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On the Approximations of Solutions to Stochastic Differential Equations Under Polynomial Condition

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Abstract. The subject of this paper is an analytic approximate method for a class of stochastic differential equations with coefficients that do not necessarily satisfy the Lipschitz and linear growth conditions but behave like a polynomials. More precisely, equations from the observed class have unique solutions with bounded moments and their coefficients satisfy polynomial condition. Approximate equations are defined on partitions of a time interval, and their coefficients are Taylor approximations of the coefficients of the initial equation. The rate of L^p convergence increases when degrees in Taylor approximations of coefficients increase. At the end of the paper, an example is provided to support the main theoretical result.

1. Introduction

On the basis of the extensive literature one can observe that most of the stochastic differential equations are not explicitly solvable. In addition, many real-life phenomena are modeled by stochastic differential equations with coefficients which are highly nonlinear. One analytic method to find their approximate solutions in an explicit form, or in a form suitable for the application of numerical methods, will be presented. The method is based on the Taylor approximations of the coefficients of the initial equations. The closeness in the L^p sense between the exact and approximate solutions will be estimated. Moreover, almost sure convergence of the sequence of the approximate solutions to the exact solution will be established.

The main motivation for this paper came from the work of Atalla [1, 2]. Following Atalla's papers, Janković and Ilić constructed approximate solutions to stochastic differential [12] and integrodifferential [13] equations, defined on a partition of a time-interval. In the corresponding approximate equations coefficients are Taylor series, up to arbitrary derivatives, of the coefficients of the initial equations. Closeness of the exact and approximate solutions is measured in the sense of the L^p-norm and with probability one. This method was appropriately extended by Milošević, Jovanović and Janković to various types of stochastic differential equations such as functional [25], pantograph with Markovian switching [23], with time-dependent delay [24] and with Poisson random measure [22]. In all of these papers the Lipschitz and linear growth conditions for the drift and diffusion coefficients are used, which guarantees the existence and uniqueness of solution of the initial equation. In a view of the cited papers one can conclude that the rate of the L^p-closeness between the exact and approximate solutions increases as degrees in Taylor expansions increase. However,

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most of coefficients of the stochastic differential equations do not satisfy these conditions. Our idea is to weaken those conditions in a way that they can satisfy some other conditions instead which guarantee boundedness of the moments (such as one-sided Lipschitz condition, monotone condition, for example) and satisfy polynomial condition. Existence and uniqueness of solutions of the initial and approximate equations are assumed, but can be easily proven if an appropriate condition is added.

In that way, we extend the results from [12] to a class of stochastic differential equations with drift coefficients which could be highly nonlinear. This extension requires the application of the technique which is slightly different than that used in the cited paper. It should be pointed out that, by this extension, the main result remains the same, that is, the rate of the L^p-closeness of the sequence of the approximate solutions to the exact solution increases as the numbers of degrees in the Taylor approximations of the coefficients of the initial equation increase. It should be stressed that, in the recent years, the existence and uniqueness of the exact solutions, development of the approximate methods, stability of the exact and approximate solutions and other qualitative and quantitative properties of the exact and approximate solutions, under highly nonlinear conditions on coefficients of the appropriate stochastic differential equations have attracted the attention of many researchers. We refer the reader, for example, to [4, 8, 17, 20, 21], among many other. So, the main aim of this paper is to provide a contribution to the analysis of stochastic differential equations with highly nonlinear drift coefficients, that is, with drifts which satisfy the polynomial condition.

As one can observe from the papers [27–29], the approximations based on the Taylor expansions are appropriately applied in the context of LIBOR modelling. Models in the cited papers are based on ordinary stochastic differential equations, as well as, on stochastic differential equations driven by Lévy processes or general semimartingales. The results from these papers suggest that the results of the present paper could be applied in the modelling of certain market parameters, bearing in mind that they are often described by stochastic differential equations with conditions which do not satisfy linear growth conditions.

Whole consideration in this paper is related to the complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfies usual conditions (it is nondecreasing, right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $W = \{W(t),\ t \geq 0\} = \{(W_1(t), ..., W_{d_1}(t))^T, t \geq 0\}$ be a d_1-dimensional Brownian motion defined on this probability space. The marks $|\cdot|$ and $\langle\cdot,\cdot\rangle$ represent the Euclidean norm of vectors or the Frobenius (trace) norm of matrices and the standard Euclidean scalar product of vectors, respectively. Let $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

The subject of consideration is stochastic differential equation of the Itô type

$$dx(t) = a(t, x(t))dt + b(t, x(t))dW(t),\ t \in [0, T],\ x(0) = x_0,$$

or, in integral form

$$x(t) = x_0 + \int_0^t a(s, x(s))ds + \int_0^t b(s, x(s))dW(s),\quad t \in [0, T], \tag{1}$$

where $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d_1}$. Let us assume that the initial condition x_0 is independent of W. This condition is not particularly restrictive.

The approximate equations will be defined on a partition of a time interval $[0, T]$. For any positive integer n consider the partition of the form

$$0 = t_0 < t_1 < \cdots < t_n = T,\quad \delta_n = \max_{0\leq k\leq n-1}(t_{k+1} - t_k). \tag{2}$$

For the completeness of the paper, we shall introduce the notion of the Fréchet derivative. Let $\mathbb{X} = (X, \|\cdot\|_1)$ and $\mathbb{Y} = (Y, \|\cdot\|_2)$ be normed spaces over the same field $\mathbb{F}$. $\mathcal{L}(X, Y)$ represents the space of all bounded linear operators from X to Y. The norm $\|\cdot\|_{1,2}$ in $\mathcal{L}(X, Y)$ is going to be defined as $\|A\|_{1,2} = \sup_{\|x\|_1\leq 1} \|Ax\|_2$, $A \in \mathcal{L}(X, Y)$. Let U be an open subset of X, let $f : U \to Y$ be a mapping, and let $x_0 \in U$.

If there exists some $F \in \mathcal{L}(X, Y)$ such that

$$\lim_{h\to 0} \frac{\|f(x_0 + h) - f(x_0) - Fh\|_2}{\|h\|_1} = 0 \tag{3}$$

holds, then F is the Fréchet derivative of f at x_0. The notation used here is $F = f'_{x_0}$. Previous limit is taken as the vector h tends to zero in X and the Fréchet derivative is unique in the case when it exists.

Now, let $f : U \to Y$ be a mapping which is Fréchet differentiable at every point $x_0 \in U$. In this case $f'_{x_0} \in \mathcal{L}(X, Y)$. We can consider the mapping $x_0 \mapsto f'_{x_0}$ from U to $\mathcal{L}(X, Y)$. If this mapping is Fréchet differentiable at x_0, then the second Fréchet derivative is $f''_{x_0} \in \mathcal{L}(X, \mathcal{L}(X, Y))$.

In the case when the second Fréchet derivative exists in some surrounding U of the vector $x_0 \in X$, we can define $F : U \to \mathcal{L}(X, Y)$ as $F(x) = f'_x$, for every $x \in U$. That way, we have

$$\lim_{h\to 0} \frac{\|F(x_0 + h) - F(x_0) - F'_{x_0}h\|_{1,2}}{\|h\|_1} = \lim_{h\to 0} \frac{\|f'_{x_0+h} - f'_{x_0} - f''_{x_0}h\|_{1,2}}{\|h\|_1} = 0. \tag{4}$$

Let $\mathbb{X}_1 = (X_1, \|\cdot\|_1), \dots, \mathbb{X}_n = (X_n, \|\cdot\|_n), \mathbb{Y} = (Y, \|\cdot\|_Y)$ be normed spaces over the same field $\mathbb{F}$, and let $B : X_1 \times \cdots \times X_n \to Y$ be a mapping, which is linear at every argument. B is called an n-linear operator from $X_1 \times \cdots \times X_n$ to Y. Such B is bounded, if there exists some constant $M \geq 0$ such that for all $(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$ the following holds:

$$\|B(x_1, \dots, x_n)\|_Y \leq M\|x_1\|_1 \cdots \|x_n\|_n. \tag{5}$$

The set of all bounded n-linear operators from $X_1 \times \cdots \times X_n$ to Y is denoted by $\mathcal{M}_n(X_1, ..., X_n; Y)$. We use shortly $\mathcal{M}_n(X, ..., X; Y) \equiv \mathcal{M}_n(X^n; Y)$.

