Filomat 35:1 (2021), 147–155
https://doi.org/10.2298/FIL2101147A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Triple Reverse Order Law of Drazin Invertible Operators

Safae Alaoui Chrifi[a], **Abdelaziz Tajmouati**[a]

[a] *Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar Al Mahraz, Laboratory of Mathematical Analysis and Applications, Fez, Morocco.*

Abstract. In this paper we study the triple reverse-order law $(ABC)^D = C^D B^D A^D$ for the Drazin invertible operators A, B and C under the commutative relations $[AB, B] = 0$, $[BC, B] = 0$ and $[AB, BC] = 0$.

1. Introduction and preliminaries

Let X and Y be two infinite dimensional Banach spaces. Denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded linear operators from X to Y. If $X = Y$, we will simply write $\mathcal{B}(X)$ instead of $\mathcal{B}(X, X)$. By $N(T)$ and $R(T)$, we denote the null space and the range of T, respectively. An operator $P \in \mathcal{B}(X)$ with the property $P^2 = P$ is called a projection. For any two operators $T, S \in \mathcal{B}(X)$, we define the commutator $[T, S]$ to be $TS - ST$.

Recall that an operator $T \in \mathcal{B}(X)$ is Drazin invertible if there exists $S \in \mathcal{B}(X)$ that satisfies the following equations

$$TS = ST, \qquad S = STS, \qquad T^{k+1}S = T^k. \tag{1}$$

The third equation in (1) means that $T - TST$ is nilpotent of index k, in this case we write $\text{ind}(T) = k$. It is worth pointing out that the Drazin inverse S of T, when it exists, it is unique. In the sequel, S will be denoted by T^D.

It is also common to cite Koliha's paper [6] as the pioneering work on generalized Drazin inverses, his definition generalizes (1) by replacing the third equation with the assumption $T - TST$ is quasi-nilpotent. Drazin invertible as well as generalized Drazin invertible operators have many suitable properties. Mainly, an operator $T \in \mathcal{B}(X)$ is Drazin invertible if and only if 0 is a pole of the resolvent and the spectral projection T^π of T corresponding to $\{0\}$ is given by $T^\pi = I - TT^D$. It is extremely useful to mention that

$$X = N(T^\pi) \oplus R(T^\pi).$$

Consequently, $T = T_1 \oplus T_2$ with $T_1 = T_{N(T^\pi)}$ is invertible and $T_2 = T_{R(T^\pi)}$ is nilpotent.

Among other things, nilpotent operators of index n are Drazin invertible with $T^D = (T^D)^{n+1}T^n = 0$. Projections P are also Drazin invertible with $P^D = P$.

2010 *Mathematics Subject Classification*. Primary 15A09; Secondary 47A08
Keywords. Reverse order law, Drazin inverse, triple product, operator matrices.
Received: 29 February 2020; Revised: 27 September 2020; Accepted: 12 October 2020
Communicated by Dijana Mosić
Email addresses: safae.alaouichrifi@usmba.ac.ma (Safae Alaoui Chrifi), abdelaziz.tajmouati@usmba.ac.ma (Abdelaziz Tajmouati)

In the literature, it is a common knowledge that if $A, B \in \mathcal{B}(X)$ are invertible then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$, this is often known as the reverse order law for ordinary inverse. However, this rule is not well-adapted to other inverses, such as Drazin inverse. In fact, if A, B and AB are Drazin invertible $(AB)^D = B^D A^D$ is meaningless. This problem was a source of interesting research as operator theorists sought to determine exactly what properties A and B must possess in order to satisfy this equality. Among the many paper which featured the aforesaid problem are [9, 11]and [10]. One can find other related results for various inverses in [2–4] and references therein.

Let H be an infinite dimensional Hilbert space, by $T^\dagger$ we denote the Moore-Penrose inverse of $T \in \mathcal{B}(H)$. With regard to the triple reverse order law for the Moore-Penrose inverses, the authors of [5] obtained necessary and sufficient conditions under which

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger,$$

where A, B, C and ABC are Hilbert space operators with closed ranges.

