Filomat 35:1 (2021), 105–124
https://doi.org/10.2298/FIL2101105G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Decomposition of the Tensor Product of Matrices

Caixing Gu[a], Jaehui Park[b]

[a] *California Polytechnic State University*
[b] *Seoul National University*

Abstract. In this paper we decompose (under unitary equivalence) the tensor product $A \oplus A$ into a direct sum of irreducible matrices, when A is a 3×3 matrix.

1. Introduction

Let H be a complex separable Hilbert space and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. A reducing subspace M for $A \in \mathcal{B}(H)$ is a closed subspace of H which is invariant for both A and A^*. An operator $A \in \mathcal{B}(H)$ is said to be irreducible if A has no nontrivial reducing subspace. A reducing subspace M for A is said to be minimal if the restriction $A|M$ is irreducible.

It is known that the set of irreducible operators is dense in $\mathcal{B}(H)$ (cf. [3]) and its complement (the set of all reducible operators) is also dense in $\mathcal{B}(H)$ (cf. [6]).

Let $H \otimes H$ be the tensor product Hilbert space, and let $A, B \in \mathcal{B}(H)$. If either A or B is reducible, then it is clear that the tensor products $A \otimes B$ and $A \otimes I + I \otimes B$ are reducible operators in $\mathcal{B}(H \otimes H)$. However, if both A and B are irreducible, we cannot guarantee that $A \otimes B$ and $A \otimes I + I \otimes B$ are irreducible (cf. [4]). We focus on the case $A = B$. Let $A \in \mathcal{B}(H)$ be irreducible, and let

$$W(A) := A \otimes A,$$
$$T(A) := A \otimes I + I \otimes A,$$

where $I = I_H$ denotes the identity operator on H. The operators $W(A)$ and $T(A)$ are always reducible. Two reducing subspaces are

$$H_s := \mathrm{Span}\{h \otimes h : h \in H\},$$
$$H_{as} := \mathrm{Span}\{h \otimes g - g \otimes h : g, h \in H\},$$

where "Span" means the closed linear span in $H \otimes H$. It is easy to see that $H \otimes H = H_s \oplus H_{as}$, and H_s and H_{as} are two reducing subspaces of both $W(A)$ and $T(A)$. Let

$$W_s(A) := W(A)|H_s, \quad W_{as}(A) := W(A)|H_{as},$$
$$T_s(A) := T(A)|H_s, \quad T_{as}(A) := T(A)|H_{as}.$$

We record the above observation as a lemma.

2010 *Mathematics Subject Classification*. Primary 47A15, 15A21; Secondary 47L40, 15A69
Keywords. irreducible operator, unitary equivalence, tensor product, symmetric tensor
Received: 28 May 2020; Accepted: 11 March 2021
Communicated by Dragan S. Djordjević
Corresponding author: Jaehui Park
Email addresses: cgu@calpoly.edu (Caixing Gu), nephenjia@snu.ac.kr (Jaehui Park)

Lemma 1.1. *If $A \in \mathcal{B}(H)$, then $T(A) = T_s(A) \oplus T_{as}(A)$ and $W(A) = W_s(A) \oplus W_{as}(A)$ on $H_s \oplus H_{as}$.*

Proof. We include a more abstract proof which indicates more general results hold for operators invariant under the permutation group on the tensor product $H \otimes \cdots \otimes H$ (cf. [2], [5]). Let σ denote the permutation of $\{1, 2\}$, i.e., $\sigma = (1\,2)$. Let U_σ be the unitary operator on $H \otimes H$ defined by $U_\sigma(h \otimes g) = g \otimes h$. Then $U_\sigma^2 = I$. The eigenvalues of U_σ are 1 and -1, and the corresponding eigenspaces are H_s and H_{as}, respectively. Since $W(A)U_\sigma = U_\sigma W(A)$, it follows that both H_s and H_{as} are reducing subspaces of $W(A)$. Similarly, both H_s and H_{as} are reducing subspaces of $T(A)$. □

The above lemma motivates the following questions.

Problem 1.2. *For which irreducible operator A are both $W_s(A)$ and $W_{as}(A)$ irreducible?*
For which irreducible operator A are both $T_s(A)$ and $T_{as}(A)$ irreducible?

For a square matrix A, the operator $T(A)$ is the Kronecker sum $A \boxplus A$ of A with itself. The decomposition of $T(A)$ when A is 3×3 matrix has been characterized in the paper [1].

Suppose that $\dim H = 3$, i.e., $H \cong \mathbb{C}^3$, where "$\cong$" stands for unitary equivalence. Then we may regard an operator $A \in \mathcal{B}(H)$ as a 3×3 matrix with complex entries. Note that H_s is the subspace of symmetric tensors and H_{as} is the subspace of anti-symmetric tensors. If $\{e_1, e_2, e_3\}$ is any orthonormal basis for H, then H_s and H_{as} have the following orthonormal bases:

$$H_s = \operatorname{Span}\left\{e_n \otimes e_n, \frac{1}{\sqrt{2}}(e_n \otimes e_m + e_m \otimes e_n) : 1 \le n \le 3,\ n < m \le 3\right\},$$
$$H_{as} = \operatorname{Span}\left\{\frac{1}{\sqrt{2}}(e_n \otimes e_m - e_m \otimes e_n) : 1 \le n \le 3,\ n < m \le 3\right\}.$$

Theorem 1.3 ([1]). *Let A be a 3×3 irreducible matrix. Then*

(i) *$T_s(A)$ is reducible if and only if A is unitarily equivalent to a matrix of the form*

$$\alpha I + \begin{bmatrix} 0 & a & 0 \\ 0 & d & a \\ 0 & 0 & 2d \end{bmatrix},$$

where $\alpha, d, a \in \mathbb{C}$ and $a \neq 0$. In this case, $T_s(A)$ has two minimal reducing subspaces H_1 and H_2 whose dimensions are 5 and 1, respectively.

(ii) *$T_{as}(A)$ is always irreducible.*

In this paper we resolve Problem 1.2 for W_s and W_{as} when A is an arbitrary 3×3 complex matrix by proving the following two theorems. For complex numbers a, b, c, and δ, let

$$J(\delta, a, b, c) = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & \delta \end{bmatrix}.$$

Theorem 1.4. *Let A be a 3×3 irreducible matrix. Assume that A is not invertible. Then*

(i) *$W_s(A)$ is reducible. Specifically, $H_s = H_1 \oplus H_2$, where H_1 and H_2 are reducing subspaces for $W_s(A)$ whose dimensions are 5 and 1, respectively.*

(ii) *$W_s(A)|H_1$ is reducible if and only if either $A \cong J(0, a, 0, c)$ or $A \cong J(\delta, a, 0, c)$ with $\delta \neq 0$ and $|a|^2 = |c|^2 + |\delta|^2$. In this case, $H_1 = K_1 \oplus K_2$, where K_1 and K_2 are minimal reducing subspaces for $W_s(A)|H_1$ whose dimensions are 3 and 2, respectively.*

(iii) *$W_{as}(A)$ is reducible. In this case, $H_{as} = K_1 \oplus K_2$, where K_1 and K_2 are minimal reducing subspaces for $W_{as}(A)$ whose dimensions are 2 and 1, respectively.*

When A is invertible, the results for $W_s(A)$ and $W_{as}(A)$ are in agreement with $T_s(A)$ and $T_{as}(A)$.

Theorem 1.5. *Let A be a 3×3 irreducible matrix. Assume that A is invertible and $\sigma(A) \neq \{\lambda, \lambda\omega, \lambda\omega^2\}$, where $\lambda \in \mathbb{C}$ and $\omega = e^{2\pi i/3}$. Then*

(i) *$W_s(A)$ is reducible if and only if for some nonzero numbers α and a, either*

$$A \cong \alpha \begin{bmatrix} 1 & a(1-\lambda) & a^2(1-\lambda)^2/2 \\ 0 & \lambda & a\lambda(1-\lambda) \\ 0 & 0 & \lambda^2 \end{bmatrix} \text{ for } \lambda \neq 1 \quad \text{or} \quad A \cong \alpha \begin{bmatrix} 1 & 2a & 2a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, $H_s = H_1 \oplus H_2$, where H_1 and H_2 are minimal reducing subspaces for $W_s(A)$ whose dimensions are 5 and 1, respectively.

(ii) *$W_{as}(A)$ is irreducible.*

Here is the outline of the paper. In Section 2, we establish the matrix representation of $W_s(A)$ and $W_{as}(A)$, and observe several lemmas. Section 3 is devoted to the proof of Theorem 1.4. Section 4 is devoted to the proof of Theorem 1.5.

2. Preliminaries

Note that $W_s(A)$ (resp. $W_{as}(A)$) is irreducible if and only if $W_s(U^*AU)$ (resp. $W_{as}(U^*AU)$) is irreducible, when U is unitary. Hence, by Schur's unitary triangularization, we can assume that A is an upper triangular irreducible matrix. If $\alpha \neq 0$, then $W_s(\alpha A) = \alpha^2 W_s(A)$, and so $W_s(A)$ is irreducible if and only if $W_s(\alpha A)$ is irreducible. This allows us to assume that one of the nonzero eigenvalues of A is 1, if it exists. We introduce some notation. Let

$$A = \begin{bmatrix} \beta & a & b \\ 0 & \gamma & c \\ 0 & 0 & \delta \end{bmatrix}, \quad W = W(A) = A \otimes A \cong \begin{bmatrix} W_s & 0 \\ 0 & W_{as} \end{bmatrix}, \quad W_s = W_s(A), \quad W_{as} = W_{as}(A),$$

$$f_1 = e_1 \otimes e_1, \quad f_2 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1), \quad f_3 = \frac{1}{\sqrt{2}}(e_1 \otimes e_3 + e_3 \otimes e_1),$$

$$f_4 = e_2 \otimes e_2, \quad f_5 = \frac{1}{\sqrt{2}}(e_2 \otimes e_3 + e_3 \otimes e_2), \quad f_6 = e_3 \otimes e_3,$$

$$g_1 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1), \quad g_2 = \frac{1}{\sqrt{2}}(e_1 \otimes e_3 - e_3 \otimes e_1), \quad g_3 = \frac{1}{\sqrt{2}}(e_2 \otimes e_3 - e_3 \otimes e_2).$$

Then $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ and $\{g_1, g_2, g_3\}$ are orthonormal bases for H_s and H_{as}, respectively. By direct computation, we have the following matrix representations of W_s and W_{as} under these bases.