It is easy to see that the $\mathcal{M}_n(X_1, \dots, X_n; Y)$ is a vector space. Moreover, if we define the norm of $B \in \mathcal{M}_n(X_1, \dots, X_n; Y)$ as the infimum of all admissible M in the inequality (5), then $\mathcal{M}_n(X_1, \dots, X_n; Y)$ is a normed space. The norm of B, obtained in a described way, is denoted by $\|B\|_{M,n}$. Multi-linear operators and bounded linear operators are in a close relation in a sense that $\mathcal{M}_n(X^n; Y)$ is isometrically isomorphic to $\underbrace{\mathcal{L}(X, \mathcal{L}(X, \dots, \mathcal{L}(X}_{n \text{ times}}, Y)...))$ with respect to standard norms on these spaces. So, if $x \in X$ then $f''_{x_0}(x) \in \mathcal{L}(X, Y)$. If $y \in X$, then $f''_{x_0}(x)(y) = f''_{x_0}(x, y) \in Y$ and the mapping

$$(x, y) \mapsto f''_{x_0}(x, y)$$

belongs to $\mathcal{M}_2(X^2; Y)$. The norm $\|f''_{x_0}\| = \|f''_{x_0}\|_{M,2}$ is the same in the space $\mathcal{L}(X, \mathcal{L}(X, Y))$ and in the space $\mathcal{M}_2(X^2; Y)$.

In the same manner we can define higher Fréchet derivatives, in the case when they exist. Thus, the n-th Fréchet derivative of the function f at x_0 is

$$f^{(n)}_{x_0} \in \mathcal{L}\underbrace{(X, \dots, \mathcal{L}(X}_{n \text{ times}}, Y)...),$$

if the function $x_0 \mapsto f^{(n-1)}_{x_0}$ is Fréchet differentiable in some neighborhood of the x_0.

Let us now recall the Taylor formula [3, 5]. Let $\mathbb{X} = (X, \|\cdot\|_1)$ and $\mathbb{Y} = (Y, \|\cdot\|_2)$ be normed spaces over the same field $\mathbb{F}$, let U be an open subset of X, and let $f : U \to Y$ be $(n + 1)$-times Fréchet differentiable. Assume that $x_0, x \in U$ such that the segment $[x_0, x] \subset U$ (that is, $x_0 + \theta(x - x_0) \in U$ for every $0 \leq \theta \leq 1$). Then the following formula holds:

$$f(x) - f(x_0) = \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}_{x_0} \underbrace{(x - x_0, \dots, x - x_0)}_{k \text{ times}} + \frac{1}{(n+1)!} f^{(n+1)}_{x_0+\theta(x-x_0)} \underbrace{(x - x_0, \dots, x - x_0)}_{n+1 \text{ times}}.$$

Notice that the k-th Fréchet derivative is a k-linear operator, so the notation $f^{(k)}_{x_0} \underbrace{(x - x_0, \dots, x - x_0)}_{k \text{ times}} \equiv f^{(k)}_{x_0}(x - x_0)^k$ will be used throughout the paper.

Bear in mind that if $t \in [0,T]$ is fixed, we can consider functions a and b from (1) as $a = a(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ and $b = b(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times d_1}$. That way, if we consider vector spaces $\mathbb{X} = \mathbb{R}^d$ and $\mathbb{Y} = \mathbb{R}^d$ (or $\mathbb{R}^{d\times d_1}$), over field $\mathbb{R}$, equipped with norms $|\cdot|$, the upper-mentioned Fréchet derivatives can easily be represented via partial derivatives. For example, if $f = (f_1,...,f_d) : \mathbb{R}^d \to \mathbb{R}^d$ is Fréchet-differentiable in $x_0 \in \mathbb{R}^d$ (enough times), than $f'_{x_0} = f^{(1)}_{x_0} = \left[\frac{\partial f_i}{\partial x_j}(x_0)\right]_{d\times d} : \mathbb{R}^d \to \mathbb{R}^d$, $f''_{x_0} = f^{(2)}_{x_0} = \left[\frac{\partial^2 f_i}{\partial x_j \partial x_k}(x_0)\right]_{d\times d\times d} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)$ or $f''_{x_0} : (\mathbb{R}^d)^2 \to \mathbb{R}^d$ (differentiating the cases $f''_{x_0} \in \mathcal{L}(X,\mathcal{L}(X,Y))$ and $f''_{x_0} \in \mathcal{M}(X^2;Y)$), and so on.

Solutions of equations

$$x^n(t) = x^n(t_k) + \int_{t_k}^t \sum_{i=0}^{m_1} \frac{a_x^{(i)}\big(s,x^n(t_k)\big)}{i!}\Big(x^n(s)-x^n(t_k)\Big)^i ds + \int_{t_k}^t \sum_{i=0}^{m_2} \frac{b_x^{(i)}\big(s,x^n(t_k)\big)}{i!}\Big(x^n(s)-x^n(t_k)\Big)^i dW(s), \qquad (6)$$
$$t \in [t_k,t_{k+1}],\ \ k \in \{0,1,\dots,n-1\},$$

in which the drift and diffusion coefficients are Taylor approximations of functions a and b, respectively, while $x^n(t_0) = x_0$ a.s, are used to approximate solution $x = \{x(t),\ t \in [0,T]\}$ of the equation (1) on the partition (2). Functions $a_x^{(i)}$ and $b_x^{(i)}$ represent i-th partial derivatives on the second argument of functions a and b, respectively. The approximate solution $x^n = \{x^n(t),\ t\in[0,T]\}$, constructed in (6) by successive connecting of the processes $\{x^n(t),\ t \in [t_k,t_{k+1}]\}$ in points t_k, $k \in \{0,1,\dots,n-1\}$, is almost surely continuous process.

One can observe from (6) that the coefficients of the approximate equations do not depend on the remainders in Taylor expansions which is not the case when the Ito-Taylor expansion is applied. The Ito-Taylor expansion is obtained by iterated applications of the Ito formula to the integrands in the integral version of the stochastic differential equations, and unified by a canonical system of repeated stochastic Ito integrals with polynomial weight functions [14]. These expansions are the basis for the well-known stochastic numerical methods such as Euler, Milstein, Wagner-Platen, which are based on Taylor expansions of zero, first and second degrees, respectively. For higher-order numerical schemes one requires adequate smoothness of the drift and diffusion coefficients, but also an appropriate information about the driving Wiener processes. This information is contained in the remainders consisting of multiple stochastic integrals with respect to the Brownian motion and theirs estimation is difficult [15, 16]. The method in the present paper is convenient when the number of degrees in Taylor expansions of the coefficients is bigger than 1.

The existence and uniqueness of solutions of equations (1) and (6) is assumed without considering any conditions which are satisfied by theirs coefficients. All Lebesgue and Itô integrals are supposed to be defined well.

We introduce the following assumptions which are necessary for proving the main results of this paper.

$\mathcal{A}_1$: Functions a and b have Taylor approximations on second argument till the orders m_1 and m_2, respectively.