The issue to be discussed in this paper concerns some reverse order law for Drazin invertible operators A, B and C under the commutative relations $[AB, B] = 0$, $[BC, B] = 0$ and $[AB, BC] = 0$. In the light of these relations, we are interested in the relationship between A, B, C and A^D, B^D, C^D. Consequently, we provide some necessary and sufficient conditions for which

$$(BCAB)^D = B^D A^D C^D B^D.$$

Additionally, we obtain several triple reverse order law corresponding to $(ABC)^D$.

2. Preparations

We drawn particular attention in this paper to 2×2 operator matrices on the Banach space $X \oplus Y$ defined by

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

where $T_1 \in \mathcal{B}(X)$, $T_2 \in \mathcal{B}(Y, X)$, $T_3 \in \mathcal{B}(X, Y)$ and $T_4 \in \mathcal{B}(Y)$. The important point to note here is that every bounded operator on $X \oplus Y$ has the aforementioned form.

We are now going to concern our self with operators $A, B, C \in \mathcal{B}(X)$. If B is Drazin invertible with ind(B) $= n$ then the Banach space X obeys the following decomposition $X = N(B^\pi) \oplus R(B^\pi)$ and A, B, C have these forms

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & N_1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}. \tag{2}$$

Such that $B_1 \in \mathcal{B}(N(B^\pi))$ is invertible, $N_1 \in \mathcal{B}(R(B^\pi))$ is nilpotent, $B^n = B_1^n \oplus 0$ and $B^D = B_1^{-1} \oplus 0$.

Before going any further we began by the following lemmas which have an adequate amount of properties required.

Lemma 2.1. *[6, 11] $A, B, C, N \in \mathcal{B}(X)$, requiring N to be nilpotent of index n.*

(1) If $[N, AN] = 0$ then AN and NA are nilpotent with $max\{ind(NA), ind(AN)\} \leq n$;

(2) If $[N, NC] = 0$ then NC and CN are nilpotent with $max\{ind(NC), ind(CN)\} \leq n$;

(3) If A, B, C are Drazin invertible and $\{A, B, C\}$ are mutual-commutative then A, B, C, A^D, B^D and C^D are all commute with

$$(ABC)^D = A^D B^D C^D = C^D B^D A^D.$$

Lemma 2.2. *[8] For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y, X)$, $C_1 \in \mathcal{B}(Y, X)$ and $C_2 \in \mathcal{B}(X, Y)$. We denote by*

$$M_{C_1} = \begin{pmatrix} A & C_1 \\ 0 & B \end{pmatrix} \qquad M_{C_2} = \begin{pmatrix} A & 0 \\ C_2 & B \end{pmatrix}$$

where the two operators M_{C_1} and M_{C_2} are in $\mathcal{B}(X \oplus Y)$.

(1) If two of M_{C_1}, A and B are Drazin invertible, then the third is also Drazin invertible;
(2) If two of M_{C_2}, A and B are Drazin invertible, then the third is also Drazin invertible;
(3) If A and B are Drazin invertible with $ind(A) = s$ and $ind(B) = t$. Then

$$M_{C_1}^D = \begin{pmatrix} A^D & X \\ 0 & B^D \end{pmatrix} \qquad M_{C_2}^D = \begin{pmatrix} A^D & 0 \\ Y & B^D \end{pmatrix}$$

where

$$X = (A^D)^2[\sum_{n=0}^{t-1}(A^D)^n C_1 B^n]B^\pi + A^\pi[\sum_{n=0}^{s-1} A^n C_1 (B^D)^n](B^D)^2 - A^D C_1 B^D;$$

and

$$Y = (B^D)^2[\sum_{n=0}^{s-1}(B^D)^n C_2 A^n]A^\pi + B^\pi[\sum_{n=0}^{t-1} B^n C_2 (A^D)^n](A^D)^2 - B^D C_2 A^D.$$

Lemma 2.3. *[7] Let $A, B \in \mathcal{B}(X)$. If AB is Drazin invertible then BA is also Drazin invertible. In this case:*

$$(AB)^D = A((BA)^D)^2 B.$$

3. Main results

Let $A, B, C \in \mathcal{B}(X)$. Suppose that B is Drazin invertible having index n. First we assume that $[B, AB] = 0$, then $[B^n, AB] = 0$. From (2) it follows that

