$\begin{cases} D^\alpha x(t) + \mu_1 a(t) f(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + q_1(t) = 0, \text{in } (0,1), \\ \\ D^\beta y(t) + \mu_2 b(t) g(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + q_2(t) = 0, \text{in } (0,1), \end{cases} \tag{9}$$

with the integral boundary conditions

$$\begin{cases} x(0) = x'(0) = ... = x^{(n-2)}(0) = 0, \ x(1) = \lambda_1 \int_0^1 x(s)ds, \\ y(0) = y'(0) = ... = y^{(n-2)}(0) = 0, \ y(1) = \lambda_2 \int_0^1 y(s)ds, \end{cases} \tag{10}$$

where $[x(t) - w_1(t)]^* = max\{x(t) - w_1(t), 0\}$, for each $t \in [0, 1]$.
Here (w_1, w_2) is solution of fractional differential equations

$$\begin{cases} D^{\alpha} w_1(t) + 2q_1(t) = 0, \ \text{ in } \ (0, 1), \\ \\ D^{\alpha} w_2(t) + 2q_2(t) = 0, \ \text{ in } \ (0, 1), \\ \end{cases} \tag{11}$$

with integral boundary conditions

$$\begin{cases} w_1(0) = w_1'(0) = ... = w_1^{(n-2)}(0) = 0, \ w_1(1) = \lambda_1 \int_0^1 w_1(s)ds, \\ w_2(0) = w_2'(0) = ... = w_2^{(n-2)}(0) = 0, \ w_2(1) = \lambda_2 \int_0^1 w_2(s)ds, \end{cases} \tag{12}$$

Using Propsition 2.6 in [3], $0 < \sigma_q^0 < \infty$ if and only if $(1 - s)^{\alpha-1} q(s) \in C(0, 1) \cap L(0, 1)$. Then by Lemma 2.4 problem (11)-(12) has a unique solution (w_1, w_2) with

$$w_1(t) = 2 \int_0^1 G_{\alpha}(t, s) q_1(s) ds, \ t \in [0, 1],$$

and

$$w_2(t) = 2 \int_0^1 G_{\beta}(t, s) q_2(s) ds, \ t \in [0, 1],$$

satisfying

$$w_1(t) \leq 2\eta_{\alpha} \sigma_{\alpha}(q_1) t^{\alpha-1} \ , \forall t \in [0, 1], \tag{13}$$

and

$$w_2(t) \leq 2\eta_{\beta} \sigma_{\beta}(q_2) t^{\beta-1} \ , \forall t \in [0, 1]. \tag{14}$$

We shall prove that there exists a solution (x, y) for the boundary value problem (9)-(10) with $x(t) \geq w_1(t)$ or $y(t) \geq w_2(t)$ for any $t \in [0, 1]$. In this case $(x - w_1, y - w_2)$ represents a nonnegative solution of the boundary value problem (1).
Next, we define the operator $T : E \longrightarrow E$ as follows

$$T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t)), \quad t \in [0, 1] \tag{15}$$

where

$$T_1(u, v)(t) = \int_0^1 G_{\alpha}(t, s)(\mu_1 a(s) f(s, [x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) + q_1(s)) ds,$$

and

$$T_2(u, v)(t) = \int_0^1 G_{\beta}(t, s)(\mu_2 b(s) g(s[x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) + q_2(s)) ds.$$

We note that our study on the problem (1) remains to the seek of fixed point of the operator T and this is due to the following lemma.

Lemma 2.7. *Suppose that (H_1)-(H_4) hold. Then $(x, y) \in C([0,1]) \times C([0,1])$ is a solution of the boundary value problem (1) if and only if $(x, y) \in C([0,1]) \times C([0,1])$ is a solution of the integral equations*

$$x(t) = \int_0^1 G_\alpha(t,s)(\mu_1 a(s) f(s, [x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) + q_1(s))ds. \tag{16}$$

$$y(t) = \int_0^1 G_\beta(t,s)(\mu_2 b(s) g(s, [x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) + q_2(s))ds. \tag{17}$$

That is (x, y) is a fixed point of the operator T defined by (15).