$\mathcal{A}_2$: Functions $a_x^{(m_1+1)}$ and $b_x^{(m_2+1)}$ are uniformly bounded, that is, there exist positive constants L_1 and L_2, such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |a_x^{(m_1+1)}(t,x)| \le L_1 \quad \text{and} \quad \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |b_x^{(m_2+1)}(t,x)| \le L_2.$$

$\mathcal{A}_3$: Functions a and b satisfy polynomial condition, i.e. there exist a positive real number D and a nonnegative integer q, such that for every $t \in [0,T]$ and $x, y \in \mathbb{R}^d$,

$$|a(t,x)-a(t,y)|^2 \vee |b(t,x)-b(t,y)|^2 \le D\Big(1+|x|^q+|y|^q\Big)|x-y|^2.$$

$\mathcal{A}_4$: Functions $a(\cdot,0)$ and $b(\cdot,0)$ are bounded on $[0,T]$. More precisely, there exist positive constants C_a and C_b such that $|a(t,0)| \le C_a$ and $|b(t,0)| \le C_b$ for every $t \in [0,T]$.

$\mathcal{A}_5$: There exist unique, almost surely continuous solutions x and x^n of equations (1) and (6) respectively, satisfying

$$E\sup_{t\in[0,T]} |x(t)|^{p(M\vee q+1)} \vee E\sup_{t\in[0,T]} |x^n(t)|^{p(M\vee q+1)^2} \le Q < \infty,$$

for $p > 2$, where $Q > 0$ is a constant independent of n and $M = m_1 \vee m_2$.

Remark 1.1. The Lipschitz condition implies that the coefficients do not change faster than a linear function of x as change in x and thus, it is too restrictive. A wide class of stochastic differential equations has coefficients which do not satisfy global Lipschitz condition or linear growth condition, which are sufficient conditions for existence and uniqueness of solution to the equation (see, for example, [6, 19]). Existence and uniqueness of solution to equation (1) is assumed, and under different conditions for coefficients of the equation can be proven, for example, if coefficients satisfy local Lipschitz and monotone conditions (Theorem 2.3.5 [19]).

The local Lipschitz condition holds if for every positive real number R, there exists a constant K_R depending only on R, such that for every $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \le R$, we have that

$$|a(t,x) - a(t,y)|^2 \vee |b(t,x) - b(t,y)|^2 \le K_R |x-y|^2, \quad t \in [0,T].$$

If there exists a positive constant $\tilde{S}$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$

$$x^T a(t,x) + \frac{1}{2}|b(t,x)|^2 \le \tilde{S}\Big(1 + |x|^2\Big), \tag{7}$$

then coefficients of equation (1) satisfy the monotone condition.

Note that one-sided Lipschitz condition for the drift, and global Lipschitz condition for the diffusion coefficient imply that monotone condition (7) holds for coefficients of the equation (1).

If there exists a positive constant $\mu > 0$ such that

$$\langle x - y, a(t,x) - a(t,y) \rangle \le \mu |x-y|^2, \tag{8}$$

for every $t \in [0,T]$, $x, y \in \mathbb{R}^d$, then function a satisfies the one-sided Lipschitz condition.

If there exists a positive constant c such that

$$|b(t,x) - b(t,y)|^2 \le c|x-y|^2, \tag{9}$$

for $t \in [0,T]$ and $x, y \in \mathbb{R}^d$, then function b satisfies the global Lipschitz condition.

Besides that, in [9] authors proved that under assumptions (8) and (9), stochastic differential equation (1) has a bounded p-th moment for $p > 2$, i.e., for every $p > 2$, there is $C = C(p,T) > 0$, such that

$$E \sup_{t \in [0,T]} |x(t)|^p \le C\Big(1 + E|x_0|^p\Big).$$

In this paper some well-known inequalities, such as Hölder and Burkholder-Davis-Gundy inequality are used in the proofs. Likewise, the elementary inequality is used in the sequel: for every $r \ge 0$ and for $a_i \in \mathbb{R},\ i \in \{1, 2, ..., n\}, n \in \mathbb{N}$,

$$\Big|\sum_{i=1}^n a_i\Big|^r \le (n^{r-1} \vee 1) \cdot \sum_{i=1}^n |a_i|^r. \tag{10}$$

The following integral Bihary type inequality plays an important role in the future analysis (see Remark 3.3 [26]).

Theorem 1.2. *Let $\bar{\mathcal{F}}$ be the class of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions:*

1) φ is nondecreasing and continuous in $\mathbb{R}^+$ and $\varphi(u) > 0$ for $u > 0$;

2) $\frac{1}{\alpha}\varphi(u) \le \varphi(\frac{u}{\alpha})$, $u \ge 0, \alpha \ge 1$.

Let $f = f(t), u = u(x)$ be a real valued nonnegative continuous functions on S, where S is any bounded open set in $\mathbb{R}$. If $g = g(x)$ is a positive, nondecreasing continuous function on S and φ belongs to class $\bar{\mathcal{F}}$ for which the following inequality

$$u(x) \le g(x) + \int_{x^0}^{x} f(t)\varphi(u(t))dt$$

holds for all $x \in S$ with $x \geq x^0 \in S$, then for $x^0 \leq x \leq x^*$,

$$u(x) \leq g(x)G^{-1}\Big(G(1) + \int_{x^0}^{x} f(t)dt\Big),$$

where

$$G(z) = \int_{z^0}^{z} \frac{ds}{\varphi(s)}, \ z \geq z^0 > 0,$$

G^{-1} is the inverse of G and x^* is chosen so that

$$G(1) + \int_{x^0}^{x} f(t)dt \in Dom(G^{-1}).$$

2. Main results

The main goal in this paper is to estimate the closeness between the solutions x and x^n, as well as the speed of convergence of the sequence $\{x^n, n \in \mathbb{N}\}$ to the solution x. In that sense the next lemma will be proved.

Lemma 2.1. *Let x^n be the solution to equation (6) and let the assumptions $\mathcal{A}_1 - \mathcal{A}_5$ hold. Then, for every $0 < r \leq p\,(M \vee q + 1)$,*

$$E|x^n(t) - x^n(t_k)|^r \leq C'\delta_n^{r/2}, \quad t \in [t_k, t_{k+1}], \quad k \in \{0, 1, \dots, n-1\},$$

where C' is a universal constant which is independent of n and δ_n.

Proof. We consider the sequence of equations (6) on the partition (2) in the form

$$x^n(t) = x^n(t_k) + \int_{t_k}^{t} A\Big(s, x^n(t_k), x^n(s)\Big)ds + \int_{t_k}^{t} B\Big(s, x^n(t_k), x^n(s)\Big)dW(s),$$

where $t \in [t_k, t_{k+1}], k \in \{0, 1, \dots, n-1\}$ and

$$\begin{aligned} A\Big(t, x^n(t_k), x^n(t)\Big) &= \sum_{i=0}^{m_1} \frac{a_x^{(i)}(t, x^n(t_k))}{i!}\Big(x^n(t) - x^n(t_k)\Big)^i, \\ B\Big(t, x^n(t_k), x^n(t)\Big) &= \sum_{i=0}^{m_2} \frac{b_x^{(i)}(t, x^n(t_k))}{i!}\Big(x^n(t) - x^n(t_k)\Big)^i. \end{aligned} \tag{11}$$

Firstly, for $r \geq 2$, by applying the elementary inequality (10) on (6) and afterwards the Hölder inequality on Lebesgue integral and Burkholder-Davis-Gundy and Hölder inequality on Itô integral, as well as Fubini theorem, for every $t \in [t_k, t_{k+1}], k \in \{0, 1, \dots, n-1\}$, one can conclude that

$$\begin{aligned} E|x^n(t) - x^n(t_k)|^r &\leq 2^{r-1}\left[E\left|\int_{t_k}^{t} A\Big(s, x^n(t_k), x^n(s)\Big)ds\right|^r + E\left|\int_{t_k}^{t} B\Big(s, x^n(t_k), x^n(s)\Big)dW(s)\right|^r\right] \\ &\leq 2^{r-1}\left[(t-t_k)^{r-1}\int_{t_k}^{t} E\Big|A(s, x^n(t_k), x^n(s))\Big|^r ds + c_r(t-t_k)^{r/2-1}\int_{t_k}^{t} E\Big|B(s, x^n(t_k), x^n(s))\Big|^r ds\right] \\ &\equiv 2^{r-1}(t-t_k)^{r/2-1}\Big[(t-t_k)^{r/2}J_1(t) + c_rJ_2(t)\Big]. \end{aligned} \tag{12}$$