Lemma 2.1. *With respect to the orthonormal bases $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ and $\{g_1, g_2, g_3\}$, we have*

$$W_s = \begin{bmatrix} \beta^2 & \sqrt{2}\beta a & \sqrt{2}\beta b & a^2 & \sqrt{2}ab & b^2 \\ 0 & \beta\gamma & \beta c & \sqrt{2}\gamma a & ac+\gamma b & \sqrt{2}bc \\ 0 & 0 & \beta\delta & 0 & \delta a & \sqrt{2}\delta b \\ 0 & 0 & 0 & \gamma^2 & \sqrt{2}\gamma c & c^2 \\ 0 & 0 & 0 & 0 & \gamma\delta & \sqrt{2}\delta c \\ 0 & 0 & 0 & 0 & 0 & \delta^2 \end{bmatrix} \quad \text{and} \quad W_{as} = \begin{bmatrix} \beta\gamma & \beta c & ac-\gamma b \\ 0 & \beta\delta & \delta a \\ 0 & 0 & \gamma\delta \end{bmatrix}.$$

Proof. The proof is a routine computation. For example,

$$\begin{aligned} W(e_1 \otimes e_2 \pm e_2 \otimes e_1) &= Ae_1 \otimes Ae_2 \pm Ae_2 \otimes Ae_1 \\ &= (\beta e_1) \otimes (ae_1 + \gamma e_2) \pm (ae_1 + \gamma e_2) \otimes (\beta e_1) \\ &= \beta a(e_1 \otimes e_1 \pm e_1 \otimes e_1) + \beta\gamma(e_1 \otimes e_2 \pm e_2 \otimes e_1), \end{aligned}$$

and so $W_s f_2 = \sqrt{2}\beta a f_1 + \beta\gamma f_2$ and $W_{as} g_1 = \beta\gamma g_1$. We omit the remaining computation. □

The following simple observation is the key lemma for the main theorems.

Lemma 2.2. *Suppose that B is reducible and $B = B_1 \oplus B_2$ on $H_1 \oplus H_2$. If λ is an eigenvalue of B, and if the eigenspace* $\ker(B - \lambda I)$ *of B corresponding to λ is not orthogonal to H_1, then λ is an eigenvalue of B_1 and* $\ker(B - \lambda I) \cap H_1 \neq \{0\}$. *In particular, if* $\ker(B - \lambda I) \not\perp H_1$ *and* $\dim \ker(B - \lambda I) = 1$, *then* $\ker(B - \lambda I) \subseteq H_1$.

Proof. Since both H_1 and H_2 are invariant for B, it follows that

$$\begin{aligned}\ker(B - \lambda I) &= [\ker(B - \lambda I) \cap H_1] \oplus [\ker(B - \lambda I) \cap H_2] \\ &= [\ker(B_1 - \lambda I) \cap H_1] \oplus [\ker(B_2 - \lambda I) \cap H_2].\end{aligned}$$

Thus if $\ker(B - \lambda I) \not\perp H_1$, then $\ker(B - \lambda I) \not\subseteq H_2$, and hence $\ker(B - \lambda I) \cap H_1 \neq \{0\}$ and $\lambda \in \sigma(B_1)$. □

Since we are dealing with a linear transformation acting on H_s and $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ is an orthonormal basis for H_s, we will denote a vector $v = \sum_{i=1}^{6} x_i f_i$ in H_s by $(x_1, \ldots, x_6)$. In other words, we will directly work with the matrix represented by $T_s(A)$. For example, when we say e_1 is in $\ker W_s$, it actually means f_1 is in $\ker W_s$.

We divide the proof of Main Theorems into three big cases according to whether A has one or two, or three distinct eigenvalues. In each big case we further divide the proof into several small cases. We have spent much time to consolidate and unify different cases, but we still have a number of cases to discuss to ensure the completeness and accuracy of our results. The following simple observation will be used repeatedly, sometimes without explicit mentioning. Let $\sigma(B)$ denote the set of (distinct) eigenvalues of B. For several subspaces $H_1, \ldots, H_k$ of H, we denote by $\bigvee_{i=1}^{k} H_i$ the smallest subspace of H containing all H_i's. An alternative notation is $\bigvee_{i=1}^{k} H_i = H_1 + H_2 + \cdots + H_k$.

We record, without proof, the following obvious characterization of one-dimensional reducing subspaces.

Lemma 2.3. *Let v be a nonzero vector in H. Then* $\mathrm{Span}\{v\}$ *is a reducing subspace of B if and only if there exists $\lambda \in \sigma(B)$ such that*

$$Bv = \lambda v \quad \text{and} \quad B^* v = \overline{\lambda} v.$$

In other words, there is a one-dimensional reducing subspace for B if and only if B and B^* have a common eigenvector.

We also need the following lemma:

Lemma 2.4. *Let*

$$A = \begin{bmatrix} \beta & a & b \\ 0 & \gamma & c \\ 0 & 0 & \delta \end{bmatrix}.$$

Then the following statements hold.

(i) *If A has three distinct eigenvalues, then A is reducible if and only if two of a, b, c are zero.*
(iia) *If $\beta = \gamma \neq \delta$, then A is reducible if and only if $a = 0$ or $b = c = 0$.*
(iib) *If $\beta \neq \gamma = \delta$, then A is reducible if and only if $c = 0$ or $a = b = 0$.*
(iic) *If $\beta = \delta \neq \gamma$, then A is reducible if and only if $(\gamma - \beta)b = ac$ or $a = c = 0$.*
(iii) *If A has one distinct eigenvalue, then A is reducible if and only if $ac = 0$.*

Proof. The proof is a routine computation, and we omit the proof. (For the detail of the proof, see [1].) □

3. Proof of Theorem 1.4

Suppose that A is a 3×3 irreducible matrix which is not invertible. By Schur's unitary triangularization, we may assume that

$$A = \begin{bmatrix} 0 & a & b \\ 0 & \gamma & c \\ 0 & 0 & \delta \end{bmatrix}.$$

We first prove the statement (iii) of Theorem 1.4: W_{as} is reducible and $H_{as} = K_1 \oplus K_2$, where K_1 and K_2 are minimal reducing subspaces for W_{as} whose dimensions are 2 and 1, respectively.

Proof. By Lemma 2.1,

$$W_{as} = \begin{bmatrix} 0 & 0 & ac-\gamma b \\ 0 & 0 & \delta a \\ 0 & 0 & \gamma\delta \end{bmatrix}.$$

It follows from Lemma 2.4 that W_{as} is reducible. Hence $H_{as} = K_1 \oplus K_2$, where K_1 and K_2 are reducing subspaces for W_{as} with $\dim K_1 = 2$ and $\dim K_2 = 1$. Assume that K_2 is not a minimal reducing subspace for W_{as}. Then W_{as} is diagonalizable, and so it is normal, i.e., $W_{as}^* W_{as} = W_{as} W_{as}^*$. By computation, we obtain $ac - \gamma b = \delta a = \gamma\delta = 0$. By using Lemma 2.4, it is easy to check that A is reducible, which is a contradiction. Hence K_1 is a minimal reducing subspace for W_{as}. This proves Theorem 1.4(iii). □

We will divide the proof of Theorem 1.4(i) and (ii) into three cases according the number of distinct eigenvalues of A. By scaling, we can assume that one of the nonzero eigenvalue of A is 1. Then we will discuss four cases

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \delta \end{bmatrix} \text{ with } \delta \neq 0, 1, \quad \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

We first deal with the case when A has three distinct eigenvalues.

Case 3.1. *Suppose that*

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \delta \end{bmatrix} \textit{ is irreducible with } \delta \neq 0, 1.$$

Then $H_s = H_1 \oplus H_2$, where H_1 and H_2 are minimal reducing subspaces for W_s with $\dim H_1 = 5$ *and* $\dim H_2 = 1$.

Proof. By Lemma 2.1, we have

$$W_s = \begin{bmatrix} 0 & 0 & 0 & a^2 & \sqrt{2}ab & b^2 \\ 0 & 0 & 0 & \sqrt{2}a & ac+b & \sqrt{2}bc \\ 0 & 0 & 0 & 0 & \delta a & \sqrt{2}\delta b \\ 0 & 0 & 0 & 1 & \sqrt{2}c & c^2 \\ 0 & 0 & 0 & 0 & \delta & \sqrt{2}\delta c \\ 0 & 0 & 0 & 0 & 0 & \delta^2 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{a}^2 & \sqrt{2}\bar{a} & 0 & 1 & 0 & 0 \\ \sqrt{2}\bar{a}\bar{b} & \overline{ac}+\bar{b} & \overline{\delta a} & \sqrt{2}\bar{c} & \bar{\delta} & 0 \\ \bar{b}^2 & \sqrt{2}\overline{bc} & \sqrt{2}\overline{\delta b} & \bar{c}^2 & \sqrt{2}\overline{\delta c} & \bar{\delta}^2 \end{bmatrix}.$$

Since $\ker W_s = \operatorname{Span}\{e_1, e_2, e_3\}$,

$$\ker W_s \cap \ker W^* = \operatorname{Span}\{v\}, \quad \text{where } v = \left(\sqrt{2}, -\bar{a}, \frac{\overline{ac}-\bar{b}}{\bar{\delta}}, 0, 0, 0\right).$$

Therefore $W_s = W_1 \oplus W_2$, where $W_1 = W_s|\mathrm{Span}\{v\}^{\perp}$ and $W_2 = W_s|\mathrm{Span}\{v\}$. We will prove that W_1 is irreducible by a contradiction. Assume $W_1 = W_3 \oplus W_4$ on $H_3 \oplus H_4$ where $\dim H_i \geq 2$ for $i = 3, 4$, since W_1 and W_1^* have no common eigenvector anymore. Since A is irreducible, one of the following holds.

(i) $ac \neq 0$, (ii) $a = 0$ and $bc \neq 0$, (iii) $c = 0$ and $ab \neq 0$.

Case 1: $ac \neq 0$. Then

$$\ker(W_s^* - \overline{\delta}^2 I) = \mathrm{Span}\{e_6\}, \quad \ker(W_s^* - \overline{\delta} I) = \mathrm{Span}\left\{\left(0,0,0,0,1,\frac{\sqrt{2}\overline{c}}{1-\overline{\delta}}\right)\right\},$$
$$\ker(W_s^* - I) = \mathrm{Span}\left\{\left(0,0,0,1,\frac{\sqrt{2}\overline{c}}{1-\overline{\delta}},\frac{\overline{c}^2}{(1-\overline{\delta})^2}\right)\right\}.$$

Without loss of generality, assume

$$\ker(W_s^* - \overline{\delta}^2 I) = \mathrm{Span}\{e_6\} \subseteq H_3. \tag{1}$$

Since $c \neq 0$, $\ker(W_s^* - \overline{\delta} I)$ is not orthogonal to $\ker(W_s^* - \overline{\delta}^2 I)$, and $\ker(W_s^* - I)$ is not orthogonal to $\ker(W_s^* - \delta^2 I)$. Therefore, by (1) and Lemma 2.2,

$$\ker(W_s^* - \overline{\lambda}^2 I) + \ker(W_s^* - \overline{\lambda} I) + \ker(W_s^* - I) \subseteq H_3, \text{ and } \mathrm{Span}\{e_4, e_5, e_6\} \subseteq H_3. \tag{2}$$

Since H_3 is reducing for W_s, so $W_s e_4 = (a^2, \sqrt{2}a, 0, 1, 0, 0, 0) \in H_4$. Since $a \neq 0$, it is easy to see that

$$\dim H_3 \geq \dim \mathrm{Span}\{e_4, e_5, e_6, W_s e_4\} = 4,$$

which is a contradiction to $\dim H_4 \geq 2$.

