Filomat 35:1 (2021), 125–146
https://doi.org/10.2298/FIL2101125K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ricci Curvature of Contact CR-Warped Product Submanifolds in Generalized Sasakian Space Forms Admitting a Trans-Sasakian Structure

Meraj Ali Khan[a], **Cenep Ozel**[b]

[a]*Department of Mathematics, University of Tabuk, KSA*
[b]*Department of Mathematics, King Abdulaziz University, KSA*

Abstract. The objective of this paper is to achieve the inequality for Ricci curvature of a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form admitting a trans-Sasakian structure in the expressions of the squared norm of mean curvature vector and warping function. We provide numerous physical applications of the derived inequalities. Finally, we prove that under a certain condition the base manifold is isometric to a sphere with a constant sectional curvature.

1. Introduction

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and ψ be a positive differentiable function on N_1. If $\pi : N_1 \times N_2 \to N_1$ and $\eta : N_1 \times N_2 \to N_2$ are the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in N_1 \times N_2$, then the *warped product manifold* is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (\psi \circ \pi)^2 g_2(\eta_* X, \eta_* Y),$$

for all $X, Y \in TM$. The function ψ is called the *warping function* of the warped product manifold [24]. If the warping function is constant, then the warped product is trivial i.e., simply Riemannian product. On the grounds that warped product manifolds admit a number of applications in Physics and theory of relativity [33], this has been a topic of extensive research. Warped products provide many basic solutions to Einstein field equations [33]. The concept of modelling of space-time near black holes adopts the idea of warped product manifolds [34]. Schwartzschild space-time is an example of warped product $P \times_r S^2$, where the base $P = R \times R^+$ is a half plane $r > 0$ and the fibre S^2 is the unit sphere. Under certain conditions, the Schwartzchild space-time becomes the black hole. A cosmological model to model the universe as a space-time known as Robertson-Walker model is a warped product [35].

Some natural properties of warped product manifolds were studied in [24]. B. Y. Chen ([1], [2]) performed an extrinsic study of warped product submanifolds in a Kaehler manifold. Since then, many geometers

2010 *Mathematics Subject Classification*. 53C25, 53C40, 53C42, 53D15
Keywords. Contact CR-submanifolds, Generalized Sasakian space form, Ricci curvature, Laplacian, Hamiltonian, Dirichlet energy.
Received: 29 February 2020; Revised: 21 May 2020; Accepted: 06 August 2020
Communicated by Mića Stanković
Email addresses: meraj79@gmail.com (Meraj Ali Khan), cenap.ozel@gmail.com (Cenep Ozel)

have explored warped product manifolds in different settings like almost complex and almost contact manifolds and various existence results have been investigated (see the survey article [10]).

In 1999, Chen [6] discovered a relationship between Ricci curvature and squared mean curvature vector for an arbitrary Riemannian manifold. On the line of Chen a series of articles have been appeared to formulate the relationship between Ricci curvature and squared mean curvature in the setting of some important structures on Riemannian manifolds (see [12], [13], [16], [17], [18], [38]). Recently Ali et al. [20] established a relationship between Ricci curvature and squared mean curvature for warped product submanifolds of a sphere and provide many physical applications.

In this paper our aim is to obtain a relationship between Ricci curvature and squared mean curvature for contact CR-warped product submanifolds in the setting of generalized Sasakian space form admitting a trans-Sasakian structure. Further, we provide some applications in terms of Hamiltonians and Euler-Lagrange equation. In the last we also worked out some applications of Obata's differential equation.

2. Preliminaries

A $(2n+1)-$dimensional $C^\infty-$manifold $\bar{M}$ is said to have an almost contact structure if there exist on $\bar{M}$ a tensor field ϕ of the type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \oplus \xi, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0, \;\; \eta(\xi) = 1. \tag{1}$$

There always exists a Riemannian metric g on an almost contact metric manifold $\bar{M}$ satisfying the following conditions

$$\eta(X) = g(X, \xi), \;\; g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

for all $X, Y \in T\bar{M}$.

An almost contact metric structure on a manifold $\bar{M}$ is called a trans-Sasakian structure if the product manifold $\bar{M} \times R$ belongs to class $\mathcal{W}_4$ [41]. J. C. Marrero [43] provided the tensorial equations characterizing the trans-Sasakian structure. D. Blair and J. A. Oubina [42] showed that an almost contact metric structure (ϕ, ξ, η, g) is trans-Sasakian structure if it satisfies the following formula

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{3}$$

for some smooth functions α and β on $\bar{M}$ and $\bar{\nabla}$ being the Levi-Civita connection of $\bar{M}$.

From the formula (3) it follows that

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{4}$$

If $(\bar{M}, \phi, \xi, \eta, \xi)$ be a trans-Sasakian manifold, then $(\bar{M} \times R, J, g)$ belongs to class $\mathcal{W}_4$ of the almost Hermitian manifolds.

In [37] P. Alegre et al. introduced the notion of generalized Sasakian space form as that an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ whose curvature tensor $\bar{R}$ satisfies

$$\begin{aligned} \bar{R}(X, Y, Z, W) =& f_1[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &- f_2[g(\phi X, Z)g(\phi Y, W) - g(\phi X, W)g(\phi Y, Z) \\ &+ 2g(\phi X, Y)g(\phi Z, W)] - f_3[\eta(Z)\{\eta(Y)g(X, W) \\ &- \eta(X)g(Y, W)\} + \eta(W)\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}] \end{aligned} \tag{5}$$

for all vector fields X, Y, Z, W and certain differentiable functions f_1, f_2, f_3 on $\bar{M}$. A generalized Sasakian space form with functions f_1, f_2, f_3 is denoted by $\bar{M}(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then $\bar{M}(f_1, f_2, f_3)$

becomes a Sasakian space form $\bar{M}(c)$ [37]. If $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ becomes a Kenmotsu space form $\bar{M}(c)$ [37] and if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\bar{M}(f_1, f_2, f_3)$ becomes a cosymplectic space form $\bar{M}(c)$ [37].

Let M be an $n-$dimensional Riemannian manifold isometrically immersed in a $m-$dimensional Riemannian manifold $\bar{M}$. Then the Gauss and Weingarten formulas are $\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$ respectively, for all $X, Y \in TM$ and $N \in T^{\perp}M$. Where ∇ is the induced Levi-civita connection on M, N is a vector field normal to M, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of the second fundamental form. The second fundamental form h and the shape operator are associated by the following formula

$$g(h(X, Y), N) = g(A_N X, Y). \tag{6}$$

The equation of Gauss is given by

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \tag{7}$$

for all $X, Y, Z, W \in TM$. Where, $\bar{R}$ and R are the curvature tensors of $\bar{M}$ and M respectively.

For any $X \in TM$ and $N \in T^{\perp}M$, ϕX and ϕN can be decomposed as follows

$$\phi X = PX + FX \tag{8}$$

and

$$\phi N = tN + fN, \tag{9}$$

where PX (resp. tN) is the tangential and FX (resp. fN) is the normal component of ϕX (resp. ϕN).

For any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space $T_x M$, the mean curvature vector $H(x)$ and its squared norm are defined as follows

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \quad \|H\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} g(h(e_i, e_i), h(e_j, e_j)), \tag{10}$$

where n is the dimension of M. If $h = 0$ then the submanifold is said to be totally geodesic and minimal if $H = 0$. If $h(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then M is called totally umbilical.