In estimating integrals $J_1(t) = \int_{t_k}^t E\Big|A(s,x^n(t_k),x^n(s))\Big|^r ds$ and $J_2(t) = \int_{t_k}^t E\Big|B(s,x^n(t_k),x^n(s))\Big|^r ds$ we use assumptions $\mathcal{A}_1 - \mathcal{A}_5$. Then, we have that

$$\begin{aligned}
J_1(t) &= \int_{t_k}^t E\Big|(a(s,x^n(s)) - a(s,0)) + a(s,0) - \Big(a(s,x^n(s)) - A(s,x^n(t_k),x^n(s))\Big)\Big|^r ds \\
&\le 3^{r-1}\left\{\int_{t_k}^t E\left|a(s,x^n(s)) - a(s,0)\right|^r ds + \int_{t_k}^t E|a(s,0)|^r ds + \int_{t_k}^t E\Big|\, a(s,x^n(s)) - A(s,x^n(t_k),x^n(s))\Big|^r ds\right\} \\
&\equiv 3^{r-1}\Big\{J_1^1(t) + J_1^2(t) + J_1^3(t)\Big\}. \qquad (13)
\end{aligned}$$

To estimate the term $J_1^1(t)$, we apply the polynomial condition $\mathcal{A}_3$, inequality (10) and assumption $\mathcal{A}_5$. Thus, we obtain

$$\begin{aligned}
J_1^1(t) &\equiv \int_{t_k}^t E\Big|\, a(s,x^n(s)) - a(s,0)\Big|^r ds = \int_{t_k}^t E\Big(\Big|\, a(s,x^n(s)) - a(s,0)\Big|^2\Big)^{r/2} ds \\
&\le \int_{t_k}^t E\Big(D\Big(1 + |x^n(s)|^q\Big)\cdot |x^n(s)|^2\Big)^{r/2} ds \\
&\le D^{r/2} 2^{(r-2)/2} \int_{t_k}^t E\Big(\Big(1 + |x^n(s)|^{rq/2}\Big)|x^n(s)|^r\Big) ds \\
&= D^{r/2} 2^{(r-2)/2} \int_{t_k}^t \Big(E|x^n(s)|^r + E|x^n(s)|^{r(1+q/2)}\Big) ds \\
&\le D^{r/2} 2^{(r-2)/2} 2 \int_{t_k}^t Q ds = QD^{r/2} 2^{r/2} (t - t_k). \qquad (14)
\end{aligned}$$

Assumption $\mathcal{A}_4$ is used for estimating the integral $J_1^2(t)$ from (13). Therefore,

$$J_1^2(t) \equiv \int_{t_k}^t E|a(s,0)|^r ds \le C_a^r (t - t_k). \qquad (15)$$

In order to estimate the integral $J_1^3(t)$ in (13), we use assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, inequality (10) and $\mathcal{A}_5$. There exists $\theta_1 \in (0,1)$ such that

$$\begin{aligned}
J_1^3(t) &\equiv \int_{t_k}^t E\left| a\Big(s,x^n(s)\Big) - A\Big(s,x^n(t_k),x^n(s)\Big)\right|^r ds \\
&\equiv \int_{t_k}^t E\left|\frac{a_x^{(m_1+1)}\Big(s,x^n(t_k) + \theta_1\Big(x^n(s) - x^n(t_k)\Big)\Big)}{(m_1+1)!}\Big(x^n(s) - x^n(t_k)\Big)^{m_1+1}\right|^r ds \\
&\le \frac{L_1^r}{\Big[(m_1+1)!\Big]^r} \cdot 2^{(m_1+1)r-1} \int_{t_k}^t \Big(E|x^n(s)|^{(m_1+1)r} + E|x^n(t_k)|^{(m_1+1)r}\Big) ds \\
&\le \frac{2^{(m_1+1)r-1} L_1^r}{\Big[(m_1+1)!\Big]^r} \int_{t_k}^t 2E \sup_{\ell\in[0,T]} |x^n(\ell)|^{(M+1)r} ds \\
&\le \frac{2^{(m_1+1)r} L_1^r Q}{[(m_1+1)!]^r}(t - t_k). \qquad (16)
\end{aligned}$$

By applying the relations (14), (15) and (16), the inequality (13) becomes

$$J_1(t) \le C_1(t - t_k), \qquad (17)$$

where C_1 is a universal constant which depends on r, Q, D, C_a, L_1 and m_1. The application of the previous procedure yields

$$J_2(t) \leq C_2(t - t_k), \tag{18}$$

where C_2 is a universal constant which depends on r, Q, D, C_b, L_2 and m_2. Finally, by replacing (17) and (18) in (12) it follows

$$\begin{aligned} E|x^n(t) - x^n(t_k)|^r &\leq 2^{r-1}(t - t_k)^{r/2}\Big[C_1(t - t_k)^{r/2} + c_r C_2\Big] \\ &\leq \tilde{C}(t - t_k)^{r/2} \leq \tilde{C}\delta_n^{r/2}, \end{aligned}$$

where $\tilde{C} = \tilde{C}(C_1, C_2, T, r)$ is a constant.

For $r \in (0, 2)$, by using the Hölder inequality with conjugate coefficients $\big(2/r, 2/(2-r)\big)$, following estimate holds by proven part of the lemma

$$E|x^n(t) - x^n(t_k)|^r \leq \Big(E|x^n(t) - x^n(t_k)|^2\Big)^{r/2} \leq \Big(\tilde{C}\delta_n^{2/2}\Big)^{r/2} = C''\delta_n^{r/2}.$$

The proof is complete with $C' = \tilde{C} \vee C''$. □

In next theorem we establish a rate of convergence for the analytic method under consideration. We show that if the degrees of the Taylor approximations of the functions a and b increase, then the rate of the closeness between solutions x and x^n increases in the sense of L^p-norm.

Theorem 2.2. *Let x and x^n be solutions to equations (1) and (6), respectively. Under the assumptions $\mathcal{A}_1$–$\mathcal{A}_5$, for $p > 0$,*

$$E \sup_{t\in[0,T]} |x(t) - x^n(t)|^p \leq K\delta_n^{\frac{(m+1)p}{2}},$$

when $n \to +\infty$ and $\delta_n \to 0$, where $m = m_1 \wedge m_2$ and K is a universal constant which is independent of n and δ_n.