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & N_1 \end{pmatrix} \quad \text{and} \quad AB = \begin{pmatrix} A_1B_1 & 0 \\ 0 & A_4N_1 \end{pmatrix}, \tag{3}$$

according to the Banach space decomposition $X = N(B^\pi) \oplus R(B^\pi)$. This gives

$$[A_1, B_1] = 0, \qquad [N_1, A_4N_1] = 0 \quad \text{and} \quad A_2N_1 = 0. \tag{4}$$

We next suppose that $[B, BC] = 0$, thus $[B^n, BC] = 0$ with respect to (2)

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & N_1 \end{pmatrix} \qquad C = \begin{pmatrix} C_1 & 0 \\ C_3 & C_4 \end{pmatrix} \qquad \text{and} \qquad BC = \begin{pmatrix} B_1C_1 & 0 \\ 0 & N_1C_4 \end{pmatrix}. \tag{5}$$

Continually on $X = N(B^\pi) \oplus R(B^\pi)$. Hence:

$$[B_1, C_1] = 0, \qquad [N_1, N_1C_4] = 0 \quad \text{and} \quad N_1C_3 = 0. \tag{6}$$

We thus get $ABC = \begin{pmatrix} A_1B_1C_1 & 0 \\ 0 & A_4N_1C_4 \end{pmatrix}$.
To sharpen these forms we further assume that $[AB, BC] = 0$, then:

$$[A_1, C_1] = 0 \qquad \text{and} \quad [A_4N_1, N_1C_4] = 0. \tag{7}$$

This yields that A_1, B_1 and C_1 are pairwise commutative. Nevertheless A, B and C are not necessary commutative (e.g. $AC \neq CA$).

The following lemma is essential to prove certain results.

Lemma 3.1. *Let $A, C, N \in \mathcal{B}(X)$, where N is nilpotent.*

(1) If $[N, AN] = 0$ and $[AN, ANC] = 0$ then CAN is also nilpotent;
(2) If $[N, NC] = 0$ and $[AN, NC] = 0$ then NCA is also nilpotent;

(3) If $[N, AN] = 0$ *(or,* $[N, NC] = 0$ *) and* $[AN, NC] = 0$ *then* ANC *is also nilpotent.*

Proof. (1) As N is nilpotent and $[N, AN] = 0$ we have AN is also nilpotent with index m (see Lemma 2.1). Further, by $[AN, ANC] = 0$, it is easily seen that $[(AN)^k, ANC] = 0$ for every $k \in \mathbb{N}$. Therefore:

$$\begin{aligned}(CAN)^m &= (CAN)^{m-2}CANCAN = (CAN)^{m-2}CANANC\\ &= (CAN)^{m-2}C(AN)^2C\\ &= (CAN)^{m-3}CANC(AN)^2C\\ &= (CAN)^{m-3}C(AN)^3C^2\\ &= \dots\\ &= C(AN)^mC^{m-1}.\end{aligned}$$

(2) From Lemma 2.1, NC is nilpotent having index n. It is clear that $[AN, (NC)^k] = 0$ and $[N, (NC)^k] = 0$ for every $k \in \mathbb{N}$, so:

$$\begin{aligned}(NCA)^n &= NCANCA(NCA)^{n-2} = ANNCCA(NCA)^{n-2}\\ &= ANCNCA(NCA)^{n-2}\\ &= A(NC)^2ANCA(NCA)^{n-3}\\ &= AAN(NC)^2CA(NCA)^{n-3}\\ &= A^2(NC)^2NCA(NCA)^{n-3}\\ &= A^2(NC)^3A(NCA)^{n-3}\\ &= \dots\\ &= A^{n-1}(NC)^nA.\end{aligned}$$

(3) In the same way we have $[N, (AN)^k] = 0$ and $[(AN)^k, NC] = 0$(or, $[N, (NC)^k] = 0$ and $[AN, (NC)^k] = 0$) for each $k \in \mathbb{N}$. Thus one can show that $(ANC)^m = (AN)^mC^m$ (or, $(ANC)^n = A^n(NC)^n$). □

We can now formulate our first main result.