Case 2: $a = 0$ and $bc \neq 0$. As in the previous case, (2) still holds since $c \neq 0$. Since H_3 is reducing for W_s, we have $W_s e_5 = (0, b, 0, \sqrt{2}c, \lambda, 0) \in H_3$. Since $b \neq 0$, it is easy to see that

$$\dim H_3 \geq \dim \mathrm{Span}\{e_4, e_5, e_6, W_s e_5\} = 4,$$

which is a contradiction to $\dim H_4 \geq 2$.

Case 3: $ab \neq 0$ and $c = 0$. Then

$$W_s = \begin{bmatrix} 0 & 0 & 0 & a^2 & \sqrt{2}ab & b^2 \\ 0 & 0 & 0 & \sqrt{2}a & b & 0 \\ 0 & 0 & 0 & 0 & \delta a & \sqrt{2}\delta b \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta^2 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{a}^2 & \sqrt{2}\overline{a} & 0 & 1 & 0 & 0 \\ \sqrt{2}\overline{a}\overline{b} & \overline{b} & \overline{\delta}\overline{a} & 0 & \overline{\delta} & 0 \\ \overline{b}^2 & 0 & \sqrt{2}\overline{\delta}\overline{b} & 0 & 0 & \overline{\delta}^2 \end{bmatrix}.$$

By a direct computation,

$$\ker(W_s^* - I) = \mathrm{Span}\{e_4\}, \quad \ker(W_s^* - \overline{\delta}^2 I) = \mathrm{Span}\{e_6\}$$
$$\ker(W_s - I) = \mathrm{Span}\left\{(a^2, \sqrt{2}a, 0, 1, 0, 0)\right\},$$
$$\ker(W_s - \delta^2 I) = \mathrm{Span}\left\{\left(\frac{b^2}{\delta^2}, 0, \frac{\sqrt{2}b}{\delta}, 0, 0, 1\right)\right\}.$$

Without loss of generality, assume

$$\ker(W_s - I) \subseteq H_3. \tag{3}$$

Since $ab \neq 0$, $\ker(W_s - \lambda^2 I)$ is not orthogonal to $\ker(W_s - I)$. Therefore, by (3) and Lemma 2.2,

$$\ker(W_s - I) + \ker(W_s - \lambda^2 I) \subseteq H_3, \text{ and } \{1, \lambda^2\} \subseteq \sigma(W_3).$$

Hence

$$\ker(W_s^* - I) + \ker(W_s^* - \overline{\lambda}^2 I) + \ker(W_s - I) + \ker(W_s - \lambda^2 I) \subseteq H_3.$$

Since $ab \neq 0$, it is easy to see the subspace on the left side of the above relation has dimension 4. Hence $\dim H_3 \geq 4$, which is a contradiction to $\dim(H_2) \geq 2$.

We conclude that W_1 is irreducible, and the proof of Case 3.1 is complete. □

We next disscuss the case when A is not invertible and A has two distinct eigenvalues. The proofs in this case are more involved. By scaling we need to discuss two cases:

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Case 3.2. *Suppose that*

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} \text{ is irreucible.}$$

Then $H_s = H_1 \oplus H_2$, where H_1 and H_2 are reducing subspaces for W_s with $\dim H_1 = 5$ and $\dim H_2 = 1$. Moreover, $W_s|H_1$ is reducible if and only if $b = 0$ and $|a|^2 = |c|^2 + 1$, in which case, $H_1 = H_3 \oplus H_4$, where $\dim H_3 = 3$, $\dim H_4 = 2$, and both H_3 and H_4 are minimal reducing subspaces for $W_s|H_1$.

Proof. By Lemma 2.1,

$$W_s = \begin{bmatrix} 0 & 0 & 0 & a^2 & \sqrt{2}ab & b^2 \\ 0 & 0 & 0 & 0 & ac & \sqrt{2}bc \\ 0 & 0 & 0 & 0 & a & \sqrt{2}b \\ 0 & 0 & 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}c \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{a}^2 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a}\overline{b} & \overline{a}\overline{c} & \overline{a} & 0 & 0 & 0 \\ \overline{b}^2 & \sqrt{2}\overline{b}\overline{c} & \sqrt{2}\overline{b} & \overline{c}^2 & \sqrt{2}\overline{c} & 1 \end{bmatrix}.$$

Since A is irreducible, we have $a \neq 0$. Hence $\ker W_s = \mathrm{Span}\{e_1, e_2, e_3\}$ and

$$v := (0, 1, -\overline{c}, 0, 0, 0) \in \ker W_s^* \cap \ker W_s$$

is a common eigenvector. Therefore $W_s = W_1 \oplus W_2$, where $W_1 = W_s|\mathrm{Span}\{v\}^\perp$ and $W_2 = W_s|\mathrm{Span}\,\{v\}$. Since A is irreducible, we have two cases:

(i) $ab \neq 0$, (ii) $b = 0$ and $ac \neq 0$.

Case 1: $a \neq 0$ and $b \neq 0$. We will prove that W_1 is irreducible by a contradiction. Assume $W_1 = W_3 \oplus W_4$ on $H_3 \oplus H_4$ where $\dim H_i \geq 2$ for $i = 3, 4$, since W_1 and W_1^* have no common eigenvector anymore. Note that

$$\ker(W_s^* - I) = \mathrm{Span}\{e_6\}, \quad \ker W_s^* \cap \mathrm{Span}\{v\}^\perp = \mathrm{Span}\{v_1, v_2\},$$

where $v_1 = (0, 0, 0, \sqrt{2}, -\overline{c}, 0)$ and $v_2 = (0, 0, 0, c, \sqrt{2}, -\overline{c}(|c|^2 + 2))$. Without loss of generality, assume

$$\ker(W_s^* - I) = \mathrm{Span}\{e_6\} \subseteq H_3.$$

Then $\sigma(W_3) = \{0,1\}$ and $\sigma(W_4) = \{0\}$. Note that v_1 is a vector in $\ker W_s^* \cap \operatorname{Span}\{v\}^\perp$ that is orthgonal to $\ker(W_s^* - I)$. Hence $v_1 \in H_4$. It follows that $v_2 \in H_3$. Let

$$w_2 = v_2 + \bar{c}(|c|^2 + 2)e_6 = (0,0,0,c,\sqrt{2},0) \in H_3.$$

Then

$$w_3 = \frac{W_s e_6 - c w_2 - e_6}{b} = (b, \sqrt{2}c, \sqrt{2}, 0,0,0) \in H_3, \quad w_4 = \frac{W_s w_2}{a} = (ac + 2b, \sqrt{2}c, \sqrt{2}, 0,0,0) \in H_3.$$

Since $\operatorname{Span}\{e_6, w_2, w_3, w_4\} \subseteq H_3$, $\{e_6, w_2, w_3, w_4\}$ is linearly dependent (otherwise $\dim H_3 \geq 4$, a contradiction). It follows that $ac + b = 0$. But then $W_s^* w_3 = \bar{a}(0,0,0,-c|a|^2, \sqrt{2}(1 + |b|^2 + |c|^2), \star) \in H_3$ and $\{e_6, w_2, w_3, W_s^* w_3\}$ is linearly independent. Thus $\dim H_3 \geq 4$, which is a contradiction.

Case 2: $b = 0$ and $ac \neq 0$. We will prove that W_1 is reducible if and only if $|a|^2 = |c|^2 + 1$. Assume $W_1 = W_3 \oplus W_4$ on $H_3 \oplus H_4$ where $\dim H_i \geq 2$ for $i = 3,4$. Assume that $\ker(W_s^* - I) = \operatorname{Span}\{e_6\} \subseteq H_3$. Then

$$\begin{aligned} v_1 &= \frac{1}{c}(W_s e_6 - e_6) = (0,0,0,c,\sqrt{2},0) \in H_3, \\ v_2 &= \frac{1}{a} W_s v_1 = (ac, \sqrt{2}c, \sqrt{2}, 0,0,0) \in H_3, \\ v_3 &= \frac{1}{\bar{a}} W_s^* v_2 = (0,0,0,|a|^2 c, \sqrt{2}(1 + |c|^2), 0) \in H_3. \end{aligned}$$

If $|a|^2 \neq 1 + |c|^2$, then $\{e_6, v_1, v_2, v_3\}$ is linearly independent, and so $\dim H_3 \geq 4$, which is a contradiction. If $|a|^2 = 1 + |c|^2$, then

$$\begin{aligned} H_3 &= \operatorname{Span}\{e_6, v_1, v_2\}, \\ H_4 &= \operatorname{Span}\Big\{(0,0,0,\sqrt{2}, -\bar{c}, 0, u_2), \Big(-\frac{\sqrt{2}(1+|c|^2)}{\overline{ac}}, c, 1, 0,0,0\Big)\Big\}. \end{aligned}$$

Similarly we can check that H_3 and H_4 are minimal reducing subspaces. We omit the details. □

It is surprising that the proof of the next case is easy even though the A in this case and the A in the above case are related in that they both have two distinct eigenvalues. This indicates that for W_s, the multiplicity of the zero eigenvalue also plays an important role.

Case 3.3. *Suppose that*

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \text{ is irreducible.}$$

Then $H_s = H_1 \oplus H_2$, where H_1 and H_2 are minimal reducing subspaces for W_s with $\dim H_1 = 5$ and $\dim H_2 = 1$.