Proof. The proof is immediate from Lemma 2.4, so we omit it. □

We call $G(t,s) = (G_\alpha(t,s), G_\beta(t,s))$ the Green's function of the problem (1).

Lemma 2.8. *Suppose that conditions (H_1)-(H_4) hold. Then $T : \Omega \longrightarrow \Omega$ is completely continuous.*

Proof. T_1 and T_2 are well defined. To prove this, let $(x, y) \in \Omega$ with $\|(x, y)\| = L$. Then we have

$$[x(s) - w_1(s)]^* \leq x(s) \leq \|(x, y)\| = L, \forall s \in [0,1],$$

$$[y(s) - w_2(s)]^* \leq y(s) \leq \|(x, y)\| = L, \forall s \in [0,1].$$

Suppose that (H_1), (H_2) and (H_3) are satisfied. Put

$$M = max\{1, \max_{t\in[0,1],x,y\in[0,L]} f(t,x,y), \max_{t\in[0,1],x,y\in[0,L]} g(t,x,y)\}.$$

Then we get for all $t \in [0,1]$

$$\begin{aligned} T_1(x,y)(t) &\leq \eta_\alpha(\mu_1 \int_0^1 K_\alpha(s) a(s)\, f(s, [x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) ds + \sigma_\alpha^0(q_1)) \\ &\leq \eta_\alpha M(\mu_1 \sigma_\alpha^0(a) + \sigma_\alpha^0(q_1)) < \infty, \end{aligned}$$

By the same manner we obtain

$$\begin{aligned} T_2(x,y)(t) &\leq \eta_\beta(\mu_2 \int_0^1 K_\beta(s) b(s)\, g(s, [x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) ds + \sigma_\beta^0(q_2)) \\ &\leq \eta_\beta M(\mu_1 \sigma_\beta^0(b) + \sigma_\beta^0(q_2)) < \infty. \end{aligned}$$

Besides, by Proposition (2.3)(v), we have $T_1(x,y)(t) \geq \lambda_\alpha^* t^{\alpha-1} \|T_1(x,y)\|$ and $T_2(x,y)(t) \geq \lambda_\beta^* t^{\beta-1} \|T_2(x,y)\|$ and so $T(x,y) \in \Omega$.

Then, by using standard arguments, we deduce that operator $T : \Omega \to \Omega$ is completely continuous. This ends the proof. □

3. Existence results

This section is devoted to the existence of positive solution for the nonlinear boundary value system (1).

Theorem 3.1. *Assume that* (H_1)-(H_4) *hold. Then there exist* $\mu_1^* > 0$ *and* $\mu_2^* > 0$ *such that for any* $0 < \mu_1 < \mu_1^*$ *and* $0 < \mu_2 < \mu_2^*$*, problem* (1) *has at least one positive solution.*

Proof. Choose

$$R > \max\{\frac{4\eta_\alpha\sigma_\alpha(q_1)}{\lambda_\alpha^*}, \frac{4\eta_\beta\sigma_\beta(q_2)}{\lambda_\beta^*}\}.$$

Let

$$M_1 = \max_{t\in[0,1],x,y\in[0,R]} f(t,x,y) \text{ and } M_2 = \max_{t\in[0,1],x,y\in[0,R]} g(t,x,y).$$