Proof. Let $t \in [0, T]$ be an arbitrary and fixed number. Bearing in mind (11), let us denote

$$\begin{aligned} A'(s) &= \sum_{k=0}^{n-1} \Big[a(s, x(s)) - A\Big(s, x^n(t_k), x^n(s)\Big)\Big] I_{[t_k, t_{k+1}\wedge t)}(s), \\ B'(s) &= \sum_{k=0}^{n-1} \Big[b(s, x(s)) - B\Big(s, x^n(t_k), x^n(s)\Big)\Big] I_{[t_k, t_{k+1}\wedge t)}(s),\ s \in [0, t]. \end{aligned}$$

Let $p \geq 2$. By using inequality (10), Hölder inequality, Burkholder-Davis-Gundy inequality and Fubini theorem, we get

$$\begin{aligned} E \sup_{s\in[0,t]} |x(s) - x^n(s)|^p &= E \sup_{s\in[0,t]} \left|\int_0^s A'(u)du + \int_0^s B'(u)dW(u)\right|^p \\ &\leq 2^{p-1}\left[E \sup_{s\in[0,t]} s^{p-1} \int_0^s \left|A'(u)\right|^p du + c_p E \left|\int_0^t \left|B'(u)\right|^2 du\right|^{p/2}\right] \\ &\leq 2^{p-1}\left[t^{p-1} \int_0^t E\left|A'(s)\right|^p ds + c_p t^{(p-2)/2} \int_0^t E\left|B'(s)\right|^p ds\right] \\ &\leq 2^{p-1}\Big[T^{p-1}S_1(t) + c_p T^{(p-2)/2} S_2(t)\Big], \end{aligned} \tag{19}$$

where

$$S_1(t) = \int_0^t E\left|A'(s)\right|^p ds \ \text{ and } \ S_2(t) = \int_0^t E\left|B'(s)\right|^p ds.$$

The estimate of $S_1(t)$ is based on the triangle inequality and inequality (10). Then,

$$\begin{aligned} S_1(t) &= \sum_{k=0}^{n-1} \int_{t_k\wedge t}^{t_{k+1}\wedge t} E\Big|a(s,x(s)) - a(s,x^n(s)) + a(s,x^n(s)) - A\Big(s,x^n(t_k),x^n(s)\Big)\Big|^p ds \\ &\le 2^{p-1}\Big[\sum_{k=0}^{n-1} \int_{t_k\wedge t}^{t_{k+1}\wedge t} E\Big|a(s,x(s)) - a(s,x^n(s))\Big|^p ds + \sum_{k=0}^{n-1} \int_{t_k\wedge t}^{t_{k+1}\wedge t} E\Big| a(s,x^n(s)) - A\Big(s,x^n(t_k),x^n(s)\Big)\Big|^p ds\Big]. \end{aligned} \tag{20}$$

One can now estimate the first integral in (20) by applying the polynomial condition $\mathcal{A}_3$, Cauchy-Schwarz-Bunyakovsky inequality and $\mathcal{A}_5$. Hence,

$$\begin{aligned} &\int_0^t E|a(s,x(s)) - a(s,x^n(s))|^p ds \\ &\quad \le D^{p/2} \int_0^t E\Big[\Big(1 + |x(s)|^q + |x^n(s)|^q\Big)^{p/2} |x(s) - x^n(s)|^p\Big] ds \\ &\quad \le D^{p/2} \int_0^t \Big[E|x(s) - x^n(s)|^p\Big]^{\frac{1}{2}} \Big[E\Big[\Big(1 + |x(s)|^q + |x^n(s)|^q\Big)^p |x(s) - x^n(s)|^p\Big]\Big]^{\frac{1}{2}} ds \\ &\quad \le D^{p/2} \int_0^t \Big[E \sup_{\ell\in[0,s]} |x(\ell) - x^n(\ell)|^p\Big]^{\frac{1}{2}} \Big[E \sup_{\ell\in[0,s]} \Big[\Big(1 + |x(\ell)|^q + |x^n(\ell)|^q\Big)^p |x(\ell) - x^n(\ell)|^p\Big]\Big]^{\frac{1}{2}} ds \\ &\quad \le D^{p/2} Q_1 \int_0^t \Big[E \sup_{\ell\in[0,s]} |x(\ell) - x^n(\ell)|^p\Big]^{\frac{1}{2}} ds, \end{aligned} \tag{21}$$

where $Q_1 = (6^p Q)^{\frac{1}{2}}$ is a constant derived via inequality (10) and Hölder inequality.

To estimate the second integral in (20) we use assumptions $\mathcal{A}_1, \mathcal{A}_2$ and Lemma 2.1. There exists $\theta_1 \in (0,1)$ such that

$$\begin{aligned} &\sum_{k=0}^{n-1} \int_{t_k\wedge t}^{t_{k+1}\wedge t} E\Bigg|\frac{a_x^{(m_1+1)}\Big(s, x^n(t_k) + \theta_1(x^n(s) - x^n(t_k))\Big)}{(m_1+1)!}\Big(x^n(s) - x^n(t_k)\Big)^{m_1+1}\Bigg|^p ds \\ &\qquad \le \frac{L_1^p}{\Big[(m_1+1)!\Big]^p} \sum_{k=0}^{n-1} \int_{t_k\wedge t}^{t_{k+1}\wedge t} E\Big|x^n(s) - x^n(t_k)\Big|^{(m_1+1)p} \\ &\qquad \le \frac{L_1^p C' T}{\Big[(m_1+1)!\Big]^p} \cdot \delta_n^{(m_1+1)p/2}. \end{aligned} \tag{22}$$

Then, on the basis of (21) and (22), (20) becomes

$$S_1(t) \le 2^{p-1}\left[D^{p/2} Q_1 \int_0^t \Big[E \sup_{\ell\in[0,s]} \Big|x(\ell) - x^n(\ell)\Big|^p\Big]^{\frac{1}{2}} ds + \frac{L_1^p C' T \delta_n^{(m_1+1)p/2}}{\Big[(m_1+1)!\Big]^p}\right]. \tag{23}$$

Analogously,

$$\begin{aligned} S_2(t) &= \sum_{k=0}^{n-1} \int_{t_k\wedge t}^{t_{k+1}\wedge t} E\Big|b(s,x(s)) - b(s,x^n(s)) + b(s,x^n(s)) - B\Big(s,x^n(t_k),x^n(s)\Big)\Big|^p ds \\ &\le 2^{p-1}\left[D^{p/2} Q_1 \int_0^t \Big[E \sup_{\ell\in[0,s]} |x(\ell) - x^n(\ell)|^p\Big]^{\frac{1}{2}} ds + \frac{L_2^p C' T \delta_n^{(m_2+1)p/2}}{\Big[(m_2+1)!\Big]^p}\right]. \end{aligned} \tag{24}$$

Now, putting (23) and (24) in (19), we compute

$$E\sup_{s\in[0,t]}|x(s)-x^n(s)|^p \le 2^{2p-2}C'T^{p/2}\left\{\frac{T^{p/2}L_1^p}{\big[(m_1+1)!\big]^p}\cdot\delta_n^{(m_1+1)p/2}+\frac{c_pL_2^p}{\big[(m_2+1)!\big]^p}\cdot\delta_n^{(m_2+1)p/2}\right\}$$
$$+\,2^{2p-2}D^{p/2}Q_1\Big(T^{p-1}+c_pT^{(p-2)/2}\Big)\int_0^t\Big[E\sup_{\ell\in[0,s]}|x(\ell)-x^n(\ell)|^p\Big]^{1/2}ds.$$

Since n is going to be large enough and δ_n is going to be small enough (close to 0), δ_n is going to be less than 1 and then

$$\delta_n^{(m_1+1)p/2}\vee\delta_n^{(m_2+1)p/2}\le\delta_n^{(m+1)p/2},$$

where $m=m_1\wedge m_2$. The last inequality becomes

$$E\sup_{s\in[0,t]}|x(s)-x^n(s)|^p\le Z_1(T)\delta_n^{(m+1)p/2}+Z_2(T)\int_0^t\Big[E\sup_{\ell\in[0,s]}|x(\ell)-x^n(\ell)|^p\Big]^{1/2}ds, \tag{25}$$

where

$$Z_1(T)=2^{2p-2}C'T^{p/2}\Big\{\frac{T^{p/2}L_1^p}{\big[(m_1+1)!\big]^p}+\frac{c_pL_2^p}{\big[(m_2+1)!\big]^p}\Big\},$$
$$Z_2(T)=2^{2p-2}D^{p/2}Q_1\Big(T^{p-1}+c_pT^{(p-2)/2}\Big).$$

To finish the proof, we apply the Bihari type inequality (Theorem 1.2) on (25), where the function φ is defined as $\varphi: z\mapsto z^{1/2}$, $z\in[0,+\infty)$ and function G is a bijection, defined for positive numbers z as $G(z)=2z^{1/2}$. Its inverse function is $G^{-1}(y)=\frac{1}{4}y^2$, $y>0$. Also, $\int_0^t f(s)ds=Z_2(T)t\le Z_2(T)T$, for every $t\in[0,T]$. Finally, for every $t\in[0,T]$,

$$E\sup_{s\in[0,t]}|x(s)-x^n(s)|^p\le Z_1(T)\delta_n^{(m+1)p/2}\frac{1}{4}\Big(2+Z_2(T)T\Big)^2=K\delta_n^{(m+1)p/2},$$

where $K=0.25Z_1(T)\Big(2+Z_2(T)T\Big)^2$ is a constant independent of n and δ_n. The last inequality holds for every $t\in[0,T]$, so

$$E\sup_{t\in[0,T]}|x(t)-x^n(t)|^p\le K\delta_n^{(m+1)p/2}. \tag{26}$$

For $0<r<2$ proof is analogous to the end of the proof of Lemma 2.1. □

Almost sure convergence of the sequence of the approximate solutions to equations (6) to the exact solution of the equation (1) is established in the next theorem.