Proof. Note that A is irreducible if and only if $c \neq 0$ and either $a \neq 0$ or $b \neq 0$. Also, by Lemma 2.1,

$$W_s = \begin{bmatrix} 0 & 0 & 0 & a^2 & \sqrt{2}ab & b^2 \\ 0 & 0 & 0 & \sqrt{2}a & ac+b & \sqrt{2}bc \\ 0 & 0 & 0 & 0 & a & \sqrt{2}b \\ 0 & 0 & 0 & 1 & \sqrt{2}c & c^2 \\ 0 & 0 & 0 & 0 & 1 & \sqrt{2}c \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{a}^2 & \sqrt{2}\bar{a} & 0 & 1 & 0 & 0 \\ \sqrt{2}\bar{a}\bar{b} & \overline{ac} + \bar{b} & \bar{a} & \sqrt{2}\bar{c} & 1 & 0 \\ \bar{b}^2 & \sqrt{2}\overline{bc} & \sqrt{2}\bar{b} & \bar{c}^2 & \sqrt{2}\bar{c} & 1 \end{bmatrix}.$$

Since $\ker W_s = \operatorname{Span}\{e_1, e_2, e_3\}$, it can be check that

$$v := \Big(1, -\frac{\bar{a}}{\sqrt{2}}, \frac{\overline{ac} - \bar{b}}{\sqrt{2}}, 0,0,0\Big) \in \ker W_s \cap \ker W_s^*$$

is the only common eigenvector (up to scalar). Therefore $W_s = W_1 \oplus W_2$ where $W_1 = W_s|\operatorname{Span}\{v\}^\perp$ and $W_2 = W_s|\operatorname{Span}\{v\}$. We will show W_1 is irreducible. Assume $W_1 = W_3 \oplus W_4$ on $H_3 \oplus H_4$ where $\dim H_i \geq 2$ for $i = 3, 4$. Without loss of generality, let

$$\ker(W_1 - I) = \ker(W_s - I) = \operatorname{Span}\{u\} \subseteq H_3, \text{ where } u = (a^2, \sqrt{2}a, 0, 1, 0, 0).$$

Since $\ker(W_1 - I)$ is one-dimensional, $\sigma(W_3) = \{1\}$ and $\sigma(W_4) = \{0\}$. Hence $u \perp \ker W_1$, where

$$\ker W_1 = \ker(W_s) \cap \operatorname{Span}\{v\}^\perp = \operatorname{Span}\left\{\left(\frac{a}{\sqrt{2}}, 1, 0, 0, 0, 0\right), \left(-\frac{ac - b}{\sqrt{2}}, 0, 1, 0, 0, 0\right)\right\}.$$

Therefore

$$a^2\frac{\bar{a}}{\sqrt{2}} + \sqrt{2}a = 0.$$

Hence $a = 0$. It follows that $u = e_4 \in H_3$. Since $W_s^* e_4 = (0, 0, 0, 1, \sqrt{2}\bar{c}, \bar{c}^2) \in H_3$, we have $(0, 0, 0, 0, \sqrt{2}, \bar{c}) \in H_3$. Since $W_s^*(0, 0, 0, 0, \sqrt{2}, c) = (0, 0, 0, 0, \sqrt{2}, 3\bar{c}) \in H_3$, we have $e_5, e_6 \in H_3$. Since $b \neq 0$, it is easy to see that $\{e_4, e_5, e_6, W_s e_6\}$ is linearly independent. Thus $\dim H_3 \geq 4$, which is a contradiction to $\dim H_4 \geq 2$. □

Finally, we deal with the case when A is irreucible, not invertible, and A has one distinct eigenvalue, i.e., $\sigma(A) = \{0\}$. By scaling, we can assume that $a = 1$.

Case 3.4. *Suppose that*

$$A = \begin{bmatrix} 0 & 1 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \text{ is irreducible with } c \neq 0.$$

Then $H_s = H_1 \oplus H_2$, where H_1 and H_2 are reducing subspaces for W_s with $\dim H_1 = 5$ and $\dim H_2 = 1$. Moreover, $W_s|H_1$ is reducible if and only if $b = 0$, in which case, $H_1 = H_3 \oplus H_4$, where $\dim H_3 = 3$, $\dim H_4 = 2$, and both H_3 and H_4 are minimal reducing subspaces for $W_s|H_1$.

Proof. By Lemma 2.1,

$$W_s = \begin{bmatrix} 0 & 0 & 0 & 1 & \sqrt{2}b & b^2 \\ 0 & 0 & 0 & 0 & c & \sqrt{2}bc \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\bar{b} & \bar{c} & 0 & 0 & 0 & 0 \\ \bar{b}^2 & \sqrt{2}\bar{b}\bar{c} & 0 & \bar{c}^2 & 0 & 0 \end{bmatrix}.$$

Since the third row and the third column of W_s are zero, e_3 is a common eigenvector of W_s and W_s^*. By an abuse of notation, $W_s = W_1 \oplus [0]$, where

$$W_1 = \begin{bmatrix} 0 & 0 & 1 & \sqrt{2}b & b^2 \\ 0 & 0 & 0 & c & \sqrt{2}bc \\ 0 & 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad W_1^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\bar{b} & \bar{c} & 0 & 0 & 0 \\ \bar{b}^2 & \sqrt{2}\bar{b}\bar{c} & \bar{c}^2 & 0 & 0 \end{bmatrix}.$$

Case 1: $b \neq 0$. We will show W_1 is irreducible by a contradiction. Assume $W_1 = W_3 \oplus W_4$ on $H_3 \oplus H_4$ where $\dim H_i \geq 2$ for $i = 3, 4$, since W_1 and W_1^* have no common eigenvector anymore:

$$\ker W_1 = \operatorname{Span}\{e_1, e_2\}, \quad \ker W_1^* = \operatorname{Span}\{e_4, e_5\}$$

Since $\sigma(W_3) = \sigma(W_4) = \{0\}$, by Lemma 2.2, $\ker W_1 \cap H_i$ and $\ker W_1^* \cap H_i$ are of dimension one for $i = 3, 4$. Without loss of generality,

$$H_3 \supseteq \mathrm{Span}\{v_1, v_2\}, \text{ where } v_1 = (1, \alpha, 0, 0, 0, 0) \text{ and } v_2 = (0, 0, 0, 0, \beta, \gamma)$$
$$H_4 \supseteq \mathrm{Span}\{u_1, u_2\}, \text{ where } u_1 = (-\overline{\alpha}, 1, 0, 0, 0, 0) \text{ and } u_2 = (0, 0, 0, 0, -\overline{\gamma}, \overline{\beta})$$

for some α and $(\beta, \gamma) \neq 0$. Note that $W_1^*(v_1) = (\star, \star, 1, \star, \star) \notin \mathrm{Span}\{v_1, v_2\}$, where $\star$ represents some quantity whose precise formula is not needed. Hence $\dim H_3 = 3$ and $\dim H_4 = 2$. We consider two cases according whether α is nonzero or not.

Case 1a: $\alpha = 0$. Note that $W_1(u_2) = (\star, \star, \overline{\beta}c^2, 0, 0) \in \mathrm{Span}\{u_1, u_2\}$ only when $\beta = 0$. But when $\beta = 0$, $W_1(u_2) = -\overline{\gamma}(\sqrt{2}b, c, 0, 0, 0)$, $W_1^* W_1(u_2) = -\overline{\gamma}(0, 0, \sqrt{2}b, \star, \star) \notin \mathrm{Span}\{u_1, u_2\}$ since $\overline{\gamma}b \neq 0$, contradicting $\dim H_4 = 2$.

Case 1b: $\alpha \neq 0$. Then $W_1^*(v_2) = (0, 0, -\overline{\alpha}, \star, \star) \notin \mathrm{Span}\{u_1, u_2\}$, again contradicting $\dim H_4 = 2$.

Case 2: $b = 0$. The desired result follows from the following computation.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|ccc} 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The proof of Case 3.4 is complete. □

4. Proof of Theorem 1.5

Suppose that A is a 3×3 irreducible matrix which is invertible. By Schur's unitary triangularization, we may assume that A is an upper triangular irreducible matrix. Thus

$$A = \begin{bmatrix} \beta & a & b \\ 0 & \gamma & c \\ 0 & 0 & \delta \end{bmatrix},$$

where $\beta\gamma\delta \neq 0$. We can easily check by using Lemma 2.4 that

$$W_{as} = \begin{bmatrix} \beta\gamma & \beta c & ac - \gamma b \\ 0 & \beta\delta & \delta a \\ 0 & 0 & \gamma\delta \end{bmatrix}$$

is irreducible. The remaining of this section is devoted to the proof of Theorem 1.5(i).

Since A is invertible, there exists a 3×3 matrix B such that $A = \exp B = \sum_{n=0}^{\infty} \frac{1}{n!} B^n$. It follows that

$$\begin{aligned} W(A) = A \otimes A &= \exp B \otimes \exp B = (\exp B \otimes I)(I \otimes \exp B) \\ &= \exp(B \otimes I) \exp(I \otimes B) = \exp(B \otimes I + I \otimes B) = \exp(T(B)). \end{aligned}$$

If $T(B)$ is reducible, then so is $\exp(T(B)) = W(A)$. By Theorem 1.3, $T_s(B)$ is reducible if and only if

$$B \cong \beta I + \begin{bmatrix} 0 & a & 0 \\ 0 & d & a \\ 0 & 0 & 2d \end{bmatrix},$$

where $\beta, d, a \in \mathbb{C}$ and $a \neq 0$. In the case $d = 0$,

$$A = \exp B \cong e^{\beta} \begin{bmatrix} 1 & a & a^2/2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}.$$

In the case $d \neq 0$,

$$A = \exp B \cong e^{\beta} \begin{bmatrix} 1 & \frac{a}{d}(\lambda - 1) & (\frac{a}{d})^2(\lambda - 1)^2/2 \\ 0 & \lambda & \frac{a}{d}(\lambda - 1)\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}, \text{ where } \lambda = e^d \neq 1.$$

From this we can guess the condition for the reducibility of $W_s(A)$.

Before starting the proof of Theorem 1.5(i), we record the following lemma.

Lemma 4.1. *Let B be an $n \times n$ matrix with $n \geq 2$ such that $\sigma(B) = \{\lambda\}$. Then there exist nonzero $v \in \ker(B - \lambda I)$ and nonzero $u \in \ker(B^* - \overline{\lambda} I)$ such that $v \perp u$.*

Proof. By Schur's unitary triangularization, there exists a unitary matrix U such that $U^*(B - \lambda I)U$ is a strictly upper triangular $n \times n$ matrix. It is easy to see that $U^*(B - \lambda I)Ue_1 = 0$ and $U^*(B^* - \overline{\lambda} I)Ue_n = 0$. Then $v = Ue_1$ and $u = Ue_n$ satisfy the desired properties. □

Let us now prove Theorem 1.5(i). We start with the case when A has one distinct nonzero eigenvalue, i.e., $\sigma(A) = \{\lambda\}$, where $\lambda \neq 0$. As in the proof of Theorem 1.4, we may assume that $\lambda = 1$.

Case 4.2. *Suppose that*

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \text{ is irreducible.}$$

Then $W_s(A)$ is reducible if and only if $|a| = |c|$ and $b = ac/2$. In this case, $H_s = H_1 \oplus H_2$, where $\dim H_1 = 5$, $\dim H_2 = 1$, and both H_1 and H_2 are minimal reducing subspaces for $W_s(A)$.