Fix now $\mu_1^* = \dfrac{R - 4\eta_\alpha\sigma_\alpha^0(q_1)}{4M_1\eta_\alpha\sigma_\alpha^0(a)}$ and $\mu_2^* = \dfrac{R - 4\eta_\beta\sigma_\beta^0(q_2)}{4M_2\eta_\beta\sigma_\beta^0(b)}$
and let $0 < \mu_1 < \mu_1^*$ and $0 < \mu_2 < \mu_2^*$.
Suppose first that there exist $(x, y) \in \partial\Omega_R$ that is $\|(x, y)\| = R$ or $\|x\| + \|y\| = R$ and suppose that there exist $\rho \in (0, 1)$ such that $(x, y) = \rho T(x, y)$ that is $x = \rho T_1(x, y)$ and $y = \rho T_2(x, y)$. Since

$$\begin{aligned}
[x(t) - w_1(t)]^* &= x(t) - w_1(t) \le x(t) \le R, \text{ if } x(t) - w_1(t) \ge 0 \textit{ and}\\
[y(t) - w_2(t)]^* &= y(t) - w_2(t) \le y(t) \le R, \text{ if } y(t) - w_2(t) \ge 0,
\end{aligned}$$

then, for all $t \in [0, 1]$, we obtain

$$\begin{aligned}
x(t) &= \rho T_1(x,y)(t) \le T_1(x,y)(t)\\
&\le \mu_1\eta_\alpha \int_0^1 K_\alpha(s)a(s)f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)ds + \eta_\alpha\sigma_\alpha^0(q_1)\\
&\le \mu_1^*\eta_\alpha M_1\sigma_\alpha^0(a) + \eta_\alpha\sigma_\alpha^0(q_1)\\
&= \frac{R}{4}
\end{aligned}$$

By the same manner we get $y(t) \le \dfrac{R}{4}$. Therefore $\|x\| \le \dfrac{R}{4}$ and $\|y\| \le \dfrac{R}{4}$. Then we have
$\|x\| + \|y\| \le \dfrac{R}{4} + \dfrac{R}{4} = \dfrac{R}{2}$ which is a contradiction.
So by Lemma 2.5, we conclude that T has a fixed point $(x_1, y_1) \in \overline{\Omega}_R$. Using Lemma 2.7 (x_1, y_1) is a nonnegative continuous solution of problem (9)-(10). Now, let us prove that $(x - w_1, y - w_2)$ is a positive solution of (1), that is $x(t) - w_1(t) > 0$ or $y(t) - w_2(t) > 0$ for all $t \in (0, 1]$.
We suppose first that $\|x\| \ge \dfrac{R}{2}$. Using (13), we obtain

$$x(t) - w_1(t) \ge t^{\alpha-1}(\lambda_\alpha^*\frac{R}{2} - 2\eta_\alpha\sigma_\alpha(q_1)) > 0, \forall t \in (0, 1].$$

If $\|y\| \ge \dfrac{R}{2}$, then by a similar approach, we deduce that

$$y(t) - w_2(t) \ge t^{\beta-1}(\lambda_\beta^*\frac{R}{2} - 2\eta_\beta\sigma_\beta(q_2)) > 0, \forall t \in (0, 1].$$

Hence, $(x - w_1, y - w_2)$ is a positive solution of problem (1). This completes the proof.
□

Theorem 3.2. *Suppose that conditions* (H_1)–(H_4) *hold. In addition, assume that the following assertions hold*

(A_1) *There exits* $\theta \in (0, \frac{1}{2})$ *such that*

$$f_\infty^* := \lim_{x+y\to+\infty} \min_{t\in J_\theta} f(t,x,y) = +\infty \ or\ g_\infty^* := \lim_{x+y\to+\infty} \min_{t\in J_\theta} g(t,x,y) = +\infty.$$

(A_2) $f^\infty := \lim_{x+y\to+\infty} \max_{t\in[0,1]} \frac{f(t,x,y)}{x+y} = 0$ *and* $g^\infty := \lim_{x+y\to+\infty} \max_{t\in[0,1]} \frac{g(t,x,y)}{x+y} = 0.$

Then there exist $\mu_1^* > 0$ *and* $\mu_2^* > 0$ *such that problem* (1) *has at least one positive solution for every* $\mu_1 > \mu_1^*$ *and* $\mu_2 > \mu_2^*$.