The scalar curvature of $\bar{M}$ is denoted by $\bar{\pi}(\bar{M})$ and is defined as

$$\bar{\pi}(\bar{M}) = \sum_{1\le p<q\le m} \bar{\kappa}_{pq}, \tag{11}$$

where $\bar{\kappa}_{pq} = \bar{\kappa}(e_p \wedge e_q)$ and m is the dimension of the Riemannian manifold $\bar{M}$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\bar{\pi}(\bar{M}) = \sum_{1\le p<q\le m} \bar{\kappa}_{pq}. \tag{12}$$

In a similar way, the scalar curvature $\bar{\pi}(L_x)$ of a $L-$plane is given by

$$\bar{\pi}(L_x) = \sum_{1\le p<q\le m} \bar{\kappa}_{pq}. \tag{13}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_xM and if e_r belongs to the orthonormal basis $\{e_{n+1}, \ldots e_m\}$ of the normal space $T^{\perp}M$, then we have

$$h^r_{pq} = g(h(e_p, e_q), e_r)) \tag{14}$$

and

$$\|h\|^2 = \sum_{p,q=1}^{n} g(h(e_p, e_q), h(e_p, e_q)). \tag{15}$$

Let κ_{pq} and $\bar{\kappa}_{pq}$ be the sectional curvatures of the plane sections spanned by e_p and e_q at x in the submanifold M^n and in the Riemannian space form $\bar{M}^m(c)$, respectively. Thus by Gauss equation, we have

$$\kappa_{pq} = \bar{\kappa}_{pq} + \sum_{r=n+1}^{m} (h^r_{pp}h^r_{qq} - (h^r_{pq})^2). \tag{16}$$

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_n\}$ on M^n is defined as

$$\bar{S}(X, Y) = \sum_{i=1}^{n} \{g(\bar{R}(e_i, Y)Y, e_i)\}, \tag{17}$$

for all $X, Y \in TM^n$. The above tensor is called the Ricci tensor. If we fix a distinct vector e_u from $\{e_1, \ldots, e_n\}$ on M^n, which is governed by χ. Then the Ricci curvature is defined by

$$Ric(\chi) = \sum_{\substack{p=1 \\ p\neq u}}^{n} \kappa(e_p \wedge e_u). \tag{18}$$

Consider the warped product submanifold $N_1 \times_{\psi} N_2$. Let X be a vector field on M_1 and Z be a vector field on M_2, then from Lemma 7.3 of [24], we have

$$\nabla_X Z = \nabla_Z X = (\frac{X\psi}{\psi})Z, \tag{19}$$

where ∇ is the Levi-Civita connection on M. For a warped product $M = M_1 \times_{\psi} M_2$ it is easy to observe that

$$\nabla_X Z = \nabla_Z X = (Xln\psi)Z, \tag{20}$$

for $X \in TM_1$ and $Z \in TM_2$.

$\nabla\psi$ is the gradient of ψ and is defined as

$$g(\nabla\psi, X) = X\psi, \tag{21}$$

for all $X \in TM$.

Let M be an $n-$dimensional Riemannian manifold with the Riemannian metric g and let $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal basis of TM. Then as a result of (21), we get

$$\|\nabla\psi\|^2 = \sum_{i=1}^{n} (e_i(\psi))^2. \tag{22}$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{n} \{(\nabla_{e_i} e_i)\psi - e_i e_i \psi\}. \tag{23}$$

The Hessian tensor for a differentiable function ψ is a symmetric covariant tensor of rank 2 and is defined as

$$\Delta\psi = -traceH^{\psi}$$

For the warped product submanifolds $N_1^{n_1} \times_{\psi} N_2^{n_2}$, we have following well known result [9]

$$\sum_{p=1}^{n_1}\sum_{q=1}^{n_2} \kappa(e_p \wedge e_q) = \frac{n_2\Delta\psi}{\psi} = n_2(\Delta ln\psi - \|\nabla ln\psi\|^2), \tag{24}$$

where n_1 and n_2 are the dimensions of the submanifolds $N_1^{n_1}$ and $N_2^{n_2}$ respectively.
Now, we state the Hopf's Lemma.

Hopf's Lemma [3]. If M is a $m-$dimensional connected compact Riemannian manifold. If ψ is a differentiable function on M s. t. $\Delta\psi \geq 0$ everywhere on M (or $\Delta\psi \leq 0$ everywhere on M), then ψ is a constant function.

For a compact orientable Riemannian manifold M with or without boundary and as a consequences of the integration theory of manifolds, we have

$$\int_M \Delta\psi dV = 0, \tag{25}$$

where ψ is a function on M and dV is the volume element of M.

3. Contact CR-warped product submanifolds of a trans-Sasakian manifold

Suppose M be a $n-$dimensional submanifold isometrically immersed in an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ such that the structure vector field ξ is tangent to M. The submanifold M is called contact CR-submanifold if it admits an invariant distribution D whose orthgonal complementary distribution $D^{\perp}$ is anti-invariant such that $TM = D \oplus D^{\perp} \oplus \langle\xi\rangle$, where $\phi D \subseteq D$, $\phi D^{\perp} \subseteq T^{\perp}M$ and $\langle\xi\rangle$ is the 1-dimensional distribution spanned by ξ. If μ is the invariant subspace of the normal bundle $T^{\perp}M$, then in the case of contact CR- submanifold, the normal bundle $T^{\perp}M$ can be decomposed as $T^{\perp}M = \mu \oplus \phi D^{\perp}$. A contact CR-submanifold is called contact CR-product submanifold if the distributions D and $D^{\perp}$ are parallel on M. As a generalization of the product manifold submanifolds one can consider warped product submanifolds. I. Hesigawa and I. Mihai [12] extended the study of Chen for the contact CR-warped product submanifolds of the Sasakian manifolds. Moreover, I. Mihai [13] obtained the estimation for the squared norm of second fundamental form in terms of the warping function for contact CR-warped product submanifolds in the setting of Sasakian space form. Further, K. Arslan et al. [14] extended the study of I. Mihai and Chen and they established an inequality for second fundamental form in terms of warping function for the contact CR-warped product submanifolds of a Kenmotsu space form. Using different techniques and methodology M. Atceken ([38], [39]) proved the inequalities for existence of contact CR-warped product submanifolds for Kenmotsu and cosymplectic space forms. Later Sibel Sular and Cihan Özgür [40] generalized these inequalities for contact CR-warped product submanifolds of generalized Sasakian space form admitting trans-Sasakian structure.

It is well known that two classes of almost contact metric manifolds namely Sasakian and Kenmotsu manifolds are quit different from each other and it has always been interesting to explore that how far a submanifold of a Sasakian manifold differ or resemble with that of Kenmotsu manifold. The setting of trans-Sasakian manifolds in a way unifies the two classes of manifolds. By studying the contact CR-warped product submanifolds of a generalized Sasakian space form admitting trans-Sasakian structure one clearly find out the deviations in the geometric behavior of a contact CR-warped product submanifold in the Sasakian and Kenmotsu space forms. Throughout, this study we consider $n-$dimensional contact

CR-warped product submanifold $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$, such that the structure vector field ξ is tangential to N_T, where n_1 and n_1 are the dimensions of the invariant and anti-invariant submanifold respectively.

Now, we have the following initial result

Lemma 3.1. *Let $M = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a trans-Sasakian manifold $\bar{M}$, then*

(i) $g(h(X,Y), \phi Z) = 0$,

(ii) $g(h(\phi X, \phi X), N) = -g(h(X,X), N)$,

for any $X, Y \in TN_T$, $Z \in TN_\perp$ and $N \in \mu$.

Proof. By using Gauss and Weingarten formulae in equation (3), we have

$$-A_{\phi Z}X - \nabla_X^\perp \phi Z - \phi \nabla_X Z - \phi h(X,Z) + \nabla_Z \phi X + h(\phi X, Z) \\ - \phi \nabla_Z X - \phi h(X,Z) = -\eta(X)\phi Z,$$

taking inner product with Y and using (6), we get the required result.

To prove (*ii*), for any $X \in TN_T$ we have

$$\bar{\nabla}_X \phi X = (\bar{\nabla}_X \phi)X + \phi \bar{\nabla}_X X,$$

using Gauss formula and (3), we get

$$\nabla_X \phi X + h(\phi X, X) = -\eta(X)\phi X + \phi \nabla_X X + \phi h(X,X),$$

taking inner product with ϕN, above equation yields

$$g(h(\phi X, X), \phi N) = g(h(X,X), N), \tag{26}$$

interchanging X by ϕX and using (4), the above equation gives

$$g(h(\phi X, X), JN) = -g(h(\phi X, \phi X), N). \tag{27}$$

From (26) and (27), we get the required result. □

By the Lemma 3.1 it is evident that the isometric immersion $N_T^{n_1} \times_\psi N_\perp^{n_2}$ into a trans-Sasakian manifold $\bar{M}$ is $D-$ minimal. The D- minimality property provides us a useful relationship between the CR-warped product submanifold $N_T \times_\psi N_\perp$ and the equation of Gauss.

Definition 3.1 The warped product $N_1 \times_\psi N_2$ isometrically immersed in a Riemannian manifold $\bar{M}$ is called N_i totally geodesic if the partial second fundamental form h_i vanishes identically. It is called N_i-minimal if the partial mean curvature vector H^i becomes zero for $i = 1, 2$.

Let $\{e_1, \ldots, e_\alpha, e_{\alpha+1} = \phi e_1 \ldots, \ldots e_{n_1-1} = \phi e_\alpha, e_{n_1} = \xi, e_{n_1+1}, \ldots, e_n\}$ be a local orthonormal frame of vector fields on the contact CR-warped product submanifold $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ such that $\{\xi, e_1, \ldots, e_{n_1}\}$ are tangent to N_T and $\{e_{n_1+1}, \ldots e_n\}$ are tangent to $N_\perp$. Moreover, $\{e_1^* = \phi e_{n_1+1}, \ldots, e_n^* = \phi e_n, e_{n+1}^*, \ldots, e_m^*\}$ is a local orthonormal frame of the normal space $T^\perp M$.