Proof. Since A is irreducible, it follows from Lemma 2.4 that $ac \neq 0$. By Lemma 2.1,

$$W_s = \begin{bmatrix} 1 & \sqrt{2}a & \sqrt{2}b & a^2 & \sqrt{2}ab & b^2 \\ 0 & 1 & c & \sqrt{2}a & ac + b & \sqrt{2}bc \\ 0 & 0 & 1 & 0 & a & \sqrt{2}b \\ 0 & 0 & 0 & 1 & \sqrt{2}c & c^2 \\ 0 & 0 & 0 & 0 & 1 & \sqrt{2}c \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & \overline{c} & 1 & 0 & 0 & 0 \\ \overline{a}^2 & \sqrt{2}\overline{a} & 0 & 1 & 0 & 0 \\ \sqrt{2}\overline{a}\overline{b} & \overline{b} + \overline{ac} & \overline{a} & \sqrt{2}\overline{c} & 1 & 0 \\ \overline{b}^2 & \sqrt{2}\overline{b}\overline{c} & \sqrt{2}\overline{b} & \overline{c}^2 & \sqrt{2}\overline{c} & 1 \end{bmatrix}.$$

Note that

$$\ker(W_s - I) = \text{Span}\left\{(1, 0, 0, 0, 0, 0), \left(0, \frac{2b - ac}{\sqrt{2}c}, -\frac{\sqrt{2}a}{c}, 1, 0, 0\right)\right\},$$

$$\ker(W_s^* - I) = \text{Span}\left\{\left(0, 0, -\frac{\sqrt{2}\overline{c}}{\overline{a}}, 1, \frac{2\overline{b} - \overline{ac}}{\sqrt{2}\overline{a}}, 0\right), (0, 0, 0, 0, 0, 1)\right\}.$$

Note also that W_s and W_s^* have a common eigenvector if and only if

$$\frac{\sqrt{2}b}{c} - \frac{a}{\sqrt{2}} = 0, \quad -\frac{\sqrt{2}a}{c} = -\frac{\sqrt{2}\overline{c}}{\overline{a}}, \quad \text{and} \quad 0 = \frac{\sqrt{2}\overline{b}}{\overline{a}} - \frac{\overline{c}}{\sqrt{2}},$$

if and only if $|a| = |c|$ and $b = ac/2$.

Case 1: $|a| \neq |c|$ or $b \neq ac/2$. We will show that W_s is irreducible. Assume to the contrary that $W_s = W_1 \oplus W_2$ on $H_1 \oplus H_2$ where $\dim H_i \geq 2$ for $i = 1, 2$. Note that

$$\sigma(W_1) = \sigma(W_2) = \{1\}.$$

By Lemma 4.1, $\ker(W_s - I) \perp \ker(W_s^* - I)$ which is a contradiction since

$$\Big(-\frac{\sqrt{2}a}{c}\Big)\Big(-\frac{\sqrt{2}c}{a}\Big) + 1 \cdot 1 = 3 \neq 0.$$

Case 2: $|a| = |c|$ and $b = ac/2$. In this case, let

$$v = \Big(0, 0, -\frac{\sqrt{2}a}{c}, 1, 0, 0\Big).$$

Then $\ker(W_s - I) \cap \ker(W_s^* - I) = \mathrm{Span}\{v\}$, and

$$\ker(W_s - I) = \mathrm{Span}\{v, e_1\} \quad \text{and} \quad \ker(W_s^* - I) = \mathrm{Span}\{v, e_6\}. \tag{4}$$

Thus $W_s = W_1 \oplus W_2$, where $W_1 = W_s|\,\mathrm{Span}\,\{v\}^\perp$ and $W_2 = W_s|\,\mathrm{Span}\{v\}$. It follows from (4) that

$$\ker(W_1 - I) = \mathrm{Span}\{e_1\}.$$

Assume to the contrary that $W_1 = W_3 \oplus W_4$ on $H_3 \oplus H_4$ where $\dim H_i \geq 1$. Then $\sigma(W_3) = \sigma(W_4) = \{1\}$, and $\ker(W_1 - I) = \ker(W_3 - I) \oplus \ker(W_4 - I)$. Hence $\dim \ker(W_1 - I) \geq 2$, which is a contradiction. Therefore, W_1 is irreducible. □

Next we look at the case when A has two distinct nonzero eigenvalues. We may assume that the eigenvalue of multiplicity 2 is 1, and arrange the eigenvalues $\{1, 1, \lambda\}$ on the diagonal of A in any desired order.

Case 4.3. *Suppose that*

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{bmatrix} \text{ is irreducible with } \lambda \neq 0, 1.$$

Then W_s is irreducible.

Proof. Since A is irreducible, Lemma 2.4 implies that one of the following holds:

(i) $ac \neq 0$, (ii) $c = 0$ and $ab \neq 0$.

By Lemma 2.1,

$$W_s = \begin{bmatrix} 1 & \sqrt{2}a & \sqrt{2}b & a^2 & \sqrt{2}ab & b^2 \\ 0 & 1 & c & \sqrt{2}a & ac+b & \sqrt{2}bc \\ 0 & 0 & \lambda & 0 & \lambda a & \sqrt{2}\lambda b \\ 0 & 0 & 0 & 1 & \sqrt{2}c & c^2 \\ 0 & 0 & 0 & 0 & \lambda & \sqrt{2}\lambda c \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & c & \overline{\lambda} & 0 & 0 & 0 \\ \overline{a}^2 & \sqrt{2}\overline{a} & 0 & 1 & 0 & 0 \\ \sqrt{2}\overline{a}\overline{b} & \overline{b}+\overline{ac} & \overline{a}\overline{\lambda} & \sqrt{2}\overline{c} & \overline{\lambda} & 0 \\ \overline{b}^2 & \sqrt{2}\overline{b}\overline{c} & \sqrt{2}\overline{b}\overline{\lambda} & \overline{c}^2 & \sqrt{2}\overline{c}\overline{\lambda} & \overline{\lambda}^2 \end{bmatrix}.$$

It can be checked by direct computation that W_s and W_s^* have no common eigenvector. We will show that W_s is irreducible by a contradiction. There is a complication when $\lambda^2 = 1$, i.e., $\lambda = -1$, since in this case $\ker(W_s^* - \overline{\lambda}^2 I)$ $(= \ker(W_s^* - I))$ is of dimension 2. We find it cumbersome and difficult to unify the proofs of $\lambda^2 = 1$ case and $\lambda^2 \neq 1$ case. So we will prove these two cases separately. Assume $W_s = W_1 \oplus W_2$ on $H_1 \oplus H_2$ with $\dim H_i \geq 2$ for $i = 1, 2$.

Case 1: $\lambda^2 \neq 1$ and $ac \neq 0$. Note that

$$\ker(W_s^* - \overline{\lambda}^2 I) = \operatorname{Span}\{e_6\}, \quad \ker(W_s^* - \overline{\lambda} I) = \operatorname{Span}\left\{\left(0,0,0,0,1,\frac{\sqrt{2}\overline{c}}{1-\overline{\lambda}}\right)\right\},$$
$$\ker(W_s^* - I) = \operatorname{Span}\left\{\left(0,0,0,1,\frac{\sqrt{2}\overline{c}}{\overline{\lambda}-1},\star\right)\right\}.$$

Without loss of generality, assume

$$\ker(W_s^* - \overline{\lambda}^2 I) = \operatorname{Span}\{e_6\} \subseteq H_1.$$

Since $c \neq 0$, $\ker(W_s^* - \overline{\lambda} I)$ is not orthogonal to $\ker(W_s^* - \overline{\lambda}^2 I)$. By Lemma 2.2,

$$\ker(W_s^* - \overline{\lambda}^2 I) + \ker(W_s^* - \overline{\lambda} I) \subseteq H_1, \quad \text{or} \quad \operatorname{Span}\{e_5, e_6\} \subseteq H_1.$$

Again, since $c \neq 0$, $\ker(W_s^* - I)$ is not orthogonal to H_1. Hence $\sigma(W_1) = \{1, \lambda, \lambda^2\}$. But either 1 or λ is in $\sigma(W_2)$. It follows from Lemma 2.2 that either $\dim\ker(W_s^* - I) \geq 2$ or $\dim\ker(W_s^* - \overline{\lambda} I) \geq 2$, which is a contradiction.

Case 2: $\lambda^2 \neq 1$, $c = 0$, and $ab \neq 0$. We prove the result by a similar argument using eigenspaces of W_s. Note that

$$W_s = \begin{bmatrix} 1 & \sqrt{2}a & \sqrt{2}b & a^2 & \sqrt{2}ab & b^2 \\ 0 & 1 & 0 & \sqrt{2}a & b & 0 \\ 0 & 0 & \lambda & 0 & \lambda a & \sqrt{2}\lambda b \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & 0 & \overline{\lambda} & 0 & 0 & 0 \\ \overline{a}^2 & \sqrt{2}\overline{a} & 0 & 1 & 0 & 0 \\ \sqrt{2}\overline{a}\overline{b} & \overline{b} & \overline{a}\overline{\lambda} & 0 & \overline{\lambda} & 0 \\ \overline{b}^2 & 0 & \sqrt{2}\overline{b\lambda} & 0 & 0 & \overline{\lambda}^2 \end{bmatrix},$$

and, since $a \neq 0$, we have

$$\ker(W_s - I) = \operatorname{Span}\{e_1\}, \quad \ker(W_s - \lambda I) = \operatorname{Span}\left\{(-\sqrt{2}b, 0, 1-\lambda, 0, 0, 0)\right\},$$
$$\ker(W_s - \lambda^2 I) = \operatorname{Span}\left\{(\star, 0, -\sqrt{2}b, 0, 0, 1-\lambda)\right\}.$$

Without loss of generality, assume that

$$\ker(W_s - I) = \operatorname{Span}\{e_1\} \subseteq H_1.$$

Since $b \neq 0$, $\ker(W_s - \lambda I)$ is not orthogonal to $\ker(W_s - I)$. By Lemma 2.2,

$$\ker(W_s - I) + \ker(W_s - \lambda I) \subseteq H_1, \quad \text{or} \quad \operatorname{Span}\{e_1, e_3\} \subseteq H_1.$$

Since $b \neq 0$, $\ker(W_s - \lambda^2 I)$ is not orthogonal to H_1. Hence $\sigma(W_1) = \{1, \lambda, \lambda^2\}$. But either $1 \in \sigma(W_2)$ or $\lambda \in \sigma(W_2)$. It follows from Lemma 2.2 that either $\dim\ker(W_s - I) \geq 2$ or $\dim\ker(W_s - \lambda I) \geq 2$, which is a contradiction.

We next deal with the case $\lambda^2 = 1$, that is, $\lambda = -1$. In this case, both $\ker(W_s - I)$ and $\ker(W_s^* - I)$ are of dimension 2.