Theorem 2.3. *Let the conditions of Theorem 2.2 be satisfied and let there exist a monotonic decreasing sequence of positive numbers* $(\lambda_n)_{n\in\mathbb{N}}$ *such that* $\lambda_n\to 0$ *when* $n\to\infty$ *and* $\sum_{n=1}^{\infty}\delta_n\lambda_n^{-2}<\infty$. *Then, the sequence* $(x^n)_{n\in\mathbb{N}}$ *of the approximate solutions of the equations (6) converges almost surely to the solution x of the equation (1).*

Proof. Chebyshev inequality and the relation (26) from the proof of the previous theorem yield

$$\sum_{n=1}^{\infty}P\Big\{\sup_{t\in[0,T]}|x(t)-x^n(t)|^{p/2}\ge\lambda_n\Big\}\le\sum_{n=1}^{\infty}E\sup_{t\in[0,T]}|x(t)-x^n(t)|^p\lambda_n^{-2}$$
$$\le K\sum_{n=1}^{\infty}\delta_n^{(m+1)p/2}\lambda_n^{-2}<\infty.$$

The Borel-Cantelli lemma implies that with probability one only finitely many events $\{\sup_{t\in[0,T]}|x(t)-x^n(t)|^{p/2}\ge\lambda_n\}$ will be realized, that is, for all large enough n $\sup_{t\in[0,T]}|x(t)-x^n(t)|<\lambda_n^{2/p}$ almost surely. Therefore, the sequence $(x^n)_{n\in\mathbb{N}}$ converges almost surely to the solution x, uniformly in $[0,T]$. □

The following example illustrates the previous theoretical findings.

Example 2.4. Let us consider an autonomous stochastic differential equation

$$dx(t) = \Big(-\alpha x^3(t) + \beta \sin x(t)\Big)dt + \sigma(1 - 2\sin x(t))dW(t),\ \ t \in [0,T], \tag{27}$$

with initial condition $x(0) = 0$ a.s., where α, β and σ are real constants.

The main goal of this example is to demonstrate the situation when the application of the proposed analytical method leads to the explicitly solvable stochastic differential equation and situation when the approximate equations are not explicitly solvable, but have unique solutions. In the second situation, the approximate equations have simpler form than Eq. (27), such that some numerical method could be applied to find theirs approximate solutions.

Both the drift coefficient $a(x) = -\alpha x^3 + \beta \sin x$ and diffusion coefficient $b(x) = \sigma(1-2\sin x)$ are continuously-differentiable and hence locally Lipschitz continuous, but the drift coefficient is not globally Lipschitz continuous whilst the diffusion coefficient is. Also, note that one-sided Lipschitz condition (8) holds for function a when it is $\alpha > 0$. Thus, in a view of Remark 1.1, all assumptions of the existence and uniqueness theorem are satisfied and the equation (27) has unique solution $x = x(t)$ which satisfies $x(0) = 0$ a.s. Conditions $\mathcal{A}_3$ and $\mathcal{A}_4$ hold and Lemma 3.2 [9] implies that $E\sup_{0\le t\le T}|x(t)|^r \le C\,(1 + E|x(0)|^r) \le Q$, for $r > 2$. Hence, condition $\mathcal{A}_5$ holds for the solution x. Notice that equation (27) is not explicitly solvable.

The approximate equations (6) will be formed by using Maclaurin approximations of functions a and b instead of Taylor approximations of those functions near the points $x(t_k)$. Because of that the subsegments $[t_k, t_{k+1}]$ of the partition (2) are transformed into $[0, t_{k+1} - t_k]$, $k \in \{0, 1, ..., n-1\}$. By the time translation $t = t_k + u$, for $k \in \{0, \ldots, n-1\}$, new Wiener process $\tilde{W}$ and unknown process $\tilde{x}$ are obtained, such that

$$\tilde{W}(u) = W(t_k + u) \text{ a.s.},\quad \tilde{x}(u) = x(t_k + u) \text{ a.s.} \tag{28}$$

Then equation (27) becomes

$$d\tilde{x}(u) = \Big(-\alpha \tilde{x}^3(u) + \beta \sin \tilde{x}(u)\Big)du + \sigma(1 - 2\sin \tilde{x}(u))d\tilde{W}(u),\quad u \in [0, t_{k+1} - t_k],\quad k \in \{0,\ldots,n-1\}. \tag{29}$$

To demonstrate the fact that the higher order of the derivatives gives the better approximation of the solution to equation (27), three types of equations are discussed bellow.

(I) Maclaurin approximations of the functions $a = a(x)$ and $b = b(x)$ up to the third and second derivative, respectively, are

$$a(x) \approx -\frac{6\alpha + \beta}{6}x^3 + \beta x,\quad b(x) \approx -2\sigma x + \sigma,\quad (x \to 0)$$

(formally $b(x) \approx 0\cdot\frac{x^2}{2} - 2\sigma x + \sigma$). Conditions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold since $\sup_x |a^{(4)}(x)| \le |\beta|$ and $\sup_x |b'''(x)| \le 2|\sigma|$. The approximate solution $\{\tilde{x}^n(u),\ u \in [0,T]\}$ is constructed successively by using the solutions of the equations

$$d\tilde{x}^n(u) = \left(-\frac{6\alpha+\beta}{6}\Big(\tilde{x}^n(u)\Big)^3 + \beta\tilde{x}^n(u)\right)du + \sigma(1 - 2\tilde{x}^n(u))d\tilde{W}(u),\quad u \in [0, t_{k+1} - t_k],\quad k \in \{0,\ldots,n-1\}. \tag{30}$$

Equations (30) are not explicitly solvable, but coefficients satisfy the same conditions as coefficients of the initial equation (27) for $\alpha \ge -\beta/6$ and

$$E\sup_{0\le u\le T}|\tilde{x}^n(u)|^r \le c_1\,(1 + E|x(0)|^r) \le Q,\quad r > 2.$$

Time translation (28) implies

$$dx^n(t) = \left(-\frac{6\alpha+\beta}{6}\Big(x^n(t)\Big)^3 + \beta x^n(t)\right)dt + \sigma(1 - 2x^n(t))dW(t),\quad t \in [t_k, t_{k+1}],\quad k \in \{0,\ldots,n-1\}. \tag{31}$$

Theorem 2.2 gives the rate of closeness in the L^p sense

$$E \sup_{t\in[0,T]} |x(t) - x^n(t)|^p \leq K\delta_n^{\frac{3}{2}p}.$$

(II) Maclaurin approximations of the functions $a = a(x)$ and $b = b(x)$ up to the second derivatives are

$$a(x) \approx \beta x, \quad b(x) \approx -2\sigma x + \sigma, \quad (x \to 0)$$