Theorem 3.2. *Let* $A, B, C \in \mathcal{B}(X)$*,* B *is Drazin invertible with* B*,* AB *and* BC *are all commute. Write*

$$\begin{aligned}\mathcal{A} =&\{ABC, BCA, CAB, ABCB, BCAB, ABCB^D, B^DABC, ABB^DC, B^DCAB, BCAB^D, CABB^D,\\ &ABCBB^D, BB^DABC\};\\ \mathcal{B} =&\{B, B^D, BB^D, AB, BC, ABC, (ABC)^D, BB^D(ABC)^D, (ABC)^DBB^D\}.\end{aligned}$$

(1) If only one element of $\mathcal{A}$ *is Drazin invertible, then all elements of* $\mathcal{A}$ *are Drazin invertible.*
(2) If only one element of $\mathcal{A}$ *is Drazin invertible, then all elements of* $\mathcal{B}$ *commute.*
(3) If only one element of $\mathcal{A}$ *is Drazin invertible, then each of the following statements hold:*

(i)

$$\begin{aligned}(ABC)^D =&(ABC)^DBB^D = BB^D(ABC)^D = (ABCB^D)^DB^D = B^D(ABCB^D)^D\\ =&(B^DABC)^DB^D = B^D(B^DABC)^D;\end{aligned}$$

(ii) $ABC(ABB^DC)^\pi$ *and* $ABC - (ABC)^2(B^DABC)^DB^D$ *are nilpotent;*
(iii) $(B^DABC)^D = (ABC)^DB = B(ABC)^D$*;*
(iv) $[(ABC)^DB, ABC(B)^D] = 0$*;*
(v) $BB^\pi(ABC)^D = (ABC)^DBB^\pi = 0$*.*

Proof. (1) Formulas (3) and (5) provided the forms of A, B, C and ABC. Note that $\{A_1, B_1, C_1\}$ are mutually commutative, $[N_1, A_4N_1] = 0$, $[N_1, N_1C_4] = 0$ and $[A_4N_1, N_1C_4] = 0$. Hence, from Lemma3.1 $A_4N_1C_4$ is nilpotent. Further

$$\begin{aligned} ABC \text{ is Drazin invertible} &\Longleftrightarrow A_1B_1C_1 \text{ is Drazin invertible} \\ &\Longleftrightarrow A_1C_1 = (A_1B_1C_1)B_1^{-1} \text{ is Drazin invertible (since } [A_1B_1C_1, B_1^{-1}] = 0). \end{aligned}$$

Also, we have $CAB = \begin{pmatrix} C_1A_1B_1 & 0 \\ C_3A_1B_1 & C_4A_4N_1 \end{pmatrix}$, and $BCA = \begin{pmatrix} B_1C_1A_1 & B_1C_1A_2 \\ 0 & N_1C_4A_4 \end{pmatrix}$.
By Lemma 3.1 $C_4A_4N_1$ and $N_1C_4A_4$ are nilpotent. Again, CAB and BCA are Drazin invertible if and only if C_1A_1 is Drazin invertible. In this case

$$(ABC)^D = \begin{pmatrix} (A_1C_1)^DB_1^{-1} & 0 \\ 0 & 0 \end{pmatrix};$$

$$(CAB)^D = \begin{pmatrix} (A_1C_1)^DB_1^{-1} & 0 \\ C_3A_1((A_1C_1)^D)^2B_1^{-1} & 0 \end{pmatrix};$$

$$(BCA)^D = \begin{pmatrix} (A_1C_1)^DB_1^{-1} & C_1((A_1C_1)^D)^2B_1^{-1}A_2 \\ 0 & 0 \end{pmatrix}.$$

We deduce that Drazin invertibility of each element of $\mathcal{A}$ lies in Drazin invertibility of A_1C_1.
(2) The set $\{B, AB, BC\}$ is commutative, then from (4), (6) and (7), the set $\{A_1, B_1, C_1\}$ is also commutative and $[N_1, A_4N_1] = [N_1, N_1C_4] = [A_4N_1, N_1C_4] = 0$. So clearly