Case 3: $\lambda = -1$ and $ac \neq 0$. Then

$$\ker(W_s^* + I) = \operatorname{Span}\left\{\left(0,0,0,0,1,\frac{\sqrt{2}\overline{c}}{2}\right)\right\},$$

$$\ker(W_s^* - I) = \operatorname{Span}\{e_6, v\}, \text{ where } v := \left(0,0,0,1,\frac{\sqrt{2}\overline{c}}{2},0\right),$$

$$\ker(W_s - I) = \operatorname{Span}\{e_1, u\}, \text{ where } u := \left(\star,\star,\star,\star,-\frac{\sqrt{2}c}{2},1\right).$$

Without loss of generality, assume

$$\ker(W_s^* + I) \subseteq H_1.$$

Since $c \neq 0$, $\ker(W_s^* - I)$ is not orthogonal to $\ker(W_s^* + I)$. By Lemma 2.2, $\sigma(W_1) = \{-1, 1\}$ and $\sigma(W_2) = \{1\}$, and thus $\dim H_1 = 4$ and $\dim H_2 = 2$. Using appropriate linear combinations of e_6 and v, we write

$$\ker(W_s^* - I) = \mathrm{Span}\{u_1, u_2\}, \text{ where } u_1 = \Big(0, 0, 0, \frac{2c}{\overline{c}(|c|^2+2)}, \frac{\sqrt{2}c}{(|c|^2+2)}, 1\Big),\ u_2 = \Big(0, 0, 0, 1, \frac{\sqrt{2}\overline{c}}{2}, -\frac{\overline{c}}{c}\Big).$$

Note that $u_1 \perp u_2$ and $u_2 \perp \ker(W_s^* + I)$. Thus $u_2 \in H_2$ and $u_1 \in H_1$. We would like to do a similar decomposition for $\ker(W_s - I)$. Since the explicit form of u is complicated, we write

$$\ker(W_s - I) = [\ker(W_s - I) \cap H_1] \oplus [\ker(W_s^* - I) \cap H_2],$$

where

$$\ker(W_s - I) \cap H_1 = \mathrm{Span}\{a_1 e_1 + a_2 u\} \quad \text{and} \quad \ker(W_s - I) \cap H_2 = \mathrm{Span}\{b_1 e_1 + b_2 u\}$$

for some constants a_1, a_2, b_1, b_2. Now we have

$$H_2 = \mathrm{Span}\{u_2, b_1 e_1 + b_2 u\}.$$

Since $\sigma(W_2) = \{1\}$, $u_2 \in \ker(W_2^* - I)$, and $b_1 e_1 + b_2 u \in \ker(W_2 - I)$, Lemma 4.1 implies that $b_1 e_1 + b_2 u \perp u_2$. But $b_1 e_1 + b_2 u$ is orthogonal to u_1. Thus $b_1 e_1 + b_2 u \perp \ker(W_s^* - I)$ and $b_1 e_1 + b_2 u = (\star, \star, \star, \star, \star, b_2) \perp e_6$. This implies that $b_2 = 0$ and $e_1 \in H_2$. Since $a \neq 0$, the set $\{u_2, e_1, W_s^* e_1\}$ is linearly indenpdent, which is a contradiction to $\dim H_2 = 2$.

Case 4: $\lambda = -1$, $c = 0$, and $ab \neq 0$. Note that

$$W_s = \begin{bmatrix} 1 & \sqrt{2}a & \sqrt{2}b & a^2 & \sqrt{2}ab & b^2 \\ 0 & 1 & 0 & \sqrt{2}a & b & 0 \\ 0 & 0 & -1 & 0 & -a & -\sqrt{2}b \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & 0 & -1 & 0 & 0 & 0 \\ \overline{a}^2 & \sqrt{2}\overline{a} & 0 & 1 & 0 & 0 \\ \sqrt{2}\overline{a}\overline{b} & \overline{b} & -\overline{a} & 0 & -1 & 0 \\ \overline{b}^2 & 0 & -\sqrt{2}\overline{b} & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\ker(W_s - I) = \mathrm{Span}\Big\{e_1, (0, 0, \sqrt{2}b, 0, 0, -2)\Big\} \quad \ker(W_s^* - I) = \mathrm{Span}\{e_4, e_6\},$$
$$\ker(W_s + I) = \mathrm{Span}\Big\{(\sqrt{2}b, 0, -2, 0, 0, 0)\Big\}, \quad \ker(W_s^* + I) = \mathrm{Span}\{e_5\}.$$

Without loss of generality, assume

$$\ker(W_s^* + I) + \ker(W_s + I) \subseteq H_1.$$

Observe that $\ker(W_s - I)$ is not orthogonal to $\ker(W_s + I)$. By Lemma 2.2, $\sigma(W_1) = \{-1, 1\}$ and $\sigma(W_2) = \{1\}$. Thus $\dim H_1 = 4$ and $\dim H_2 = 2$. If a_1 and a_2 are constants and $a_1 e_1 + a_2(0, 0, \sqrt{2}b, 0, 0, -2) \perp \ker(W_s + I)$, then $a_1 \overline{b} = 2a_2 b$. Thus

$$H_2 = \mathrm{Span}\Big\{\Big(b, 0, \frac{\sqrt{2}}{2}|b|^2, 0, 0, -\overline{b}\Big), b_1 e_4 + b_2 e_6\Big\}$$

for some constants b_1, b_2. Then $-\overline{b_2} e_4 + \overline{b_1} e_6 \perp H_2$, which implies $b_1 = 0$. It follows that

$$H_2 = \mathrm{Span}\{u, e_6\}, \quad \text{where } u := (\sqrt{2}, 0, \overline{b}, 0, 0, 0).$$

Thus $W_s e_6 \in H_2$, but then $\{u, e_6, W_s e_6\}$ is a linearly independent subset of H_2, which is a contradiction.

Therefore W_s is irreducible, and the proof is complete. □

Next look at the case when A has three distinct nonzero eigenvalues β, γ, δ:

$$A = \begin{bmatrix} \beta & a & b \\ 0 & \gamma & c \\ 0 & 0 & \delta \end{bmatrix}, \quad W_s = \begin{bmatrix} \beta^2 & \sqrt{2}\beta a & \sqrt{2}\beta b & a^2 & \sqrt{2}ab & b^2 \\ 0 & \beta\gamma & \beta c & \sqrt{2}\gamma a & ac+\gamma b & \sqrt{2}bc \\ 0 & 0 & \beta\delta & 0 & \delta a & \sqrt{2}\delta b \\ 0 & 0 & 0 & \gamma^2 & \sqrt{2}\gamma c & c^2 \\ 0 & 0 & 0 & 0 & \gamma\delta & \sqrt{2}\delta c \\ 0 & 0 & 0 & 0 & 0 & \delta^2 \end{bmatrix}.$$

Then $\sigma(W_s) = \{\beta^2, \beta\gamma, \beta\delta, \gamma^2, \gamma\delta, \delta^2\}$. There are complications when W_s has an eigenvalue of multiplicity 2. Next we discuss when this happens. There are two choices that will reduce our algebra (sometimes greatly). First we may arrange $\{\beta, \gamma, \delta\}$ on the diagonal of A in any order desired. Second, we can scale one of $\{\beta, \gamma, \delta\}$ to be 1. Through these two choices, one of the following statements holds.

(i) $\sigma(W_s)$ consists of 6 distinct numbers; then we can assume $\beta = 1$.
(ii) $\sigma(W_s)$ consists of 5 distinct numbers; we can assume $\beta = 1$ and either $\gamma = -1$ or $\delta = \gamma^2$.
(iii) $\sigma(W_s)$ consists of 4 distinct numbers; we can assume $\beta = 1$, $\gamma = i$, and $\delta = -1$.
(iv) $\sigma(W_s)$ consists of 3 distinct numbers; we can assume $\{\beta, \gamma, \delta\} = \{1, \omega, \omega^2\}$, where $\omega = e^{2\pi i/3}$.

Note that if A is irreducible, then one of the following holds.

(i) $ac \neq 0$, (ii) $c = 0$ and $ab \neq 0$, (iii) $a = 0$ and $bc \neq 0$.

Case 4.4. *Suppose that $\sigma(W_s)$ consists of 6 distinct numbers. Assume that $\beta = 1$. Then W_s is irreducible.*

Proof. By Lemma 2.1,

$$W_s = \begin{bmatrix} 1 & \sqrt{2}a & \sqrt{2}b & a^2 & \sqrt{2}ab & b^2 \\ 0 & \gamma & c & \sqrt{2}\gamma a & ac+\gamma b & \sqrt{2}bc \\ 0 & 0 & \delta & 0 & \delta a & \sqrt{2}\delta b \\ 0 & 0 & 0 & \gamma^2 & \sqrt{2}\gamma c & c^2 \\ 0 & 0 & 0 & 0 & \gamma\delta & \sqrt{2}\delta c \\ 0 & 0 & 0 & 0 & 0 & \delta^2 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\bar{a} & \bar{\gamma} & 0 & 0 & 0 & 0 \\ \sqrt{2}\bar{b} & \bar{c} & \bar{\delta} & 0 & 0 & 0 \\ \bar{a}^2 & \sqrt{2}\overline{a\gamma} & 0 & \bar{\gamma}^2 & 0 & 0 \\ \sqrt{2}\bar{a}\bar{b} & \bar{b}\bar{\gamma} + \overline{ac} & \bar{a}\bar{\delta} & \sqrt{2}\overline{c\gamma} & \bar{\gamma}\bar{\delta} & 0 \\ \bar{b}^2 & \sqrt{2}\overline{bc} & \sqrt{2}\overline{b\delta} & \bar{c}^2 & \sqrt{2}\bar{c}\bar{\delta} & \bar{\delta}^2 \end{bmatrix}.$$

It can be checked that W_s and W_s^* have no common eigenvector. We will show that W_s is irreducible by a contradiction. Assume $W_s = W_1 \oplus W_2$ on $H_1 \oplus H_2$ with $\dim H_i \geq 2$ for $i = 1, 2$.

Case 1: $ac \neq 0$. Note that

$$\ker(W_s - I) = \mathrm{Span}\{e_1\}, \quad \ker(W_s - \gamma I) = \mathrm{Span}\left\{(\sqrt{2}a, \gamma - 1, 0, 0, 0, 0)\right\},$$
$$\ker(W_s - \delta I) = \mathrm{Span}\left\{(\star, c, \delta - \gamma, 0, 0, 0)\right\}.$$

Since $a \neq 0$, $\ker(W_s - \gamma I)$ is not orthogonal to $\ker(W_s - I)$, and since $c \neq 0$, $\ker(W_s - \delta I)$ is not orthogonal to $\ker(W_s - \gamma I)$. By Lemma 2.2, without loss of generality, we may assume that

$$\ker(W_s - I) + \ker(W_s - \gamma I) + \ker(W_s - \delta I) \subseteq H_1, \quad \text{or} \quad \mathrm{Span}\{e_1, e_2, e_3\} \subseteq H_1.$$

Now $W_s^* e_2, W_s^* e_3 \in H_1$. Since $\sqrt{2}a\gamma \neq 0$ and $a\delta \neq 0$, the dimension of H_1 is at least 5, which is a contradiction.

Case 2: $c = 0$ and $ab \neq 0$. In this case,

$$\ker(W_s - \delta I) = \mathrm{Span}\left\{(\sqrt{2}b, 0, \delta - 1, 0, 0, 0)\right\},$$

so $\ker(W_s - \delta I)$ is not orthogonal to $\ker(W_s - I)$. The rest of the argument is the same as in Case 1.