From Lemma 3.1, it is easy to conclude that

$$\sum_{r=n+1}^{m} \sum_{i,j=1}^{n_1} g(h(e_i, e_j), e_r) = 0. \tag{28}$$

Thus it follows that the trace of h due to N_T becomes zero. Hence in view of the Definition 3.1, we obtain the following important result.

Theorem 3.2. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a trans-Sassakian manifold. Then M^n is $D-$ minimal.*

So, it is easy to conclude the following

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{m} (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}), \tag{29}$$

where $\|H\|^2$ is the squared mean curvature.

4. Ricci curvature for contact CR-warped product submanifold

In this section, we investigate Ricci curvature in terms of the squared norm of mean curvature and the warping function as follows

Theorem 4.1. *Let $M = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a trans-Sasakian structure . Then for each orthogonal unit vector field $\chi \in T_xM$ orthogonal to ξ, either tangent to N_T or $N_\perp$ we have*

(1) The Ricci curvature satisfies the following inequality.

(i) If χ is tangent to $N_T^{n_1}$, then

$$\begin{aligned} Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 - \frac{n_2 \Delta\psi}{\psi} + (n + n_1 n_2 - 1) f_1 + \frac{3f_2}{2} \\ - (n_2 + 1) f_3. \end{aligned} \tag{30}$$

(ii) χ is tangent to $N_\perp^{n_2}$, then

$$\begin{aligned} Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 - \frac{n_2 \Delta\psi}{\psi} + (n + n_1 n_2 - 1) f_1 \\ - (n_2 + 1) f_3. \end{aligned} \tag{31}$$

(2) If $H(x) = 0$ for each $x \in M^n$, then there is a unit vector field X which satisfies the equality case of (1) if and only if M^n is mixed totally geodesic and χ lies in the relative null space N_x at x.

(3) For the equality case we have

(a) The equality case of (30) holds identically for all unit vector fields tangent to N_T at each $x \in M^n$ if and only if M^n is mixed totally geodesic and $D-$totally geodesic contact CR-warped product submanifold in $\bar{M}^m(f_1, f_2, f_3)$.

(b) The equality case of (31) holds identically for all unit vector fields tangent to $N_\perp$ at each $x \in M^n$ if and only if M is mixed totally geodesic and either M^n is $D^\perp$- totally geodesic contact CR-warped product or M^n is a $D^\perp$ totally umbilical in $\bar{M}^m(f_1, f_2, f_3)$ with dim $D^\perp = 2$.

(c) The equality case of (1) holds identically for all unit tangent vectors to M^n at each $x \in M^n$ if and only if either M^n is totally geodesic submanifold or M^n is a mixed totally geodesic totally umbilical and $D-$ totally geodesic submanifold with dim $N_\perp = 2$.

Where n_1 and n_2 are the dimensions of $N_T^{n_1}$ and $N_\perp^{n_2}$ respectively.

Proof. Suppose that $M = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold of a generalized Sasakian space form. From Gauss equation, we have

$$n^2 \|H\|^2 = 2\pi(M^n) + \|h\|^2 - 2\bar{\pi}(M^n). \tag{32}$$

Let $\{e_1, \ldots, e_{n_1}, e_{n_1+1}, \ldots, e_n\}$ be a local orthonormal frame of vector fields on M^n such that $\{e_1, \ldots, e_{n_1}\}$ are tangent to N_T and $\{e_{n_1+1}, \ldots, e_n\}$ are tangent to $N_\perp$. So, the unit tangent vector $\chi = e_A \in \{e_1, \ldots, e_n\}$ can be expanded (32) as follows

$$n^2\|H\|^2 = 2\pi(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h^r_{11} + \cdots + h^r_{nn} - h^r_{AA})^2 + (h^r_{AA})^2\}$$

$$-\sum_{r=n+1}^{m}\sum_{1\le p\ne q\le n} h^r_{pp}h^r_{qq} - 2\bar{\pi}(M^n). \tag{33}$$

□

The above expression can be written as follows

$$\begin{aligned} n^2\|H\|^2 = 2\pi(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h^r_{11} + \cdots + h^r_{nn})^2 \\ + (2h^r_{AA} - (h^r_{11} + \cdots + h^r_{nn}))^2\} + 2\sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(h^r_{pq})^2 \\ - 2\sum_{r=n+1}^{m}\sum_{1\le p<q\le n} h^r_{pp}h^r_{qq} - 2\bar{\pi}(M^n). \end{aligned}$$

In view of the Lemma 3.1, the preceding expression takes the form

$$\begin{aligned} n^2\|H\|^2 = \sum_{r=n+1}^{m}\{(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2 + +(2h^r_{AA} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2\} \\ + 2\pi(M^n) + \sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{1\le p<q\le n} h^r_{pp}h^r_{qq} + \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\ne A}}(h^r_{aA})^2 \\ + \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}} h^r_{pp}h^r_{qq} - 2\bar{\pi}(M^n). \end{aligned} \tag{34}$$

By equation (16), we have

$$\begin{aligned} \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}} h^r_{pp}h^r_{qq} \\ = \sum_{\substack{1\le p<q\le n\\ p,q\ne A}}\bar{\kappa}_{p,q} - \sum_{\substack{1\le p<q\le n\\ p,q\ne A}}\kappa_{p,q} \end{aligned} \tag{35}$$

Substituting the values of equation (35) in (34), we discover

$$\begin{aligned} n^2\|H\|^2 = 2\pi(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2 \\ + \sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{1\le p<q\le n} h^r_{pp}h^r_{qq} - 2\bar{\pi}(M^n) \\ + \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\ne A}}(h^r_{aA})^2 + \sum_{\substack{1\le p<q\le n\\ p,q\ne A}}\bar{\kappa}_{p,q} - \sum_{\substack{1\le p<q\le n\\ p,q\ne A}}\kappa_{p,q}. \end{aligned} \tag{36}$$

Since, $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$, then from (13), the scalar curvature of M^n can be defined as follows

$$\begin{aligned}\pi(M^n) &= \sum_{1\le p<q\le n} \kappa(e_p \wedge e_q)\\ &= \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} \kappa(e_i \wedge e_j) + \sum_{1\le i<k\le n_1} \kappa(e_i \wedge e_k) + \sum_{n_1+1\le l<o\le n} \kappa(e_l \wedge e_o)\end{aligned} \tag{37}$$

The usage of (13) and (24), we derive

$$\pi(M^n) = \frac{n_2\Delta\psi}{\psi} + \pi(N_T^{n_1}) + \pi(N_\perp^{n_2}) \tag{38}$$

Utilizing (38) together with (5) in (36), we have

$$\begin{aligned}\frac{1}{2}n^2\|H\|^2 =& \frac{n_2\Delta\psi}{\psi} + \sum_{\substack{1\le p<q\le n\\ p,q\neq A}} \bar{\kappa}_{p,q} + \bar{\pi}(N_T^{n_1}) + \bar{\pi}(N_\perp^{n_2})\\ &+ \sum_{r=n+1}^{m}\Big\{\sum_{1\le p<q\le n}(h_{pq}^r)^2 - \sum_{\substack{1\le p<q\le n\\ p,q\neq A}} h_{pp}^r h_{qq}^r\Big\}\\ &+ \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\neq A}}(h_{aA}^r)^2 + \sum_{r=n+1}^{m}\sum_{1\le i\neq j\le n_1}(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2)\\ &+ \sum_{r=n+1}^{m}\sum_{n_1+1\le s\neq t\le n}(h_{ss}^r h_{tt}^r - (h_{st}^r)^2)\\ &+ \frac{1}{2}\sum_{r=n+1}^{m}(2h_{AA}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r))^2\\ &- \{f_1(n(n-1)) + f_2(3(n_1-1)) - f_3(2(n-1))\}.\end{aligned} \tag{39}$$

Considering unit tangent vector $\chi = e_a$, we have two choices: χ is either tangent to the base manifold $N_T^{n_1}$ or to the fibre $N_\perp^{n_2}$.