$$N_1A_4N_1C_4 = A_4N_1N_1C_4 = A_4N_1C_4N_1,$$

that is $[N_1, A_4N_1C_4] = 0$ and, in consequence, $[B, ABC] = 0$. Similarly,

$$A_4N_1A_4N_1C_4 = A_4A_4N_1N_1C_4 = A_4N_1C_4A_4N_1,$$

which means that $[A_4N_1C_4, A_4N_1] = 0$, hence $[ABC, AB] = 0$. Besides this, $[ABC, BC] = 0$ as well. On the other hand all the element of $\mathcal{B}$ can be written as diagonal matrix forms, and this imply that all the elements of $\mathcal{B}$ commute.
(3) Observe that, $ABCB^D = AB^DBC = ABB^DC$

$$ABCB^D = \begin{pmatrix} A_1C_1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (ABCB^D)^D = \begin{pmatrix} (A_1C_1)^D & 0 \\ 0 & 0 \end{pmatrix}.$$

In addition, $ABC(ABB^DC)^\pi = \begin{pmatrix} A_1B_1C_1(A_1C_1)^\pi & 0 \\ 0 & A_4N_1C_4 \end{pmatrix}$ is nilpotent. Finally, we can verify by a simple computation the other equalities. □

The following theorem gives a partial solution of the reverse order law for the triple product ABC.

Theorem 3.3. *Let $A, B, C \in \mathcal{B}(X)$. If B, AB, BC, C are Drazin invertible and B, AB, BC are all commute, then ABC is Drazin invertible and the following reverse order laws conditions are equivalent.*

(i) $(ABC)^D = C^D(AB)^D$;
(ii) $((AB)^DABC)^D = C^D(AB)^DAB$;
(iii) $(ABC)^DAB = C^D(AB)^DAB$.

Proof. If B is Drazin invertible and $\{B, AB, BC\}$ are mutually commutative, then by (3) and (5):

$$AB = \begin{pmatrix} A_1B_1 & 0 \\ 0 & A_4N_1 \end{pmatrix} \qquad C = \begin{pmatrix} C_1 & 0 \\ C_3 & C_4 \end{pmatrix} \quad \text{and} \quad ABC = \begin{pmatrix} A_1B_1C_1 & 0 \\ 0 & A_4N_1C_4 \end{pmatrix}.$$

From the proof of [11, Theorem 3.1] AB is Drazin invertible if and only if A_1 is Drazin invertible. In this case

$$(AB)^D = \begin{pmatrix} A_1^D B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Also the Drazin invertibility of BC implies that C_1 is Drazin invertible. Now since C and C_1 are Drazin invertible then by Lemma 2.2 C_4 is also Drazin invertible.

By assuming that $\text{ind}(C_1) = s$ and $\text{ind}(C_4) = t$, we can assert that $C^D = \begin{pmatrix} C_1^D & 0 \\ Y & C_4^D \end{pmatrix}$, where

$$Y = (C_4^D)^2[\sum_{n=0}^{s-1}(C_4^D)^n C_3 C_1^n]C_1^\pi + C_4^\pi[\sum_{n=0}^{t-1} C_4^n C_3 (C_1^D)^n](C_1^D)^2 - C_4^D C_3 C_1^D.$$

Also, from Lemma 3.1, $A_4N_1C_4$ is nilpotent $\{A_1, B_1, C_1\}$ are mutually commutative and A_1, B_1, C_1 are all Drazin invertible. Hence, ABC is also Drazin invertible and

$$(ABC)^D = \begin{pmatrix} A_1^D B_1^{-1} C_1^D & 0 \\ 0 & 0 \end{pmatrix}.$$

Now let's mention that

$$(AB)^D ABC = \begin{pmatrix} A_1^D B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1B_1C_1 & 0 \\ 0 & A_4N_1C_4 \end{pmatrix} = \begin{pmatrix} A_1^D A_1 C_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$((AB)^D ABC)^D = \begin{pmatrix} A_1^D A_1 C_1^D & 0 \\ 0 & 0 \end{pmatrix}, \text{(since } [C_1, A_1A_1^D] = 0 \text{ and } A_1A_1^D \text{is a projection)}$$

$$C^D(AB)^D = \begin{pmatrix} C_1^D & 0 \\ Y & C_4^D \end{pmatrix}\begin{pmatrix} A_1^D B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1^D A_1^D B_1^{-1} & 0 \\ Y A_1^D B_1^{-1} & 0 \end{pmatrix},$$

$$C^D(AB)^D AB = \begin{pmatrix} C_1^D A_1^D A_1 & 0 \\ Y A_1^D A_1 & 0 \end{pmatrix}.$$

We can deduce that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow YA_1^D = 0$. □

A similar observation gives the following theorem and its proof will be omitted.