Proof. First suppose that (A_1) holds, then for $A = max\{\frac{1}{\gamma_\alpha \sigma_\alpha^\theta(a)}, \frac{1}{\gamma_\beta \sigma_\beta^\theta(b)}\}$ there exists $R_0 > 0$ such that

$$f(t,x,y) \geq A,\ \forall t \in J_\theta,\ \forall x+y \geq R_0, \tag{18}$$

or

$$g(t,x,y) \geq A,\ \forall t \in J_\theta,\ \forall x+y \geq R_0. \tag{19}$$

First suppose that $f_\infty^* = \infty$, then (18) holds.
Fix $R_1 > max\{\frac{4R_0}{\lambda_\alpha^* \theta^{\alpha-1}}, \frac{4R_0}{\lambda_\beta^* \theta^{\beta-1}}, \frac{8\eta_\alpha \sigma_\alpha(q_1)}{\lambda_\alpha^*}, \frac{8\eta_\beta \sigma_\beta(q_2)}{\lambda_\beta^*}\}$.
Define $\mu_1^* = \frac{R_1}{\gamma_\alpha \sigma_\alpha^\theta(a) A} > 0$ and $\mu_2^* = \frac{R_1}{\gamma_\beta \sigma_\beta^\theta(b) A} > 0$. Let $\mu_1 > \mu_1^*$ and $\mu_2 > \mu_2^*$.
Then, for each $(x,y) \in \partial\Omega_{R_1}$, we have $\|(x,y)\| = R_1$, so $\|x\| \geq \frac{R_1}{2}$ or $\|y\| \geq \frac{R_1}{2}$. We suppose that $\|x\| \geq \frac{R_1}{2}$.
Then for each $t \in [0,1]$, we have

$$\begin{aligned}
x(t) - w_1(t) &\geq x(t) - 2\eta_\alpha \sigma_\alpha(q_1) t^{\alpha-1} \\
&\geq x(t) - \frac{2\eta_\alpha \sigma_\alpha(q_1)}{\lambda_\alpha^*} \frac{x(t)}{\|x\|} \\
&\geq x(t)\left(1 - \frac{4\eta_\alpha \sigma_\alpha(q_1)}{\lambda_\alpha^* R_1}\right) \\
&\geq \frac{1}{2} x(t) \geq 0.
\end{aligned}$$

So, for $(x,y) \in \partial\Omega_{R_1}$ and $t \in J_\theta$, we get

$$[x(t) - w_1(t)]^* + [y(t) - w_2(t)]^* \geq \frac{1}{2} x(t) \geq \frac{1}{4} \lambda_\alpha^* \theta^{\alpha-1} R_1 > R_0.$$

Then for any $(x,y) \in \partial\Omega_{R_1}$ and $t \in J_\theta$, we obtain

$$f(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) \geq A.$$

It follows that for any $x \in \partial\Omega_{R_1}$ and $t \in J_\theta$

$$\begin{aligned}
T_1(x,y)(t) &\geq \mu_1 \gamma_\alpha \int_\theta^{1-\theta} K_\alpha(s)(a(s) f(s, [x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*)) ds \\
&\geq \mu_1 \gamma_\alpha A \int_\theta^{1-\theta} K_\alpha(s) a(s) ds \\
&= R_1.
\end{aligned}$$

Thus

$$\|T_1(x,y)\| \geq \|(x,y)\|,\ \forall (x,y) \in \partial\Omega_{R_1}.$$

Therefore, we get

$$\|T(x,y)\| \geq \|(x,y)\|, \ \forall (x,y) \in \partial\Omega_{R_1}. \tag{20}$$