Case i: If e_a is tangent to $N_T^{n_1}$, then fix a unit tangent vector from $\{e_1, \ldots, e_{n_1}\}$ suppose $\chi = e_a = e_1$, then from (39) and (18), we find

$$\begin{aligned}Ric(\chi) \le& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} - \frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \ldots h_{nn}^r))^2\\ &- \sum_{r=n+1}^{m}\sum_{1\le p<q\le n_1}(h_{pq}^r)^2 + \sum_{r=n+1}^{m}\Big[\sum_{1\le i<j\le n_1}(h_{ij}^r)^2 - \sum_{1\le i<j\le n_1} h_{ii}^r h_{jj}^r\Big]\\ &+ \sum_{r=n+1}^{m}\sum_{n_1+1\le s<t\le n}(h_{st}^r)^2 + \sum_{r=n+1}^{m}\Big[\sum_{n_1+1\le s<t\le n}(h_{ij}^r)^2 - \sum_{n_1+1\le s<t\le n} h_{ss}^r h_{tt}^r\Big]\\ &+ \sum_{r=n+1}^{m}\sum_{2\le p<q\le n} h_{pp}^r h_{qq}^r + f_1(n(n-1)) + f_2(3(n_1-1)) - f_3(2(n-1))\\ &- \sum_{2\le p<q\le n}\bar{\kappa}_{p,q} - \bar{\pi}(N_T^{n_1}) - \bar{\pi}(N_\perp^{n_2}).\end{aligned} \tag{40}$$

From (5), (13) and (14), we have

$$\sum_{2\leq p<q\leq n}\bar{\kappa}_{p,q}=\frac{f_1}{2}((n-1)(n-2))+\frac{f_2}{2}(3(n_1-2))-\frac{f_3}{2}(2(n-2)),\tag{41}$$

$$\bar{\pi}(N_T^{n_1})=\frac{f_1}{2}((n_1(n_1-1))+\frac{f_2}{2}(3(n_1-1))-\frac{f_3}{2}(2(n_1-1)),\tag{42}$$

$$\bar{\pi}(N_T^{n_1})=\frac{f_1}{2}((n_2(n_2-1))\tag{43}$$

Using in (40), we have

$$\begin{aligned}Ric(\chi)\leq&\frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta\psi}{\psi}+(n+n_1n_2-1)f_1+\frac{3f_2}{2}-(n_2+1)f_3\\&-\frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11}-(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}))^2\\&-\sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(h^r_{pq})^2+\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2+\sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h^r_{st})^2]\\&-\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}h^r_{ii}h^r_{jj}+\sum_{n_1+1\leq s<t\leq n}h^r_{ss}h^r_{tt}]\\&+\sum_{r=n+1}^{m}\sum_{2\leq p<q\leq n}h^r_{pp}h^r_{qq}.\end{aligned}\tag{44}$$

Further, the seventh and eighth terms on right hand side of the above inequality can be written as

$$\begin{aligned}\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2+\sum_{n_1+1\leq s<t\leq n}(h^r_{st})^2]-\sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(h^r_{pq})^2\\=-\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}h^r_{ii}h^r_{jj}+\sum_{n_1+1\leq s\neq t\leq n}h^r_{ss}h^r_{tt}-\sum_{2\leq p<q\leq n}h^r_{pp}h^r_{qq}]\\=\sum_{r=n+1}^{m}[\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n}h^r_{pp}h^r_{qq}-\sum_{j=2}^{n_1}h^r_{11}h^r_{jj}].\end{aligned}$$

Utilizing above two values in (44), we get

$$\begin{aligned}Ric(\chi)\leq&\frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta\psi}{\psi}+(n+n_1n_2-1)f_1+\frac{3f_2}{2}-(n_2+1)f_3\\&-\frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11}-(h^r_{n_1+1n_1+1}+\ldots h^r_{nn}))^2\\&-\sum_{r=n+1}^{m}[\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2+\sum_{b=2}^{n_1}h^r_{11}h^r_{bb}-\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n}h^r_{pp}h^r_{qq}].\end{aligned}\tag{45}$$

Since $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ is $N_T^{n_1}$-minimal then we can observe the following

$$\sum_{r=n+1}^{m} \sum_{p=2}^{n_1} \sum_{q=n_1+1}^{n} h_{pp}^r h_{qq}^r = -\sum_{r=n+1}^{m} \sum_{q=n_1+1}^{n} h_{11}^r h_{qq}^r \tag{46}$$

and

$$\sum_{r=n+1}^{m} \sum_{b=2}^{n_1} h_{11}^r h_{bb}^r = -\sum_{r=n+1}^{m} (h_{11}^r)^2. \tag{47}$$

Simultaneously, we can conclude

$$\begin{aligned}\frac{1}{2}\sum_{r=n+1}^{m} (2h_{11}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r))^2 + \sum_{r=n+1}^{m} \sum_{q=n_1+1}^{n} h_{11}^r h_{qq}^r \\ = 2\sum_{r=n+1}^{m} (h_{11}^r)^2 + \frac{1}{2} n^2 \|H\|^2.\end{aligned} \tag{48}$$

Using (46) and (47) in (45), after the assessment of (48), we finally get

$$\begin{aligned}Ric(\chi) \leq & \frac{1}{2} n^2 \|H\|^2 - \frac{n_2 \Delta\psi}{\psi} + (n + n_1 n_2 - 1) f_1 + \frac{3f_2}{2} - (n_2+1) f_3 \\ & - \frac{1}{4}\sum_{r=n+1}^{m} \sum_{q=n_1+1}^{n} (h_{qq}^r)^2 - \sum_{r=n+1}^{m} \{(h_{11}^r)^2 - \sum_{q=n_1+1}^{n} h_{11}^r h_{qq}^r \\ & + \frac{1}{4}(h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r)^2\}.\end{aligned} \tag{49}$$

Further, using the fact that $\sum_{r=n+1}^{m} (h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r) = n^2 \|H\|^2$, we get

$$\begin{aligned}Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 - \frac{n_2 \Delta\psi}{\psi} + (n + n_1 n_2 - 1) f_1 + \frac{3f_2}{2} - (n_2+1) f_3 \\ - \frac{1}{4}\sum_{r=n+1}^{m} (2h_{11}^r - \sum_{q=n_1+1}^{n} h_{qq}^r)^2.\end{aligned} \tag{50}$$

From the above inequality, we can conclude the inequality (30).

Case ii: If e_a is tangent to $N_\perp^{n_2}$, then we choose the unit vector from $\{e_{n_1+1}, \ldots, e_n\}$, suppose that the unit vector is e_n i.e., $\chi = e_n$. Then from (5), (13) and (14), we have

$$\sum_{1 \leq p < q \leq n-1} \bar{\kappa}_{p,q} = \frac{f_1}{2}((n-1)(n-2)) + \frac{f_2}{2}(3(n_1-1)) - \frac{f_3}{2}(2(n-2)). \tag{51}$$

$$\bar{\pi}(N_T^{n_1}) = \frac{f_1}{2}(n_1(n_1-1)) + \frac{f_2}{2}(3(n_1-1)) - \frac{f_3}{2}(2(n-1)). \tag{52}$$

$$\bar{\pi}(N_\perp^{n_2}) = \frac{f_1}{2}(n_2(n_2-1)). \tag{53}$$

Now, in a similar way as in case i using (51), we have

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} - \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{nn}) - 2h^r_{nn})^2 \\ & - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n_1}(h^r_{pq})^2 + \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2 - \sum_{1\leq i<j\leq n_1}h^r_{ii}h^r_{jj}] \\ & + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h^r_{st})^2 + \sum_{r=n+1}^{m}[\sum_{n_1+1\leq s<t\leq n}(h^r_{ij})^2 - \sum_{n_1+1\leq s<t\leq n}h^r_{ss}h^r_{tt}] \\ & + \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n-1}h^r_{pp}h^r_{qq} + (n + n_1n_2 - 1)f_1 - (n_2+1)f_3. \end{aligned} \tag{54}$$

Using similar steps of case i, the above inequality takes the form

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2+1)f_3 \\ & - \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{nn}) - 2h^r_{nn})^2 \\ & - \sum_{r=n+1}^{m}[\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2 + \sum_{b=n_1+1}^{n-1}h^r_{nn}h^r_{bb} - \sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n-1}h^r_{pp}h^r_{qq}]. \end{aligned} \tag{55}$$

By the Lemma 3.1, one can observe that

$$\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n-1}h^r_{pp}h^r_{qq} = 0. \tag{56}$$

Utilizing this in (55), we get

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2+1)f_3 \\ & - \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{nn}) - 2h^r_{nn})^2 \\ & - \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n-1}h^r_{nn}h^r_{bb}. \end{aligned} \tag{57}$$

The last term of the above inequality can be written as

$$-\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n-1}h^r_{nn}h^r_{bb} = -\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n}h^r_{nn}h^r_{bb} + \sum_{r=n+1}^{m}(h^r_{nn})^2$$