Theorem 3.4. *Let $A, B, C \in \mathcal{B}(X)$. If A, B, AB, BC are Drazin invertible and B, AB, BC are all commute, then ABC is Drazin invertible and the following reverse order laws conditions are equivalent.*

(i) $(ABC)^D = (BC)^D A^D$;
(ii) $(ABC(BC)^D)^D = (BC)^D(BC)A^D$;
(iii) $BC(ABC)^D = (BC)^D(BC)A^D$.

Theorem 3.5. *Let $A, B, C \in \mathcal{B}(X)$. If A, B, C, AB, BC are Drazin invertible and B, AB, BC are all commute, then the following reverse order law conditions are equivalent:*

(i) $(BCAB)^D = B^D A^D C^D B^D$;
(ii) $(ABB^DC)^D = BB^D A^D C^D B^D B$;
(iii) $B(BCAB)^D B = BB^D A^D C^D B^D B$.

Proof. The Drazin invertibility of A, B, C, AB, BC combined with the commutativity conditions of B, AB, BC provided the following matrix forms

$$A^D = \begin{pmatrix} A_1^D & X \\ 0 & A_4^D \end{pmatrix} \qquad C^D = \begin{pmatrix} C_1^D & 0 \\ Y & C_4^D \end{pmatrix}, \tag{8}$$

with

$$X=(A_1^D)^2[\sum_{n=0}^{t_1-1}(A_1^D)^nA_2A_4^n]A_4^\pi+A_1^\pi[\sum_{n=0}^{s_1-1}A_1^nA_2(A_4^D)^n](A_4^D)^2-A_1^DA_2A_4^D,$$

$$Y=(C_4^D)^2[\sum_{n=0}^{s_2-1}(C_4^D)^nC_3C_1^n]C_1^\pi+C_4^\pi[\sum_{n=0}^{t_2-1}C_4^nC_3(C_1^D)^n](C_1^D)^2-C_4^DC_3C_1^D.$$

Here $\mathrm{ind}(A_1)=s_1$, $\mathrm{ind}(A_4)=t_1$, $\mathrm{ind}(C_1)=s_2$ as well as $\mathrm{ind}(C_4)=t_2$. Also $BCAB=\begin{pmatrix}C_1(B_1)^2A_1 & 0\\ 0 & N_1C_4A_4N_1\end{pmatrix}$. Certainly, $N_1C_4A_4N_1$ is nilpotent and $(BCAB)^D=\begin{pmatrix}C_1^D(B_1^{-1})^2A_1^D & 0\\ 0 & 0\end{pmatrix}$. Moreover, $ABB^DC=\begin{pmatrix}A_1C_1 & 0\\ 0 & 0\end{pmatrix}$ and $(ABB^DC)^D=\begin{pmatrix}A_1^DC_1^D & 0\\ 0 & 0\end{pmatrix}$. By a simple calculation, we can obtain the following:

$$B^DA^DC^DB^D=\begin{pmatrix}A_1^D(B_1^{-1})^2C_1^D+B_1^{-1}XYB_1^{-1} & 0\\ 0 & 0\end{pmatrix},$$

$$BB^DA^DC^DB^DB=\begin{pmatrix}A_1^DC_1^D+XY & 0\\ 0 & 0\end{pmatrix},$$

$$B(BCAB)^DB=\begin{pmatrix}C_1^DA_1^D & 0\\ 0 & 0\end{pmatrix}.$$

This gives the following equivalences $(i)\Longleftrightarrow(ii)\Longleftrightarrow(iii)\Longleftrightarrow XY=0$. □

In the following theorem, we get a first glimpse of $(ABC)^D=C^DB^DA^D$.