Case 3: $a = 0$ and $bc \neq 0$. In this case,

$$\ker(W_s - I) = \text{Span}\{e_1\}, \quad \ker(W_s - \gamma I) = \text{Span}\{e_2\},$$
$$\ker(W_s - \delta I) = \text{Span}\left\{\left(\frac{\sqrt{2}b(\delta - \gamma)}{\delta - 1}, c, \delta - \gamma, 0, 0, 0\right)\right\}.$$

Since $b \neq 0$, $\ker(W_s - I)$ is not orthogonal to $\ker(W_s - \delta I)$, and since $c \neq 0$, $\ker(W_s - \gamma I)$ is not orthogonal to $\ker(W_s - \delta I)$. By Lemma 2.2, without loss of generality, we may assume that

$$\ker(W_s - I) + \ker(W_s - \gamma I) + \ker(W_s - \delta I) \subseteq H_1, \quad \text{or} \quad \text{Span}\{e_1, e_2, e_3\} \subseteq H_1.$$

Now $W_s^* e_2 = (\star, \star, \star, \star, \overline{b\gamma}, \star)$ and $W_s^* e_3 = (\star, \star, \star, \star, 0, \sqrt{2}\overline{b\delta})$ belong to H_1. Thus the dimension of H_1 is at least 5, which is a contradiction. □

Case 4.5. *Suppose that $\sigma(W_s)$ consists of 5 distinct numbers. Assume $\beta = 1$ and $\gamma = -1$. Then W_s is irreducible.*

Proof. Note that $\delta^4 \neq 1$. By Lemma 2.1, we have

$$W_s = \begin{bmatrix} 1 & \sqrt{2}a & \sqrt{2}b & a^2 & \sqrt{2}ab & b^2 \\ 0 & -1 & c & -\sqrt{2}a & ac - b & \sqrt{2}bc \\ 0 & 0 & \delta & 0 & \delta a & \sqrt{2}\delta b \\ 0 & 0 & 0 & 1 & -\sqrt{2}c & c^2 \\ 0 & 0 & 0 & 0 & -\delta & \sqrt{2}\delta c \\ 0 & 0 & 0 & 0 & 0 & \delta^2 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & -1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & \overline{c} & \overline{\delta} & 0 & 0 & 0 \\ \overline{a}^2 & -\sqrt{2}\overline{a} & 0 & 1 & 0 & 0 \\ \sqrt{2}\overline{ab} & \overline{ac} - \overline{b} & \overline{\delta a} & -\sqrt{2}\overline{c} & -\overline{\delta} & 0 \\ \overline{b}^2 & \sqrt{2}\overline{bc} & \sqrt{2}\overline{b\delta} & \overline{c}^2 & \sqrt{2}\overline{c\delta} & \overline{\delta}^2 \end{bmatrix}.$$

Then

$$\ker(W_s^* + \overline{\delta} I) = \text{Span}\left\{\left(0, 0, 0, 0, 1, -\frac{\sqrt{2}\overline{c}}{1 + \overline{\delta}}\right)\right\}, \quad \ker(W_s^* - \overline{\delta}^2 I) = \text{Span}\{e_6\},$$
$$\ker(W_s^* - \overline{\delta} I) = \text{Span}\left\{(0, 0, 2, 0, \overline{a}, \star)\right\}.$$

It can be checked that W_s and W_s^* have no common eigenvector. We will show that W_s is irreducible by a contradiction. Assume $W_s = W_1 \oplus W_2$ on $H_1 \oplus H_2$ with $\dim H_i \geq 2$ for $i = 1, 2$.

Case 1: $ac \neq 0$. Assume $\ker(W_s^* + \overline{\delta} I) \subseteq H^1$. By Lemma 2.2,

$$\ker(W_s^* + \overline{\delta} I) + \ker(W_s^* - \overline{\delta} I) + \ker(W_s^* - \overline{\delta}^2 I) \subseteq H_1, \quad \text{or} \quad \text{Span}\{e_3, e_5, e_6\} \subseteq H_1.$$

Now $W_s e_3, W_s e_5 \in H_1$. Since $c \neq 0$, the dimension of H_1 is at least 5, which is a contradiction.

Case 2: $c = 0$ and $ab \neq 0$. In this case,

$$\ker(W_s^* + \overline{\delta} I) = \{e_5\} \quad \text{and} \quad \ker(W_s^* - \overline{\delta} I) = \left\{\left(0, 0, 2, 0, \overline{a}, \frac{2\sqrt{2}\overline{b}}{1 - \overline{\delta}}\right)\right\}.$$

The rest of the argument is the same as in Case 1.

Case 3: $a = 0$ and $bc \neq 0$. In this case,

$$\ker(W_s^* + \overline{\delta} I) = \text{Span}\left\{\left(0, 0, 0, 0, 1, -\frac{\sqrt{2}\overline{c}}{1 + \overline{\delta}}\right)\right\}, \quad \ker(W_s^* - \overline{\delta}^2 I) = \text{Span}\{e_6\},$$
$$\ker(W_s^* - \overline{\delta} I) = \left\{\left(0, 0, 1, 0, 0, \frac{\sqrt{2}\overline{b}}{1 - \overline{\delta}}\right)\right\}.$$

Since $bc \neq 0$, $\ker(W_s^* + \overline{\delta}^2 I)$ is not orthogonal to both $\ker(W_s^* + \overline{\delta} I)$ and $\ker(W_s^* - \overline{\delta} I)$. By Lemma 2.2, without loss of generality, we may assume that

$$\ker(W_s^* + \overline{\delta} I) + \ker(W_s^* - \overline{\delta} I) + \ker(W_s^* - \overline{\delta}^2 I) \subseteq H_1, \quad \text{or} \quad \text{Span}\{e_3, e_5, e_6\} \subseteq H_1.$$

Now $W_s e_3 = (\star, c, \star, 0, 0, 0)$ and $W_s e_5 = (\star, \star, \star, -\sqrt{2}c, \star, 0)$ belong to H_1. Thus the dimension of H_1 is at least 5, which is a contradiction. □

Case 4.6. *Suppose that $\sigma(W_s)$ consists of 5 distinct numbers and Assume $\beta = 1$ and $\delta = \gamma^2$. Then W_s is reducible if and only if $ac = 2\gamma b$ and $|c| = |\gamma a|$, in which case, $H_s = H_1 \oplus H_2$, where H_1 and H_2 are minimal reducing subspaces for W_s whose dimensions are 5 and 1, respectively.*

Proof. Note that $\beta^4 \neq 1$ and $\beta^3 \neq 1$. By Lemma 2.1, we have

$$W_s = \begin{bmatrix} 1 & \sqrt{2}a & \sqrt{2}b & a^2 & \sqrt{2}ab & b^2 \\ 0 & \gamma & c & \sqrt{2}\gamma a & ac + \gamma b & \sqrt{2}bc \\ 0 & 0 & \gamma^2 & 0 & \gamma^2 a & \sqrt{2}\gamma^2 b \\ 0 & 0 & 0 & \gamma^2 & \sqrt{2}\gamma c & c^2 \\ 0 & 0 & 0 & 0 & \gamma^3 & \sqrt{2}\gamma^2 c \\ 0 & 0 & 0 & 0 & 0 & \gamma^4 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & \overline{\gamma} & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & \overline{c} & \overline{\gamma}^2 & 0 & 0 & 0 \\ \overline{a}^2 & \sqrt{2}\overline{a\gamma} & 0 & \overline{\gamma}^2 & 0 & 0 \\ \sqrt{2}\overline{a}\overline{b} & \overline{\gamma}\overline{b} + \overline{ac} & \overline{\gamma}^2\overline{a} & \sqrt{2}\overline{\gamma c} & \overline{\gamma}^3 & 0 \\ \overline{b}^2 & \sqrt{2}\overline{b}\overline{c} & \sqrt{2}\overline{\gamma}^2\overline{b} & \overline{c}^2 & \sqrt{2}\overline{\gamma}^2\overline{c} & \overline{\gamma}^4 \end{bmatrix}.$$

Then

$$\ker(W_s - I) = \text{Span}\{e_1\}, \quad \ker(W_s - \gamma I) = \text{Span}\left\{\left(\frac{\sqrt{2}a}{\gamma - 1}, 1, 0, 0, 0, 0\right)\right\},$$

$$\ker\left(W_s^* - \overline{\gamma}^4 I\right) = \text{Span}\{e_6\}, \quad \ker\left(W_s^* - \overline{\gamma}^3 I\right) = \text{Span}\left\{\left(0, 0, 0, 0, 1, \frac{\sqrt{2}\overline{c}}{(1 - \overline{\gamma})\overline{\gamma}}\right)\right\},$$

$$\ker(W_s - \gamma^2 I) = \text{Span}\left\{\left(\frac{a^2}{(\gamma - 1)^2}, \frac{\sqrt{2}a}{\gamma - 1}, 0, 1, 0, 0\right), \left(\frac{\sqrt{2}(\gamma^2 b - \gamma b + ac)}{\gamma(\gamma - 1)(\gamma^2 - 1)}, \frac{c}{\gamma(\gamma - 1)}, 1, 0, 0, 0\right)\right\},$$

$$\ker(W_s^* - \overline{\gamma}^2 I) = \text{Span}\left\{\left(0, 0, 1, 0, \frac{\overline{a}}{1 - \overline{\gamma}}, \frac{\sqrt{2}(\overline{ac} + \overline{b} - \overline{\gamma}\overline{b})}{(1 - \overline{\gamma}^2)(1 - \overline{\gamma})}\right), \left(0, 0, 0, 1, \frac{\sqrt{2}\overline{c}}{(1 - \overline{\gamma})\overline{\gamma}}, \frac{\overline{c}^2}{(1 - \overline{\gamma})^2\overline{\gamma}^2}\right)\right\}.$$

Note that W_s and W_s^* have a common eigenvector if and only if

$$\dim[\ker(W_s - \gamma^2 I) + \ker(W_s^* - \overline{\gamma}^2 I)] \leq 3,$$

if and only if the vectors

$$\left(\gamma^2 b - \gamma b + ac,\ c,\ \overline{\gamma a},\ (\overline{ac} + \overline{b} - \overline{\gamma}\overline{b})\overline{\gamma}^2\right) \quad \text{and} \quad \left(a^2\gamma(\gamma + 1),\ 2\gamma a,\ 2\overline{c},\ \overline{c}^2(1 + \overline{\gamma})\right)$$

are linearly dependent, if and only if $ac = 2\gamma b$ and $|c| = |\gamma a|$. In that case, a common eigenvector is

$$v := (0, 0, \sqrt{2}\gamma a, -c, 0, 0).$$

Case 1: $ac = 2\gamma b \neq 0$ and $|c| = |\gamma a|$. Let $W_s = W_1 \oplus W_2$, where $W_1 = W_s|\text{Span}\{v\}^\perp$ and $W_2 = W_s|\text{Span}\{v\}$. We will prove that W_1 is irreducible by a contradiction. Assume $W_1 = W_3 \oplus W_4$ on $H_3 \oplus H_4$ where $\dim H_i \geq 2$ for $i = 3, 4$, since W_1 and W_1^* have no common eigenvector anymore. By Lemma 2.2, without loss of generality, we may assume that

$$\ker(W_s - I) + \ker(W_s - \gamma I) \subseteq H_3, \quad \text{or} \quad \text{Span}\{e_1, e_2\} \subseteq H_3.$$

It follows that $\text{Span}\{e_1, e_2, W_1^* e_1, W_1^* e_2\} \subseteq H_3$, and so $\dim H_3 \geq 4$, which is a contradiction. Therefore H_3 and H_4 are minimal reducing subspaces for W_s.