Moreover, the fifth term of (57) can be expanded as

$$\begin{aligned} -\frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}) - 2h^r_{nn})^2 = & \\ & - \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2 \\ & - 2\sum_{r=n+1}^{m}(h^r_{nn})^2 + \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj}. \end{aligned} \tag{58}$$

Using last two values in (57), we have

$$\begin{aligned}Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3\\ & - \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots h^r_{nn})^2 - 2\sum_{r=n+1}^{m}(h^r_{nn})^2\\ & + 2\sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj} - \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2\\ & - \sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n}h^r_{nn}h^r_{bb} + \sum_{r=n+1}^{m}(h^r_{nn})^2,\end{aligned} \tag{59}$$

or equivalently

$$\begin{aligned}Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3\\ & - \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots h^r_{nn})^2 - \sum_{r=n+1}^{m}(h^r_{nn})^2\\ & + \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj} - \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2\end{aligned} \tag{60}$$

On applying similar techniques as in the proof of case i, we arrive

$$\begin{aligned}Ric(\chi) \leq & \frac{1}{4}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + (n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3\\ & - \frac{1}{4}\sum_{r=n+1}^{m}(h^r_{nn} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2,\end{aligned} \tag{61}$$

which gives the inequality (31).

Next, we explore the equality cases of the inequality (30). First, we redefine the notion of the relative null space $\mathcal{N}_x$ of the submanifold M^n in the generalized Sasakian space form $\bar{M}^m(f_1, f_2, f_3)$ at any point $x \in M^n$, the relative null space was defined by B. Y. Chen [6], as follows

$$\mathcal{N}_x = \{X \in T_xM^n : h(X, Y) = 0, \forall Y \in T_xM^n\}.$$

For $A \in \{1, \dots, n\}$ a unit vector field e_A tangent to M^n at x satisfies the equality sign of (30) identically if and only if

$$(i)\ \sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}h^r_{pq} = 0\ (ii)\ \sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n}h^r_{bA} = 0\ (iii)\ 2h^r_{AA} = \sum_{q=n_1+1}^{n}h^r_{qq}, \tag{62}$$

such that $r \in \{n+1, \dots m\}$ the condition (i) implies that M^n is mixed totally geodesic contact CR-warped product submanifold. Combining statements (ii) and (iii) with the fact that M^n is contact CR-warped product submanifold, we get that the unit vector field $\chi = e_A$ belongs to the relative null space $\mathcal{N}_x$. The converse is trivial, this proves statement (2).

For a contact CR-warped product submanifold, the inequality sign of (30) holds identically for all unit tangent vector belong to N_T at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0 \ \ (ii)\sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n_1} h^r_{bA} = 0 \ \ (iii)\ 2h^r_{pp} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{63}$$

where $p \in \{1,\dots,n_1\}$ and $r \in \{n+1,\dots,m\}$. Since M^n is contact CR-warped product submanifold, the third condition implies that $h^r_{pp} = 0,\ \ p \in \{1,\dots,n_1\}$. Using this in the condition (*ii*), we conclude that M^n is $D-$totally geodesic contact CR-warped product submanifold in $\bar{M}^m(f_1, f_2, f_3)$ and mixed totally geodesicness follows from the condition (*i*). Which proves (*a*) in the statement (3).

For a contact CR-warped product submanifold, the equality sign of (30) holds identically for all unit tangent vector fields tangent to $N_\perp$ at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ (ii)\sum_{b=1}^{n}\sum_{\substack{A=n_1+1\\ b\neq A}}^{n} h^r_{bA} = 0\ (iii)\ 2h^r_{KK} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{64}$$

such that $K \in \{n_1+1,\dots,n\}$ and $r \in \{n+1,\dots,m\}$. From the condition (*iii*) two cases emerge, that is

$$h^r_{KK} = 0,\ \ \forall K \in \{n_1+1,\dots,n\} \ \text{ and } \ r \in \{n+1,\dots,m\} \ \text{ or } \ dim\, N_\perp = 2. \tag{65}$$

If the first case of (64) satisfies, then by virtue of condition (*ii*), it is easy to conclude that M^n is a $D^\perp-$ totally geodesic contact CR-warped product submanifold in $\bar{M}^m(c)$. This is the first case of part (*b*) of statement (3).

For the other case, assume that M^n is not $D^\perp-$totally geodesic contact CR-warped product submanifold and dim $N^\perp = 2$. Then condition (*ii*) of (64) implies that M^n is $D^\perp-$ totally umbilical contact CR-warped product submanifold in $\bar{M}^m(f_1, f_2, f_3)$, which is second case of this part. This verifies part (*b*) of (3).

To prove (*c*) using parts (*a*) and (*b*) of (3), we combine (63) and (64). For the first case of this part, assume that $dimN_\perp \neq 2$. Since from parts (*a*) and (*b*) of statement (3) we conclude that M^n is $D-$totally geodesic and $D^\perp-$ totally geodesic submanifold in $\bar{M}^m(f_1, f_2, f_3)$. Hence M^n is a totally geodesic submanifold in $\bar{M}^m(c)$.

For another case, suppose that first case does not satisfy. Then parts (*a*) and (*b*) provide that M^n is mixed totally geodesic and $D-$ totally geodesic submanifold of $\bar{M}^m(f_1, f_2, f_3)$ with $dimN_\perp = 2$. From the condition (*b*) it follows that M^n is $D^\perp-$totally umbilical contact CR-warped product submanifold and from (*a*) it is $D-$totally geodesic, which is part (*c*). This proves the theorem.

In view of (24), we have another version of the theorem 4.1 as follows

Theorem 4.2. *Let $M = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form $\bar{M}^m(f_1, f_2, f_3)$ admitting a admitting a trans-Sasakian structure. Then for each orthogonal unit vector field $\chi \in T_xM$ orthogonal to ξ, either tangent to N_T or $N_\perp$. Then the Ricci curvature satisfies the following inequality.*

(i) If χ is tangent to N_T, then

$$\begin{aligned} Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 - n_2\Delta ln\psi + n_2\|\nabla ln\psi\|^2 + (n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} \\ - (n_2+1)f_3. \end{aligned} \tag{66}$$

(ii) If χ is tangent to $N_\perp$, then

$$\begin{aligned} Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 - n_2\Delta ln\psi + n_2\|\nabla ln\psi\|^2 + (n + n_1n_2 - 1)f_1 \\ - (n_2+1)f_3. \end{aligned} \tag{67}$$

The equality cases are similar as in Theorem 4.1.

5. Some geometric applications in Mechanics

In this section, we investigate some applications of our attained inequalities, this section is divided in different subsections as follows

5.1. Application of Hopf's Lemma

In this subsection, we shall consider that the submanifold M^n is a compact such that $\partial M = \phi$. In the next theorem, we will see the application of Hopf's lemma for contact CR-warped product submanifold

Theorem 5.1. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form $\bar{M}^m(f_1, f_2, f_3)$ admitting a admitting a trans-Sasakian structure. If the unit tangent vector χ orthogonal to ξ is tangent to either N_T or $N_\perp$, then M^n is a simply Riemannian product submanifold if the Ricci curvature satisfy one of the following inequalities.*

(i) the unit vector field χ is tangent to N_T and

$$Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 + (n + n_1 n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2 + 1)f_3. \tag{68}$$

(ii) the unit vector field χ is tangent to $N_\perp$ and

$$Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 + (n + n_1 n_2 - 1)f_1 - (n_2 + 1)f_3. \tag{69}$$

Proof. Suppose that inequality (68) holds then from (30), we get $\frac{\Delta\psi}{\psi} \leq 0$, which implies $\Delta\psi \leq 0$, on using Hopf's Lemma, we observe that the warping function is constant and the submanifold M^n is Riemannian product. Similar result can be proved by using inequality (69). □

5.2. First eigenvalue of the warping function

The lower bound of Ricci curvature contains numerous geometric properties. Suppose the submanifold M^n is complete non-compact and x be a any arbitrary point on M^n. For the Riemannian manifold M^n, $\lambda_1(M^n)$ denotes the first eigenvalue of the following Dirichlet boundary value problem for a smooth function τ on M^n

$$\Delta\tau = \lambda\tau \text{ in } M^n \text{ and } \tau = 0 \text{ on } \partial M^n, \tag{70}$$

where Δ denotes the Laplacian on M^n and defined as $\Delta\tau = -div(\nabla\phi)$. By the principle of monotonicity one has $r < t$ which indicates that $\tau_1(M_r^n) > \lambda_1(M_t^n)$ and $Lim_{r\to\infty}\lambda_1(D_r)$ exists and first eigenvalue is defined as

$$\lambda_1(M) = Lim_{r\to\infty}\lambda_1(D_r).$$

Several geometers have been worked on the analysis of first eigenvalue of the Laplacian operator ([26], [11], [21]). For a non-constant warping function the maximum (minimum) principle on the eigenvalue λ_1, we have ([6], [32])

$$\lambda_1 \int_{M^n} \tau^2 dV \leq \int_{M^n} \|\nabla\tau\|^2 dV. \tag{71}$$

The equality holds if and only if $\Delta\tau = \lambda_1\tau$.