If $\|y\| \geq \dfrac{R_1}{2}$, then by the same manner we get relation (20).
Now, we suppose that $g^*_\infty = \infty$ then (19) holds. By a similar way as above we get relation (20).
Now, since $f^\infty = g^\infty = 0$, then for $\varepsilon = min\{\frac{1}{2\mu_1\eta_\alpha\sigma^0_\alpha(a)}, \frac{1}{2\mu_2\eta_\beta\sigma^0_\beta(b)}\} > 0$, there exists $B > 0$ such that for each $t \in [0,1]$, $x + y \geq B$, we have

$$f(t,x,y) \leq \varepsilon(x+y) \text{and } g(t,x,y) \leq \varepsilon(x+y).$$

Therefore, $\forall t \in [0,1], \forall x, y \geq 0$, we obtain

$$f(t,x,y) \leq M_1 + \varepsilon(x+y) \ and \ g(t,x,y) \leq M_2 + \varepsilon(x+y).$$

where $M_1 = \max\limits_{\substack{t\in[0,1]\\ x+y\leq B}} f(t,x,y)$ and $M_2 = \max\limits_{\substack{t\in[0,1]\\ x+y\leq B}} g(t,x,y)$.
Let $M = \max\{1, M_1, M_2\}$ and choose

$$\begin{aligned} R_2 \ > \ & \max\{2R_1, \mu_1\eta_\alpha M\sigma^0_\alpha(a)(\frac{1}{4} - \mu_1\sigma^0_\alpha(a)\eta_\alpha\varepsilon)^{-1}, \\ & \mu_2\eta_\beta M\sigma^0_\beta(b)(\frac{1}{4} - \mu_2\sigma^0_\beta(b)\eta_\beta\varepsilon)^{-1}, 4\eta_\alpha M\sigma^0_\alpha(q_1), 4\eta_\beta M\sigma^0_\beta(q_2)\}. \end{aligned}$$

It follows that for any $(x,y) \in \partial\Omega_{R_2}$ and $t \in [0,1]$

$$\begin{aligned} T_1(x,y)(t) \ &\leq \ \mu_1\eta_\alpha \int_0^1 K_\alpha(s)a(s)f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)ds + \eta_\alpha\sigma^0_\alpha(q_1) \\ &\leq \ \mu_1\eta_\alpha M_1\sigma^0_\alpha(a) + \mu_1\eta_\alpha\varepsilon \int_0^1 K_\alpha(s)a(s)([x(s)-w_1(s)]^* + [y(s)-w_2(s)]^*)ds + \eta_\alpha\sigma^0_\alpha(q_1) \\ &\leq \ \mu_1\eta_\alpha M\sigma^0_\alpha(a) + \mu_1\eta_\alpha\sigma^0_\alpha(a)\varepsilon R_2 + \eta_\alpha M\sigma^0_\alpha(q_1) \\ &\leq \ R_2(\frac{1}{4} - \mu_1\sigma^0_\alpha(a)\eta_\alpha\varepsilon) + \mu_1\eta_\alpha\sigma^0_\alpha(a)\varepsilon R_2 + \eta_\alpha M\sigma^0_\alpha(q_1) = \frac{\|(x,y)\|}{2}. \end{aligned}$$

So, we get

$$\|T_1(x,y)\| \leq \frac{\|(x,y)\|}{2}, \forall (x,y) \in \partial\Omega_{R_2}.$$

Similarly, we prove

$$\|T_2(x,y)\| \leq \frac{\|(x,y)\|}{2}, \forall (x,y) \in \partial\Omega_{R_2}.$$

Thus we obtain

$$\|T(x,y)\| \leq \|(x,y)\|, \forall (x,y) \in \partial\Omega_{R_2}. \tag{21}$$

Thus, using (20) and (21) by Lemma 2.6 we deduce that the operator T has a fixed point in $\overline{\Omega_{R_2}}\backslash\Omega_{R_1}$.
Therefore by Lemma 2.7, (x,y) is a nonnegative continuous solution of problem (9)-(10) satisfying

$$R_1 < \|(x,y)\| \leq R_2. \tag{22}$$

Thus, we deduce that $(x - w_1, y - w_2)$ is nonnegative solution of problem (1).
The positivity of the solution is shown as in proof of the previous Theorem.