(formally $a(x) \approx 0 \cdot \frac{x^2}{2} + \beta x$ and $b(x) \approx 0 \cdot \frac{x^2}{2} - 2\sigma x + \sigma$). The approximate solution $\{\tilde{x}^n(u),\ u \in [0,T]\}$ is constructed using the equations

$$d\tilde{x}^n(u) = \beta\tilde{x}^n(u)du + \sigma(1 - 2\tilde{x}^n(u))d\tilde{W}(u), \quad u \in [0, t_{k+1} - t_k], \quad k \in \{0,\dots,n-1\}. \tag{32}$$

Equations (32) are inhomogeneous linear stochastic differential equations with multiplicative noise and theirs coefficients satisfy global Lipschitz and linear growth conditions. Hence, equations (32) are explicitly solvable (see [14], p. 119), theirs solutions are

$$\tilde{x}^n(u) = \tilde{x}_0^n e^{(\beta-2\sigma^2)u-2\sigma(\tilde{W}(u)-\tilde{W}(0))} + 2\sigma^2\int_0^u e^{-(\beta-2\sigma^2)(u-s)-2\sigma(\tilde{W}(u)-\tilde{W}(s))}ds + \sigma\int_0^u e^{-(\beta-2\sigma^2)(u-s)+2\sigma(\tilde{W}(u)-\tilde{W}(s))}d\tilde{W}(s),$$
$$u \in [0, t_{k+1} - t_k],\ k \in \{0,\dots,n-1\},$$

and $E\sup_{0\leq t\leq T}|x^n(t)|^r = E\sup_{0\leq u\leq T}|\tilde{x}^n(u)|^r \leq c_2\left(1 + 3^{r-1}E|x(0)|^r\right) \leq Q$, for $r > 2$ (Theorem 2.4.4 [19]). On the basis of the time translation (28) we obtain

$$x^n(t) = x^n(t_k)e^{(\beta-2\sigma^2)(t-t_k)-2\sigma(W(t)-W(t_k))} + 2\sigma^2\int_{t_k}^t e^{-(\beta-2\sigma^2)(t-s)-2\sigma(W(t)-W(s))}ds + \sigma\int_{t_k}^t e^{-(\beta-2\sigma^2)(t-s)-2\sigma(W(t)-W(s))}dW(s), \tag{33}$$
$$t \in [t_k, t_{k+1}], \quad k \in \{0,\dots,n-1\}.$$

Theorem 2.2 gives the same rate of closeness in the L^p sense as in the previous case.

(III) Maclaurin approximations of the functions $a = a(x)$ and $b = b(x)$ up to the derivatives of the order two and zero, respectively, are

$$a(x) \approx \beta x, \quad b(x) \approx \sigma, \quad (x \to 0).$$

Then the equations

$$d\tilde{x}^n(u) = \beta\tilde{x}^n(u)du + \sigma d\tilde{W}(u), \quad u \in [0, t_{k+1} - t_k], \quad k \in \{0,\dots,n-1\}. \tag{34}$$

are explicitly solvable (see [14], p. 118) and theirs solutions are

$$\tilde{x}^n(u) = e^{\beta u}\left(\tilde{x}_0^n + \sigma\int_0^u e^{-\beta s}d\tilde{W}(s)\right), \quad u \in [0, t_{k+1} - t_k], \quad k \in \{0,\dots,n-1\}.$$

By applying time translation we get

$$x^n(t) = e^{\beta(t-t_k)}\Big(x^n(t_k) + \sigma\int_{t_k}^t e^{-\beta(s-t_k)}dW(s)\Big), \quad t \in [t_k, t_{k+1}], \quad k \in \{0,\dots,n-1\}. \tag{35}$$

Theorem 2.2 gives the rate of closeness in the L^p sense

$$E \sup_{t\in[0,T]} |x(t) - x^n(t)|^p \leq K\delta_n^{\frac{1}{2}p}.$$

It should be pointed out that in equations (29), (30), (32) and (34), for $k = 0$ the initial condition is $\tilde{x}(0) = 0$ a.s. and for $k \in \{1,\dots,n-1\}$ the initial conditions are determined successively as the values of the process

$\tilde{x}(u)$ in the points $t_k - t_{k-1}$. Moreover, by successive connecting of the processes $\{x^n(t),\ t \in [t_k, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$, which represent the solutions of the equations (31), (33) and (35), in the partition points almost surely continuous solution $\{x^n(t),\ t \in [0, T]\}$ is constructed.

The most commonly used method for approximating the solutions of stochastic differential equations is numerical Euler-Maruyama (EM) method. It is shown in [9] that this method, which is based on Taylor expansion of zero degrees, has order 1/2. We can compare the approximate solutions of the initial equation (27) obtained by the numerical EM method and analytical method described in this paper by applying Taylor expansions of second and zero degrees for different values of the parameters α, β and σ.

The EM method applied to initial equation (27) computes approximations $X_k \approx x(t_k)$, where $X_0 = 0$,

$$\begin{aligned} X_{k+1} &= X_k + \Big(-\alpha X_k^3 + \beta \sin X_k\Big)(t_{k+1} - t_k) + \sigma(1 - 2\sin X_k)(W(t_{k+1}) - W(t_k)), \\ \bar{X}(t) &= X_k + (t - t_k)\Big(-\alpha X_k^3 + \beta \sin X_k\Big) + \sigma(1 - 2\sin X_k)(W(t) - W(t_k)), \end{aligned} \tag{36}$$

for $t \in [t_k, t_{k+1}]$ and EM solution is defined by $X(t) = \bar{X}(t)$ for $t \in [t_k, t_{k+1}]$.

The result of this comparison can be seen in Figures 1 and 2. According to Theorem 2.2 the sequence of the approximate solutions (33) has greater order of the L^p convergence comparing to solutions (35) and (36).

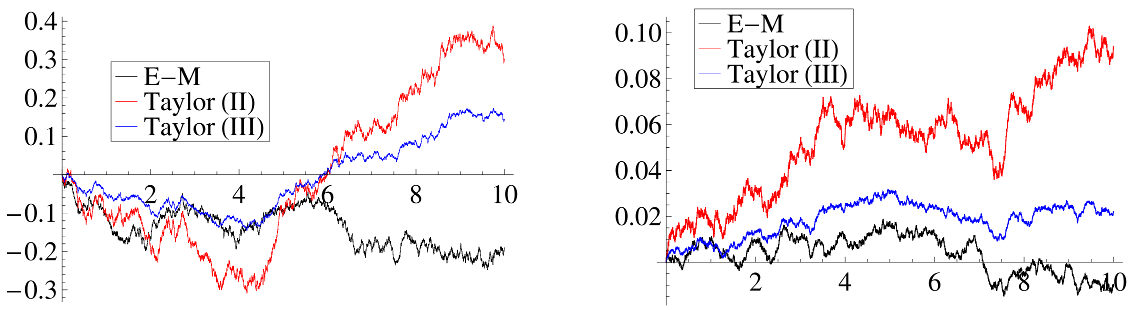

Figure 1: Trajectories of the solutions to the equations (36), (33) and (35) for $x_0 = 0$ and: $\alpha = 0.3$, $\beta = 0.3\pi^3/108$, $\sigma = 0.05$ (left), $\alpha = 0.1$, $\beta = 0.1$, $\sigma = 0.01$ (right), for $\delta_{10\,000} = 0.001$ on time interval $[0, 10]$