The relation between Ricci curvature and first eigenvalue is derived in the following theorem

Theorem 5.2. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form $\bar{M}^m(f_1, f_2, f_3)$ admitting a trans-Sasakian structure. Suppose that the warping function $\ln\psi$ is an eigen function of the Laplacian of M^n associated to the first eigenvalue $\lambda_1(M^n)$ of the problem* (70)*, then the following inequalities hold* *(i) If the unit vector field χ is tangent to N_T then*

$$\int_{M^n} Ric(\chi)dV \leq \frac{1}{4}n^2 \int_{M^n} \|H\|^2 dV + n_2\lambda_1 \int_{M^n} (ln\psi)^2 dV + [(n + n_1n_2 - 1)f_1 - \frac{3f_2}{2} - (n_2 + 1)f_3]Vol(M^n). \tag{72}$$

(ii) If the unit vector field χ is tangent to $N_\perp$ then

$$\int_{M^n} Ric(\chi)dV \leq \frac{1}{4}n^2 \int_{M^n} \|H\|^2 dV + n_2\lambda_1 \int_{M^n} (ln\psi)^2 dV + [(n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3]Vol(M^n). \tag{73}$$

The equality cases are same as in Theorem (4.1).

Proof. Since M^n is compact that mean it has lower and upper bounds. Let $\lambda_1 = \lambda_1(M)$ and $ln\psi$ be a solution of Dirichlet boundary problem corresponding to the first eigenvalue $\lambda_1(M^n)$. Suppose $\chi \in TN_T$, then the inequality (66) can be written as follows

$$Ric(\chi) - n_2\|\nabla ln\psi\|^2 \leq \frac{1}{4}n^2\|H\|^2 - n_2\Delta ln\psi + (n + n_1n_2 - 1)f_1 - \frac{3f_2}{2} - (n_2 + 1)f_3. \tag{74}$$

Integrating above inequality with respect to volume element dV, we find

$$\int_{M^n} Ric(\chi)dV - n_2\int_{M^n} \|\nabla ln\psi\|^2 dv \leq \frac{n^2}{4}\int_{M^n} \|H\|^2 dV - (n_2 + 1)f_3 Vol(M^n) + [(n + n_1n_2 - 1)f_1 - \frac{3f_2}{2}]Vol(M^n). \tag{75}$$

Since λ_1 is an eigenvalue of the eigen function $ln\psi$, such that $\Delta ln\psi = \lambda_1 ln\psi$, then equality in (71) holds for $\tau = ln\psi$,

$$\int_{M^n} \|\nabla ln\psi\|^2 dV = \lambda_1 \int_{M^n} (ln\psi)^2 dV, \tag{76}$$

using in (75), we obtain

$$\int_{M^n} Ric(\chi)dV - n_2\lambda_1\int_{M^n} (ln\psi)^2 dV \leq \frac{n^2}{4}\int_{M^n} \|H\|^2 dV + (n_2 + 1)f_3 Vol(M^n) + [(n + n_1n_2 - 1)f_1 - \frac{3f_2}{2}]Vol(M^n). \tag{77}$$

Which proves the part (i). Similarly, one can proves the part (ii). □

5.3. *Dirichlet energy and Lagrangian for the warping function*

Let M^n be a compact Riemannian manifold and ϕ be a positive differentiable function on M^n. Then formula for Dirichlet energy of a function τ is given by [23]

$$E(\tau) = \frac{1}{2}\int_{M^n} \|\nabla\tau\|^2 dV, \tag{78}$$

where dV is the volume element of M^n and formula for Lagrangian of the function τ on M^n is given in [23]

$$L_\tau = \frac{1}{2}\|\nabla\tau\|^2. \tag{79}$$

The Euler-Lagrange equation for L_τ is given by

$$\Delta\tau = 0. \tag{80}$$

Considering that the contact CR-warped product submanifold $M^n = N_T^{n_1} \times_\psi N_\perp^{n_1}$ is a compact orientable without boundary such that $\partial M^n = \phi$. Then in the following theorem we have a relation between Dirichlet energy, Ricci curvature and mean curvature vector

Theorem 5.3. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be contact CR-warped product submanifold of a generalized Sasakian space form admitting a admitting a trans-Sasakian structure. Then we have the following inequalities for the Dirichlet energy of the warping function $ln\psi$*

(i) If the unit vector field χ is tangent to N_T then

$$\begin{aligned} E(ln\psi) \geq & \frac{1}{2n_2}\int_{M^n} Ric(\chi)dV - \frac{n^2}{8n_2}\int_{M^n} \|H\|^2 dV \\ & - \frac{1}{2n_2}\Big[(n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2+1)\Big]Vol(M^n). \end{aligned} \tag{81}$$

(ii) If the unit vector field χ is tangent to $N_\perp$ then

$$\begin{aligned} E(ln\psi) \geq & \frac{1}{2n_2}\int_{M^n} Ric(\chi)dV - \frac{n^2}{8n_2}\int_{M^n} \|H\|^2 dV \\ & - \frac{1}{2n_2}\Big[(n + n_1n_2 - 1)f_1 - (n_2+1)\Big]Vol(M^n). \end{aligned} \tag{82}$$

The equality cases are similar as in Theorem 4.1.

Proof. For a positive valued differentiable function τ defined on a compact orientable Riemannian manifold without boundary, by theory of integration on Riemannian manifold we have $\int_{M^n} \Delta\phi dV = 0$. On applying this fact for the warping function $ln\psi$, we have

$$\int_{M^n} \Delta ln\psi dV = 0. \tag{83}$$

Integrating inequality (30) with respect to volume element dV on contact CR-warped product submanifold M^n, which is compact and orientable without boundary, we get

$$\begin{aligned} \int_{M^n} Ric(\chi)dV \leq & \frac{n^2}{4}\int_{M^n} \|H\|^2 dV + n_2\int_{M^n} \|\nabla ln\psi\|^2 dV - n_2\int_{M^n} \Delta ln\psi dV \\ & + \frac{1}{n_2}\Big[(n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2+1)\Big]Vol(M^n). \end{aligned} \tag{84}$$

□

Using the formula (78) and after some computation, the required inequality is derived. In a similar method, we can prove the inequality (81)

Further, in the following theorem we will compute the Lagrangian for the warping function $ln\psi$

Theorem 5.4. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a compact orientable contact CR-warped submanifold isometrically immersed in a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a trans-Sasakian structure such that the warping function $ln\psi$ satisfies the Euler-Lagrangian equation, then* *(i) If the unit vector field χ is tangent to N_T, then*

$$L_{ln\psi} \geq \frac{1}{2n_2}Ric(\chi) - \frac{n^2}{8n_2}||H||^2 - \frac{1}{2n_2}[(n+n_1n_2-1)f_1 + \frac{3f_2}{2} - (n_2+1)f_3]. \tag{85}$$

(ii) If the unit vector field χ is tangent to $N_\perp$, then

$$L_{ln\psi} \geq \frac{1}{2n_2}Ric(\chi) - \frac{n^2}{8n_2}||H||^2 - \frac{1}{2n_2}[(n+n_1n_2-1)f_1 - (n_2+1)f_3] \tag{86}$$

Where $L_{ln\psi}$ is the Lagrangian of the warping function defined in (79). The equality cases are same as theorem. 4.1

Proof. The proof follows immediately on using (79) and (80) in theorem 30. □

Further, the Hamiltonian for a local orthonormal frame at any point $x \in M^n$ is expressed as follows [23]

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{n} p(e_i)^2. \tag{87}$$

On replacing p by a differential operator $d\phi$, then from (22), we get

$$H(d\phi,x) = \frac{1}{2}\sum_{i=1}^{n} d\phi(e_i)^2 = \frac{1}{2}\sum_{i=1}^{n} e_i(\phi)^2 = \frac{1}{2}||\nabla\phi||^2. \tag{88}$$

In the next result we obtain a relation between Hamiltonian of warping function, Ricci curvature and squared norm of mean curvature vector