Theorem 3.6. *Let $A,B,C\in\mathcal{B}(X)$. If A,B,C,AB,BC are Drazin invertible and B,AB,BC are all commute, then ABB^D,B^DBC,ABC are all Drazin invertible. Furthermore, the following reverse order law conditions are equivalent:*

1. $(ABC)^D=C^DB^DA^D$;
2. $C^D(AB)^D=C^DB^DA^D=(BC)^DA^D$;
3. $BB^DC^DB^DA^D=C^DB^DA^D=C^DB^DA^DBB^D$;
4. $(ABB^D)^DB^D(B^DBC)^D=C^DB^DA^D$;
5. $A^DB^DBC^DB^DA^DABB^D=C^DB^DA^D$;
6. $B^\pi C^DB^DA^D=BB^\pi C^DB^DA^D$ *and* $C^DB^DA^DB^\pi=C^DB^DA^DB^\pi B$.

Proof. AB,BC and ABC have the matrix forms:

$$AB=\begin{pmatrix}A_1B_1 & 0\\ 0 & A_4N_1\end{pmatrix},\qquad BC=\begin{pmatrix}B_1C_1 & 0\\ & N_1C_4\end{pmatrix}\quad\text{and}\quad ABC=\begin{pmatrix}A_1B_1C_1 & 0\\ 0 & A_4N_1C_4\end{pmatrix}.$$

Of course, A_4N_1, N_1C_4 and $A_4N_1C_4$ are nilpotent. Moreover, A and AB are Drazin invertible (*resp*, C and BC) then A_1 and A_4 (*resp*, C_1 and C_4) are Drazin invertible. Hence, it is easy to verify that ABC is Drazin invertible. In this case, we obtain

$$(AB)^D=\begin{pmatrix}A_1^DB_1^{-1} & 0\\ 0 & 0\end{pmatrix},\qquad (BC)^D=\begin{pmatrix}B_1^{-1}C_1^D & 0\\ & 0\end{pmatrix}\quad\text{and}\quad (ABC)^D=\begin{pmatrix}A_1^DB_1^{-1}C_1^D & 0\\ 0 & 0\end{pmatrix}.$$

On the other hand A^D, C^D can be written as in (8). So we get

$$C^DB^DA^D=\begin{pmatrix}A_1^DB_1^{-1}C_1^D & C_1^DB_1^{-1}X\\ YB_1^{-1}A_1^D & YB_1^{-1}X\end{pmatrix}.$$

Equivalent conditions of $(ABC)^D = C^DB^DA^D$ are: $\begin{cases} C_1^DX &= 0 \\ YA_1^D &= 0 \\ YB_1^{-1}X &= 0 \end{cases}$. Note that

$$C^D(AB)^D = C^DB^DA^DBB^D = \begin{pmatrix} C_1^DA_1^DB_1^{-1} & 0 \\ YA_1^DB_1^{-1} & 0 \end{pmatrix},$$

and

$$(BC)^DA^D = BB^DC^DB^DA^D = \begin{pmatrix} B_1^{-1}C_1^DA_1^D & C_1^DB_1^{-1}X \\ 0 & 0 \end{pmatrix}.$$

Therefore, (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow \begin{cases} C_1^DX &= 0 \\ YA_1^D &= 0 \\ YB_1^{-1}X &= 0 \end{cases}$. Also it is easy to show that

$$(ABC)^D = (ABB^D)^DB^D(B^DBC)^D = A^DB^DBC^DB^DA^DABB^D.$$

So, (1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5). Finally,

$$B^\pi C^DB^DA^D = \begin{pmatrix} 0 & 0 \\ YB_1^{-1}A_1^D & YB_1^{-1}X \end{pmatrix},$$
$$C^DB^DA^DB^\pi = \begin{pmatrix} 0 & C_1^DB_1^{-1}X \\ 0 & YB_1^{-1}X \end{pmatrix},$$
$$BB^\pi C^DB^DA^D = \begin{pmatrix} 0 & 0 \\ N_1YB_1^{-1}A_1^D & N_1YB_1^{-1}X \end{pmatrix},$$
$$C^DB^DA^DB^\pi B = \begin{pmatrix} 0 & C_1^DB_1^{-1}XN_1 \\ 0 & YB_1^{-1}XN_1 \end{pmatrix}.$$

Thus, (6) $\Longleftrightarrow \begin{cases} (I-N_1)C_1^DB_1^{-1}X &= 0 \\ YB_1^{-1}A_1^D(I-N_1) &= 0 \\ (I-N_1)YB_1^{-1}X &= 0 \end{cases} \Longleftrightarrow \begin{cases} C_1^DX &= 0 \\ YA_1^D &= 0 \\ YB_1^{-1}X &= 0 \end{cases}$. B_1 and $I - N_1$ are invertible (because N_1 is nilpotent). This is the desired conclusion. □

Inserting the revers order law of AB in Theorem3.3 yields the following corollary.