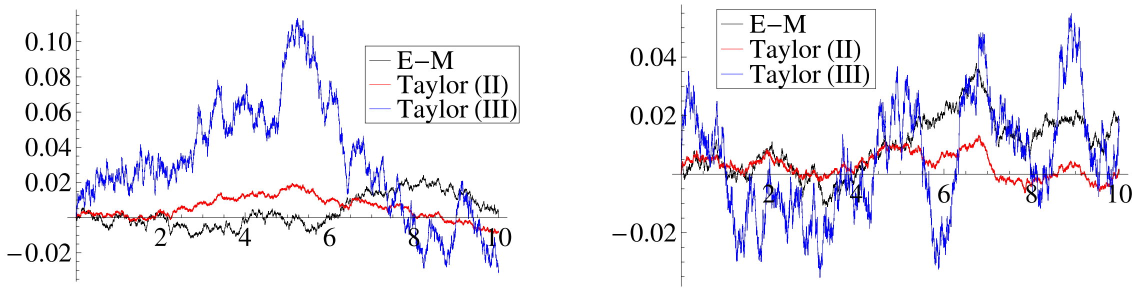

Figure 2: Trajectories of the solutions to the equations (36), (33) and (35) for $x_0 = 0$, $\alpha = 0.5$, $\beta = -0.2$ and: $\sigma = 0.01$ (left), $\sigma = -0.01$ (right), for $\delta_{10\,000} = 0.001$ on time interval $[0, 10]$

3. Conclusion

The goal of this paper was to construct the approximate solutions to stochastic differential equation defined on a partition of a time interval. The coefficients of initial equation are approximated by theirs Taylor series up to arbitrary derivatives in the case when they behave like polynomials and moment bounds are available. The closeness of the initial and approximate solutions is estimated in the sense of the L^p-norm and with probability one.

The stochastic numerical methods such as Euler, Milstein, Wagner-Platen have order of convergence 1/2, 1, 3/2, respectively, which is exactly the case with analytic approximations described in this paper, obtained by applying Taylor expansions of zero, first and second degrees, respectively. The authors in [9, 10, 18], for example, under non-Lipschitz and polynomial conditions for the coefficients of the stochastic differential equation, proved that Euler-Maruyama solution converges strongly at the rate one half. Milstein-type [7, 11, 30] schemes, under the same conditions, may achieve a strong convergence order greater than that of Euler-type schemes and additional computational effort is required to approximate the iterated Ito integrals for every time step. This will enable these schemes to lose their advantage over Euler-type schemes in computational efficiency. These facts indicate that numerical approximations based on Taylor expansions of higher degrees could be improved by combining them with the presented analytic approximations.

It should be stressed that in the present case we obtain bigger error of approximation than in the cases when the remainders are included, but with the appropriate choice of the number of steps n that error could be made satisfactory small.

Very often in the applications, when it is more suitable to deal with the polynomials comparing to cases when the coefficients of equations are complex nonlinear functions, it is useful to apply the analytical approximations.

The presented method could be appropriately extended to different types of stochastic differential equations. Besides that, some other conditions for coefficients of the equation can also provide the application of analytical approximation to coefficients that do not behave necessarily as polynomials.

References

[1] Atalla, M. A.: Finite-difference approximations for stochastic differential equations, Probabilistic Methods for the Investigation of Systems with an Infinite number of Degrees of freedom. Inst. of Math. Acad. of Science USSR, Kiev, 11-16 (1986)(in Russian).

[2] Atalla, M. A.: On one approximating method for stochastic differential equations, Asymptotic Methods for the Theory of Stochastic processes. Inst. of Math. Acad. of Science USSR, Kiev, 15-21 (1987) (in Russian).

[3] Collatz, L.: Functional analysis and numerical mathematics. Academic Press, New York - San Francisco - London, 1966.

[4] Feng, L., Li, S.: The Almost Sure Asymptotic Stability and Boundedness of Stochastic Functional Differential Equations with Polynomial Growth Condition. Abstract and Applied Analysis Volume 2014, Article ID 629426, 11 pages.

[5] Flett, T. M.: Differential analysis. Cambridge University Press, Cambridge, 1980.

[6] Gikhman I.I., Skorokhod A.V. Stochastic differential equations. Springer, Berlin, 1972.

[7] Guo, Q., Liu, W., Mao, X., Yue, R.: The truncated Milstein method for stochastic differential equations. J. of Comp. and Appl. Math., 338 (2018) 298–310.

[8] Han, M., Ma, Q., Ding, X.: The projected explicit Itô-Taylor methods for stochastic differential equations under locally Lipschitz conditions and polynomial growth conditions. J. Comp. Appl. Math. 348 (2019) 161–180.

[9] Higham, D. J., Mao, X., Stuart, A. M.: Strong convergence of Euler-type methods for nonlinear stochastic differential equations. SIAM J. Numer. Anal., 40(3)(2002) 1041–1063.

[10] Hutzenthaler, M., Jentzen, A., Kloeden, P. E.: Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. Ann. Appl. Probab. 22 (4) (2012) 1611–1641.

[11] Izgi B., Cetin, C.: Milstein-type semi-implicit split-step numerical methods for nonlinear SDE with locally Lipschitz drift terms. Thermal Science, 23 (2019) S1–S12.

[12] Janković, S., Ilić, D.: An analytic approximation of solutions of stochastic differential equations. Computers & Math. with Appl, 47 (2004) 903–912.

[13] Janković, S., Ilić, D.: An analytic approximate method for solving stochastic integrodifferential equations. J. Math. Anal. Appl., 320 (2006) 230–245.

[14] Kloeden, P. E., Platen E.: Numerical Solution of Stochastic Differential Equation. Springer, Berlin, Heidelberg, 1999.

[15] Kloeden, P. E., Platen E., Wright, W.: The approximation of the multiple stochastic integrals. Stoch. Anal. and Appl., 10 (1992) 431–441.

[16] Kloeden, P. E., Shardlow, T.: The Milstein scheme for stochastic delay differential equations without using anticipative calculus. Stoch. Anal. Appl. 30(2) (2012) 181–202.

[17] Kumar, C., Sabanis, S.: Strong Convergence of Euler Approximations of Stochastic Differential Equations with Delay Under Local Lipschitz Condition. Stochastic Analysis and Application 32 (2014) 207–228.

[18] Liu, W., Mao, X.: Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations. Appl. Math. and Comp., 223 (2013) 389–400.

[19] Mao X.: Stochastic differential equations and applications. Horwood Publishing, Chichester, 2008.

[20] Mao, X., Rassias, M.: Khasminskii-Type Theorems for Stochastic Differential Delay Equations. Stochastic Analysis and Applications 23 (2005) 1045–1069.

[21] Mao, X., Szpruch, L.: Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients. J. Comput. Appl. Math. 238 (2013) 14–28.

[22] Milošević, M.: On the approximations of solutions to stochastic differential delay equations with Poisson random measure via Taylor series. Filomat, 27(1) (2013) 201–214.

[23] Milošević, M, Jovanović M.: A Taylor polynomial approach in approximations of solution to pantograph stochastic differential equations with Markovian switching. Math. and Comp. Modell., 53(1-2) (2011) 280–293.

[24] Milošević, M., Jovanović, M.: An application of Taylor series in the approximation of solutions to stochastic differential equations with time-dependent delay. J. Comput. Appl. Math., 235(15) (2011) 4439–4451.

[25] Milošević, M., Jovanović, M., Janković, S.: An approximate method via Taylor series for stochastic functional differential equations. J. Math. Anal. Appl, 363 (2010) 128–137.

[26] Oguntuase, J. A.: On integral inequalities of Gronwall-Bellman-Bihari type in several variables. J. Ineq. Pure Appl. Math., 1(2) (2000).

[27] Papapantoleon A.: Old and new approaches to LIBOR modeling, Statistica Neederlanica, 64(3) (2010) 257–275.

[28] Papapantoleon A., Siopacha M.: Strong Taylor approximation of stochastic differential equations and application to the Levy LIBOR model, Proceedings of the Actuarial and Financial Mathematical Conference, Brussels, Belgium, 2011, 47–62.

[29] Siopacha M., Teichmann J.: Weak and strong Taylor methods for numerical solutions of stochastic differential equations, Quantitative Finance, 11:4 (2011) 517-528.

[30] Wang X., Gan S.: The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. J. Differ. Equ. Appl. 19 (3) (2013) 466–490.