Theorem 5.5. *Let $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form admitting a trans-Sasakian structure, then the Hamiltonian of the warping function satisfy the following inequalities*

(i) If $\chi \in TN_T$, then

$$H(dln\psi,x) \geq \frac{1}{2n_2}\{Ric(\chi) - \frac{n^2}{4}||H||^2 - (n+n_1n_2-1)f_1 - \frac{3f_2}{2} + (n_2+1)f_3 \tag{89}$$

(ii) If $\chi \in TN_\perp$, then

$$H(dln\psi,x) \geq \frac{1}{2n_2}\{Ric(\chi) - \frac{n^2}{4}||H||^2 - (n+n_1n_2-1)f_1 + (n_2+1)f_3 \tag{90}$$

The equality cases are same as theorem 4.1

Proof. By the application of (88) in theorem 4.1, we get the required results.
□

5.4. Application of Obata's differential equation

This subsection is based on the study of Obata [22]. Basically, Obata characterized a Riemannian manifolds by a specific ordinary differential equation and derived that an $n-$dimensional complete and connected Riemannian manifold (M^n, g) to be isometric to the $n-$sphere S^n if and only if there exists a non-constant smooth function τ on M^n that is the solution of the differential equation $H^\tau = -c\tau g$, where H^τ is the Hessian of τ. Inspired by the work of Obata [22], we obtain the following characterization

Theorem 5.6. *Suppose $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a compact orientable contact CR-warped product submanifold isometrically immersed in a generalized Sasakian space form $M^m(f_1, f_2, f_3)$ admitting a trans-Sasakian structure with positive Ricci curvature and satisfying one of the following relation*

(i) $\chi \in TN_T$ orthogonal to ξ and

$$||Hess\tau||^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}||H||^2 - \frac{3\lambda_1}{n_1n_2}\Big[(n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2 + 1)f_3\Big] \tag{91}$$

(ii) $\chi \in TN_\perp$ and

$$||Hess\tau||^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}||H||^2 - \frac{3\lambda_1}{n_1n_2}\Big[(n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3\Big], \tag{92}$$

where $\lambda_1 > 0$ is an eigenvalue of the warping function $\tau = ln\psi$. Then the base manifold $N_T^{n_1}$ is isometric to the sphere $S^{n_1}(\frac{\lambda_1}{n_1})$ with constant sectional curvature $\frac{\lambda_1}{n_1}$.

Proof. Let $\chi \in TN_T$. Consider that $\tau = ln\psi$ and define the following relation as

$$||Hess\tau - t\tau I||^2 = ||Hess\tau||^2 + t^2\tau^2||I||^2 - 2t\tau g(Hess\tau, I). \tag{93}$$

But we know that $||I||^2 = trace(II^*) = p$ and

$$g(Hess(\tau), I^*) = trace(Hess\tau, I^*) = traceHess(\tau).$$

Then equation (93) transform to

$$||Hess\tau - t\tau I||^2 = ||Hess\tau||^2 + pt^2\tau^2 - 2t\tau\Delta\tau. \tag{94}$$

Assuming λ_1 is an eigenvalue of the eigen function τ then $\Delta\tau = \lambda_1\tau$. Thus we get

$$||Hess\tau - t\tau I||^2 = ||Hess\tau||^2 + (pt^2 - 2t\lambda)\tau^2. \tag{95}$$

On the other hand, we obtain $\Delta\tau^2 = 2\tau\Delta\tau + ||\nabla\tau||^2$ or $\lambda_1\tau^2 = 2\lambda_1\tau^2 + ||\nabla\tau||^2$ which implies that $\tau^2 = -\frac{1}{\lambda_1}||\nabla\tau||^2$, using this in equation (95), we have

$$||Hess\tau - t\tau I||^2 = ||Hess\tau||^2 + (2t - \frac{pt^2}{\lambda_1})||\nabla\tau||^2. \tag{96}$$

In particular $t = -\frac{\lambda_1}{n_1}$ on (96) and integrating with respect to dV

$$\int_{M^n} ||Hess\tau + \frac{\lambda_1}{n_1}\tau I||^2 dV = \int_{M^n} ||Hess\tau||^2 dV - \frac{3\lambda_1}{n_1}\int_{M^n} ||\nabla\tau||^2 dV. \tag{97}$$

Integrating the inequality (66) and using the fact $\int_{M^n} \Delta\phi dV = 0$, we have

$$\begin{aligned}\int_{M^n} Ric(\chi)dV \leq &\frac{n^2}{4}\int_{M^n} ||H||^2 dV + n_2\int_{M^n} ||\nabla\tau||^2 dV + \\ &+ [(n + n_1n_2 - 1)f_1 - \frac{3f_2}{2} - (n_2 + 1)f_3]Vol(M^n).\end{aligned} \tag{98}$$

From (97) and (98) we derive

$$\begin{aligned}\frac{1}{n_2}\int_{M^n} Ric(\chi)dV \leq &\frac{n^2}{4n_2}\int_{M^n} ||H||^2 dV - \frac{n_1}{3\lambda_1}\int_{M^n} ||Hess\tau + \frac{\lambda_1}{n_1}\tau I||^2 dV \\ &+ \frac{n_1}{3\lambda_1}\int_{M^n} ||Hess\tau||^2 dV + \frac{1}{n_2}[(n + n_1n_2 - 1)f_1 \\ &+ \frac{3f_2}{2} - (n_2 + 1)f_3]Vol(M^n).\end{aligned} \tag{99}$$

According to assumption $Ric(\chi) \geq 0$, the above inequality gives

$$\begin{aligned}\int_{M^n} \|Hess\tau + \frac{\lambda_1}{n_1}\phi I\|^2 dV \leq & \frac{3n^2\lambda_1}{4n_1n_2}\int_{M^n} \|H\|^2 dV + \int_{M^n} \|Hess\tau\|^2 dV \\ & - \frac{3\lambda_1}{n_1n_2}[(n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} \\ & - (n_2 + 1)f_3]Vol(M^n).\end{aligned} \tag{100}$$

From (91), we get

$$\int_{M^n} \|Hess\tau + \frac{\lambda_1}{n_1}\tau I\|^2 dV \leq 0. \tag{101}$$

but we know that

$$\int_{M^n} \|Hess\tau + \frac{\lambda_1}{n_1}\tau I\|^2 dV \geq 0. \tag{102}$$

Combining last two statements, we get

$$\int_{M^n} \|Hess\tau + \frac{\lambda_1}{n_1}\tau I\|^2 dV = 0 \Rightarrow Hess\tau = -\frac{\lambda_1}{n_1}\tau I. \tag{103}$$

Since the warping function $\tau = ln\psi$ is not constant function on M^n so equation (103) is Obata's [22] differential equation with constant $c = \frac{\lambda_1}{n_1} > 0$. As $\lambda_1 > 0$ and therefore the base submanifold $N_T^{n_1}$ is isometric to the sphere $S^{n_1}(\frac{\lambda_1}{n_1})$ with constant sectional curvature $\frac{\lambda_1}{n_1}$. This proves the theorem. □

In [21] Rio et al. studied another version of Obata's differential equation in the characterization of Euclidean sphere. Basically, they proved that if τ be a real valued non constant function on a Riemannian manifold satisfying $\Delta\tau + \lambda_1\tau = 0$ such that $\lambda < 0$, then M^n is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function τ is the solution of the following differential equation

$$\frac{d^2\tau}{dt^2} + \lambda_1\tau = 0. \tag{104}$$

Motivated by the study of Rio et al [21] and Ali et al. [20] we obtain the following characterization.