Corollary 3.7. *Let $A, B, C \in \mathcal{B}(X)$ be such that A, B, C, AB, BC are Drazin invertible and B, AB, BC are all commute. If $(AB)^D = B^DA^D$ then the following reverse order law conditions are equivalent:*

(i) $(ABC)^D = C^DB^DA^D$;
(ii) $((AB)^DABC)^D = C^DB^DA^DAB$;
(iii) $(ABC)^DAB = C^DB^DA^DAB$.

Proof. The reverse order law condition $(AB)^D = B^DA^D$ is equivalent to $X = 0$. Thus $C^DB^DA^D = \begin{pmatrix} A_1^DB_1^{-1}C_1^D & 0 \\ YB_1^{-1}A_1^D & 0 \end{pmatrix}$, and the equality $(ABC)^D = C^DB^DA^D$ is equivalent to $YA_1^D = 0$.

$$C^DB^DA^DAB = \begin{pmatrix} C_1^DA_1^DA_1 & 0 \\ YA_1^DA_1 & 0 \end{pmatrix} \quad \text{and} \quad ((AB)^DABC)^D = \begin{pmatrix} A_1^DA_1C_1^D & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $((AB)^DABC)^D = C^DB^DA^DAB \Longleftrightarrow YA_1^D = 0$.
Also, $(ABC)^DAB = C^DB^DA^DAB \Longleftrightarrow YA_1^D = 0$. Which complete the proof. □

In a similar pattern using the reverse order law of BC in Theorem 3.4, we obtain:

Corollary 3.8. *Let $A, B, C \in \mathcal{B}(X)$ be such that A, B, C, AB, BC are Drazin invertible and B, AB, BC are all commute. If $(BC)^D = C^D B^D$ then the following reverse order law conditions are equivalent:*

(i) $(ABC)^D = C^D B^D A^D$;

(ii) $(ABC(BC)^D)^D = BCC^D B^D A^D$;

(iii) $BC(ABC)^D = BCC^D B^D A^D$.

Acknowledgments We gratefully acknowledge the judicious comments and suggestions provided by the anonymous referees.

References

[1] P. Aiena, Fredholm and local spectral theory II with application to Weyl-type theorems, Springer, 2018.
[2] Dijana Mosić, Reverse order laws for the generalized Drazin inverse in Banach algebras, J. Math. Anal. Appl, 429 (2015) 461–477.
[3] Dijana Mosić, Reverse order laws on the conditions of the commutativity up to a factor, RACSAM. 111 (2017) 685–695.
[4] D.S. Djordjević,Unified approach to the reverse order rule for generalized inverses, Acta Sci. Math. (Szeged) 167 (2001) 761–776.
[5] N.C. Dinčić, D.S. Djordjević, Hartwig's triple reverse order law revisited, Lin. Multi. Algebra 62 (2014) 918–924.
[6] J.J. Koliha, A generalized Drazin inverse, Glasgow Math.J 38 (1996) 367-381.
[7] Liao YH, Chen JL, Cui J, Cline's formula for the generalized Drazin inverse, B. Malays. Math. Sci. So 37 (2014) 37–42.
[8] Meyer CD, Rose NJ, The Index and the Drazin Inverse of Block Triangular Matrices, SIAM. J.Appl. Math 33 (19777) 1-7.
[9] H. Wang, J. Huang, Reverse Order Law of Drazin Inverse for Bounded Linear Operators, Filomat 32 (2018) 4857–4864.
[10] H. Wang, J. Huang, Reverse Order Law for the Drazin Inverse in Banach Spaces, Bull. Iran. Math. Soc 45 (2019) 1443–1456.
[11] X. N. Wang, A. Q. Yu, T. F. Li, C. Y. Deng, Reverse order laws for the Drazin inverses, J. Math. Anal. Appl 444 (2016) 672-689.