□

4. Examples

In this section, we present some examples in order to illustrate our results. We remark that in the following examples, it is immediate to verify that conditions (H_1)-(H_4) hold.

Example 4.1. *We consider the following nonlinear fractional differential equations*

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu_1 \frac{1}{t}(u+v)^2 - \frac{1}{1-t} = 0, & in\ (0,1), \\ D^{\frac{5}{2}}v(t) + \mu_2 \frac{1}{t}(u+v)^3 - \frac{1}{(1-t)^{\frac{3}{2}}} = 0, & in\ (0,1), \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 u(s)ds, \\ v(0) = v'(0) = 0, \ v(1) = \int_0^1 v(s)ds. \end{cases} \tag{23}$$

Let $f(t,u,v) = (u+v)^2, g(t,u,v) = (u+v)^3$ $a(t) = b(t) = \frac{1}{t}$, $\lambda_1 = \lambda_2 = 1$, $q_1(t) = \frac{1}{1-t}$ *and* $q_2(t) = \frac{1}{(1-t)^{\frac{3}{2}}}$. *By direct calculation, we get* $\sigma^0_\alpha(a) = \sigma^0_\beta(b) \approx 0.3009$, $\sigma^0_\alpha(q_1) \approx 0.2006$, $\sigma^0_\beta(q_2) \approx 0.37613$, $\sigma_1 \approx 0.5015$ *and* $\sigma_2 \approx 0.7522$. *Choose* $r = 15$, *then using the expressions of* μ_1^* *and* μ_2^* *given in Theorem 3.1, we get* $\mu_1^* \approx 1.510^{-2}$ *and* $\mu_2^* \approx 5.0110^{-4}$. *Then using Theorem 3.1, problem* (23) *has at least one positive solution for every* $0 < \mu_1 < 1.5 \times 10^{-2}$ *and* $0 < \mu_2 < 5.01 \times 10^{-4}$.

Example 4.2. *Consider the following boundary value problem*

$$\begin{cases} D^{\frac{7}{3}}u(t) + \mu_1 \dfrac{1}{t}(200 + \dfrac{1}{2+\sqrt{u+v}}) - \dfrac{1}{1-t} & in\ (0,1), \\ D^{\frac{7}{3}}u(t) + \mu_2 \dfrac{1}{t}(100 + \dfrac{1}{2+(u+v)^{\frac{1}{3}}}) - \dfrac{1}{1-t} & in\ (0,1), \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 u(s)ds, \\ v(0) = v'(0) = 0, \ v(1) = \int_0^1 v(s)ds. \end{cases} \tag{24}$$

Let $f(t,u,v) = 200 + \dfrac{1}{2+\sqrt{u+v}}$, $g(t,u,v) = 100 + \dfrac{1}{1+(u+v)^{\frac{1}{3}}}$, $a(t) = b(t) = \frac{1}{t}$ *and* $q_1(t) = q_2(t) = \dfrac{1}{1-t}$. *By direct calculation, we obtain* $f^\infty = g^\infty = 0$ *and for* $\theta = \dfrac{1}{4}$ *we have* $f^*_\infty = 200$ *and* $g^*_\infty = 100$. *We also obtain* $\sigma^0_\alpha = \sigma^0_\beta \approx 0.35995$, $\sigma^0_\alpha(a) = \sigma^0_\beta(b) \approx 0.26996$, $\sigma_1 = \sigma_2 \approx 0.62991$ *and* $\sigma^\theta_\alpha(a) = \sigma^\theta_\beta(b) \approx 0.16979$. *Choose* $R_1 = 500$ *and* $R_2 = 1001$. *A simple calculs yields to* $\mu_1^* = 284.01$ *and* $\mu_2^* = 568.03$. *So Theorem 3.2 ensures the existence of solution of problem (24) for every* $\mu_1 > 284.01$ *and* $\mu_2 > 586.03$ *such that* $500 < \|(u+w_1, v+w_2)\| < 1001$.

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