Theorem 5.7. *Suppose $M^n = N_T^{n_1} \times_\psi N_\perp^{n_2}$ be a compact orientable contact CR- warped product submanifold isometrically immersed in generalized Sasakian space form admitting a trans-Sasakian structure with positive Ricci curvature and satisfying one of the following statement*

(i) $\chi \in TN_T$ and

$$\|Hess\tau\|^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}\|H\|^2 - \frac{3\lambda_1}{n_1n_2}\Big[(n + n_1n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2 + 1)f_3\Big] \tag{105}$$

(ii) $\chi \in TN_\perp$ and

$$\|Hess\tau\|^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}\|H\|^2 + \frac{3\lambda_1}{n_1n_2}\Big[(n + n_1n_2 - 1)f_1 - (n_2 + 1)f_3\Big], \tag{106}$$

where $\lambda_1 < 0$ is a negative eigenvalue of the eigen function $\tau = ln\psi$. Then $N_T^{n_1}$ is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function $\tau = ln\psi$ satisfies the differential equation

$$\frac{d^2\tau}{dt^2} + \lambda_1\tau = 0. \tag{107}$$

Proof. Since we assumed that the Ricci curvature is positive then by the Myers's theorem according to which, a complete Riemannian manifold with positive Ricci curvature is compact that mean M^n is compact contact CR-warped product submanifold with free boundary [28]. Then by (99)

$$\begin{aligned}\frac{1}{n_2}\int_{M^n} Ric(\chi)dV \leq &\frac{n^2}{4n_2}\int_{M^n}||H||^2 dV - \frac{n_1}{3\lambda_1}\int_{M^n}||Hess\tau + \frac{\lambda_1}{n_1}\tau I||^2 dV\\ &+ \frac{n_1}{3\lambda_1}\int_{M^n}||Hess\tau||^2 dV + \frac{1}{n_2}[(n + n_1 n_2 - 1)f_1\\ &+ \frac{3f_2}{2} - (n_2+1)f_3]Vol(M^n).\end{aligned} \tag{108}$$

According to hypothesis Ricci curvature is positive $Ric(\chi) > 0$, then we have

$$\begin{aligned}\int_{M^n}||Hess\tau + \frac{\lambda_1}{n_1}\tau I||^2 dV < &\frac{3n^2\lambda_1}{4n_1 n_2}\int_{M^n}||H||^2 dV + \int_{M^n}||Hess\phi||^2 dV\\ &+ \frac{3\lambda_1}{n_1 n_2}\Big[(n + n_1 n_2 - 1)f_1 + \frac{3f_2}{2} - (n_2+1)f_3\Big]\end{aligned} \tag{109}$$

If equation (105) holds, then from last inequality we get $||Hess\phi + \frac{\lambda_1}{n_1}\psi I||^2 < 0$, which is not possible hence $||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 = 0$. Since $\lambda < 0$, then by result of [21], the submanifold $N_T^{n_1}$ is isometric to a warped product of the Euclidean line and a complete Riemannian manifold, where the warping function on R is the solution of the differential equation (107). This proves the theorem. Similarly by assuming (106), we can also prove the theorem.

□

Acknowledgment: The authors are highly thankful to anonymous referee for his/her valuable suggestions and comments which have improved the contents of the paper.

References

[1] B. Y. Chen, CR-submanifolds of a Kaehler manifold I, J. Differential Geometry, 16(1981), 305 - 323.
[2] B. Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds I, Monatsh Math., 133(2001), 177 - 195.
[3] B. Y. Chen, Pseudo-Riemannian Geometry, $\delta-$invariants and Applications, World Scientific Publishing Company, Singapore, 2011.
[4] B.Y. Chen, A general inequality for submanifolds in complex space forms and its applications, Arch. Math., 67 (1996), 519 - 528.
[5] B.Y. Chen, Mean curvature and shape operator of isometric immersions in real space forms, Glasg. Math. J., 38(1996), 87-97.
[6] B.Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension, Glasg. Math. J., 41 (1999),33-41.
[7] B.Y. Chen, On isometric minimal immersions from warped products into real space forms, Proc. Edinb. Math. Soc., 45(03) (2002), 579-587.
[8] B.Y. Chen, Differential Geometry of Warped Product Manifolds and Submanifolds, World Scientific, 2017.
[9] B.Y. Chen, F. Dillen, L. Verstraelen, L. Vrancken, Characterization of Riemannian space forms, Einstein spaces and conformally flate spaces, Proc. Amer. Math. Soc., 128 (2) (199), 589–598.
[10] B. Y. Chen, A survey on geometry of warped product submanifolds, arXiv:1307.0236, arxiv.org, 2013.
[11] B. Palmer, The Gauss map of a spacelike constant mean curvature hypersurface of Minkowski space, Comment. Math. Helv., 65 (1990), 52-57.
[12] K. Arslan, R. Ezentas, I. Mihai, C. Ozgur, Certain inequalities for submanifolds in $(k,\mu)-$contact space form, Bull. Aust. Math. Soc., 64 (2001), 201-212.
[13] K. Arslan, R. Ezentas, I. Mihai, C. Ozgur, Ricci curvature of submanifolds in Kenmotsu space forms, Int. J. Math. Mathematical Sci., 29(12) (2002), 719-726.
[14] K. Arslan, R. Ezentas, I. Mihai, C. Ozgur, Contact CR-warped product submanifolds in Kenmotsu space forms, J. Korean Math. Soc., 42(5) (2005), 1101-1110.
[15] F. R. Al-Solamy, M. A. Khan, Application of Hopf's Lemma on contact CR-warped product submanifolds of a nearly Kenmotsu manifold, 43(1) (2017), 95-107
[16] D. Cioroboiu, B. Y. Chen, Inequalities for semi-slant submanifolds in Sasakian space forms, Int. J. Mathematics and Mathematical Sciences, 27 (2003), 1731-1738.
[17] A. Mihai, C. Ozgur, Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection, Taiwanese J. Math., 14(2010), 1465-1477.

[18] D. W. Yoon, Inequality for Ricci curvature of slant submanifolds in cosymplectic space forms, Turk. J. Math., 30(2006), 43-56.
[19] K. Kenmotsu, Class of almost contact Riemannian manifolds, Tohoku Mathematical Journal, 24 (1972), 93-103.
[20] A. Ali, Piscoran Laurian-Ioan, Ali H. Al-Khalidi, Ricci curvature on warped product submanifolds in spheres with geometric applications, Journal of Geometry and Physics, 2019, In press.
[21] E. Garcia-Rio, D.N. Kupeli, B. Unal, On a differential equation characterizing Euclidean sphere, J. Differential Equations, 194 (2003), 287 - 299.
[22] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan, 14 (1962), 333-340.
[23] O. Calin, D.C. Chang, Geometric Mechanics on Riemannian Manifolds: Applications to Partial Differential Equations, Springer Science & Business Media, 2006.
[24] R.L. Bishop, B. O'Neil, Manifolds of negative curvature, Trans. Amer. Math. Soc., 145 (1969), 1-9.
[25] R.H. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom., 17 (1982), 255-306.
[26] S.S. Cheng, Pectrum of the Laplacian and its Applications to Differential Geometry (Ph.D. Dissertation), Univ. of California, Berkeley, 1974.
[27] S.Y. Cheng, Eigenvalue comparison theorem and its geometric applications, Math. Z., 143 (1975), 289-297.
[28] S.B. Myers, Riemannian manifolds with positive mean curvature, Duke Math. J., 8 (2) (1941), 401-404.
[29] S. Nolker, Isometric immersions of warped products, Differential Geom. Appl. 6 (1996), 1-30.
[30] N. Ejiri, Some compact hypersurfaces of constant scalar curvature in a sphere, J. Geom., 19 (2) (1982), 197-199.
[31] K. Sekigawa, Some CR-submanifolds in a 6-dimensional sphere, Tensor (N.S.), 41 (1) (1984), 13-20.
[32] M. Berger, Les Varietes riemanniennes ($\frac{1}{4}$)-pinces, Ann. Sc. Norm. Super. Pisa CI. Sci., 14(4) (1960), 161-170.
[33] J. K. Beem, P. Ehrlich, T. G. Powell, Warped product manifolds in relativity. Selected studies, North-Holland, Amsterdam-New York, 1982.
[34] S. W. Hawkings, G. F. R. Ellis, The large scale structure of space-time, Cambridge Univ. Press, Cambridge, 1973.
[35] B. O'Neill, Semi-Riemannian Geometry with application to Relativity, Academic Pres., 1983.
[36] D. E. Blair, Contact manifolds in Riemannian Geometry Lecture Notes in Math. 509, Berlin: Springer-Verlag, 1976.
[37] P. Alegre, D. E. Blair, A. Carriazo, Generalized Sasakian space forms, Israel J. Math., 141 (2004),157-183.
[38] M. Atceken, Contact CR-warped product submanifolds in Kenmotsu space forms, Bull. of the Iranian Math. Soc., 39 (3)(2013), 415-429.
[39] M. Atceken, Contact CR-warped product submanifolds in Cosymplectic space forms, Collect. Math. , 62(2011), 17-26.
[40] Sibel Sular, Cihan Özgür, Contact CR-warped product submanifolds in generalized Sasakian space forms, Turk. J. Math., 36(2012), 485-497.
[41] A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura. Appl., 4(1980), 35-58.
[42] D. E. Blair, J. Oubina, Conformal and related changes of metric on the product of two almost contact metric manifolds, Publications Matematiques, 34 (1990), 199-207.
[43] J.C. Marrero, The local structure of trans-Sasakian manifolds, Ann. Mat. Pura Appl., 162(1992), 77- 86.