In the remaining cases, W_s and W_s^* have no common eigenvector. We will show that W_s is irreducible. Assume to the contrary that $W_s = W_1 \oplus W_2$ on $H_1 \oplus H_2$ where $\dim H_i \geq 2$ for $i = 1, 2$.

Case 2: $ac \neq 0$ and either $ac \neq 2\gamma b$ or $|c| \neq |\gamma a|$. By Lemma 2.2, we may assume that

$$\ker(W_s - I) + \ker(W_s - \gamma I) \subseteq H_1, \quad \text{or} \quad \mathrm{Span}\{e_1, e_2\} \subseteq H_1.$$

It follows that $\mathrm{Span}\{e_1, e_2, W_s^* e_1, W_s^* e_2\} \subseteq H_1$. If $b \neq 0$, by using Lemma 2.2, we can show that $\ker\left(W_s^* - \overline{\gamma}^3 I\right) \subseteq H_1$ and $\ker\left(W_s^* - \overline{\gamma}^4 I\right) \in H_1$. Similarly, if $b = 0$, then $e_5, e_6 \in H_1$. Thus $\mathrm{Span}\{e_1, e_2, W_s^* e_1, W_s^* e_2, e_5, e_6\} \subseteq H_1$, so $\dim H_1 \geq 5$, which is a contradiction.

Case 3: $c = 0$ and $ab \neq 0$. Assume $\ker(W_s - I) \subseteq H_1$. Since $\ker(W_s - \gamma I)$ is not orthogonal to $\ker(W_s - I)$, it follows Lemma 2.2 that $e_1, e_2 \in H_1$. Then $W_s^* e_1 \in H_1$. It follows that neither $\ker(W_s^* - \overline{\gamma}^4 I)$ nor $\ker(W_s^* - \overline{\gamma}^3 I)$ is orthogonal to H_1. Hence $\{e_1, e_2, e_5, e_6, W_s^* e_1\} \subseteq H_1$. Then $\dim H_1 \geq 5$, which is a contradiction.

Case 4. $a = 0$ and $bc \neq 0$. Assume that H_1 contains $\ker(W_s^* - \overline{\gamma}^4 I)$. By a similar argument in Cases 2, we can show that $\{e_5, e_6, W_s e_6, e_1, e_2\} \subseteq H_1$. Since $\{e_1, e_2, e_5, e_6, W_s e_6\}$ is linearly independent, it follows that $\dim H_1 \geq 5$, which is a contradiction.

Therefore W_s is irreducible, and the proof is complete. □

When $\sigma(W_s)$ consists of 4 distinct numbers, the proof for the irreducibility of W_s is rather difficult since we have tried a number of orthogonality conditions without success. In this case when $\sigma(W_s)$ consists of four distinct numbers, we can assume that $\beta = 1$, $\gamma = i$, and $\delta = i^2 = -1$.

Case 4.7. *Suppose that $\sigma(W_s)$ consists of 4 distinct numbers. Assume $\beta = 1$, $\gamma = i$, and $\delta = -1$. Then W_s is reducible if and only if $ac = 2ib$ and $|c| = |a|$, in which case, $H_s = H_1 \oplus H_2$, where H_1 and H_2 are minimal reducing subspaces for W_s whose dimensions are 5 and 1, respectively.*

Proof. By Lemma 2.1, we have

$$W_s = \begin{bmatrix} 1 & \sqrt{2}a & \sqrt{2}b & a^2 & \sqrt{2}ab & b^2 \\ 0 & i & c & \sqrt{2}ia & ac + ib & \sqrt{2}bc \\ 0 & 0 & -1 & 0 & -a & -\sqrt{2}b \\ 0 & 0 & 0 & -1 & \sqrt{2}ic & c^2 \\ 0 & 0 & 0 & 0 & -i & -\sqrt{2}c \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad W_s^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & -i & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & \overline{c} & -1 & 0 & 0 & 0 \\ \overline{a}^2 & -\sqrt{2}i\overline{a} & 0 & -1 & 0 & 0 \\ \sqrt{2}\overline{a}\overline{b} & \overline{ac} - i\overline{b} & -\overline{a} & -\sqrt{2}i\overline{c} & i & 0 \\ \overline{b}^2 & \sqrt{2}\overline{b}\overline{c} & -\sqrt{2}\overline{b} & \overline{c}^2 & -\sqrt{2}\overline{c} & 1 \end{bmatrix}.$$

Then

$$\ker(W_s - I) = \mathrm{Span}\left\{e_1, u_1 := \left(0, \frac{(1-i)bc + iac^2}{2\sqrt{2}}, \frac{(1-i)ac - 2b}{2\sqrt{2}}, \frac{-ic^2}{2}, \frac{(i-1)c}{\sqrt{2}}, 1\right)\right\},$$

$$\ker(W_s^* - I) = \mathrm{Span}\left\{e_6, u_2 := \left(1, \frac{(1-i)\overline{a}}{\sqrt{2}}, \frac{(1-i)\overline{ac} + 2\overline{b}}{2\sqrt{2}}, \frac{-i\overline{a}^2}{2}, \frac{(1-i)\overline{a}\overline{b} - i\overline{a}^2\overline{c}}{2\sqrt{2}}, 0\right)\right\},$$

$$\ker(W_s + I) = \mathrm{Span}\left\{\left(\frac{ia^2}{2}, \frac{-(1+i)a}{\sqrt{2}}, 0, 1, 0, 0\right), \left(\frac{(1-i)ac - 2b}{2\sqrt{2}}, \frac{(i-1)c}{2}, 1, 0, 0, 0\right)\right\},$$

$$\ker(W_s^* + I) = \mathrm{Span}\left\{\left(0, 0, 1, 0, \frac{(1-i)\overline{a}}{2}, \frac{(1-i)\overline{ac} + 2\overline{b}}{2\sqrt{2}}\right), \left(0, 0, 0, 1, \frac{(1+i)\overline{c}}{\sqrt{2}}, \frac{i\overline{c}^2}{2}\right)\right\},$$

$$\ker(W_s + iI) = \mathrm{Span}\left\{\left(\frac{(1+i)ab - a^2c}{2\sqrt{2}}, \frac{(3-i)ac - 2b}{4}, \frac{(1+i)a}{-2}, \frac{(i-1)c}{\sqrt{2}}, 1, 0\right)\right\},$$

$$\ker(W_s^* + iI) = \mathrm{Span}\left\{\left(0, 1, \frac{(1+i)\overline{c}}{2}, \frac{(1-i)\overline{a}}{\sqrt{2}}, \frac{(3-i)\overline{ac} + 2\overline{b}}{4}, \frac{\overline{ac}^2 + (1+i)\overline{b}\overline{c}}{2\sqrt{2}}\right)\right\},$$

$$\ker(W_s - iI) = \mathrm{Span}\left\{\left(\frac{(1+i)a}{-\sqrt{2}}, 1, 0, 0, 0, 0\right)\right\}, \quad \ker(W_s^* - iI) = \mathrm{Span}\left\{\left(0, 0, 0, 0, 1, \frac{(1+i)\overline{c}}{\sqrt{2}}\right)\right\}.$$

Note that $\dim[\ker(W_s - I) + \ker(W_s^* - I)] = 4$, and

$$\dim[\ker(W_s + I) + \ker(W_s^* + I)] \leq 3$$

if and only if $ac = 2ib$ and $|c| = |a|$. These are the necessary and sufficient condition for W_s and W_s^* to have a common eigenvector. In that case, a common eigenvector is

$$v := (0, 0, \sqrt{2}a, ic, 0, 0).$$

Case 1: $ac = 2ib \neq 0$ and $|c| = |a|$. Let $W_s = W_1 \oplus W_2$, where $W_1 = W_s|\operatorname{Span}\{v\}^\perp$ and $W_2 = W_s|\operatorname{Span}\{v\}$. We will prove that W_1 is irreducible. Assume $W_1 = W_3 \oplus W_4$ on $H_3 \oplus H_4$ where $\dim H_i \geq 2$ for $i = 3, 4$, since W_1 and W_1^* have no common eigenvector anymore. Without loss of generality, we may assume that $\ker(W_s - iI) \subseteq H_3$. Note that

$$\ker(W_s + iI) = \operatorname{Span}\left\{\Big(\frac{(1-i)ab}{2\sqrt{2}}, \frac{3ib}{2}, \frac{(1+i)a}{-2}, \frac{(i-1)c}{\sqrt{2}}, 1, 0\Big)\right\}.$$

Since $abc \neq 0$, $\ker(W_s + iI)$ is not orthogonal to $\ker(W_s - iI)$, and so $\ker(W_s + iI) \subseteq H_3$ by Lemma 2.2. By the same argument as above,

$$\ker(W_s - iI) + \ker(W_s + iI) + \ker(W_s^* + iI) + \ker(W_s^* - iI) \subseteq H_3, \quad \text{and so} \quad \dim H_3 \geq 4,$$

which is a contradiction. Therefore H_3 and H_4 are minimal reducing subspaces for W_s.

In the remaining cases, W_s and W_s^* have no common eigenvector. We will show that W_s is irreducible. Assume to the contrary that $W_s = W_1 \oplus W_2$ on $H_1 \oplus H_2$ where $\dim H_i \geq 2$ for $i = 1, 2$.

Case 2: $ac \neq 0$ and either $ac \neq 2ib$ or $|c| \neq |a|$. Without loss of generality, assume

$$\ker(W_s - iI) \subseteq H_1.$$

Since $\ker(W_s^* + iI)$ is not orthogonal to $\ker(W_s - iI)$, it follows that $\ker(W_s^* + iI) \subseteq H_1$. Since $ac \neq 0$, if $\ker(W_s - iI) \perp \ker(W_s + iI)$ and $\ker(W_s^* - iI) \perp \ker(W_s^* + iI)$, then

$$\frac{2b}{ac} + i = \frac{|a|^2 + 3}{|a|^2 + 1} = -\frac{|c|^2 + 3}{|c|^2 + 1},$$

which is a contradiction. Thus, by using Lemma 2.2, we can show that

$$\ker(W_s - iI) + \ker(W_s^* + iI) + \ker(W_s + iI) + \ker(W_s^* - iI) \subseteq H_1.$$

Since $a \neq 0$, neither $\ker(W_s - I)$ nor $\ker(W_s + I)$ is orthogonal to $\ker(W_s - iI)$. Thus $\sigma(W_1) = \{1, -1, i, -i\}$ by Lemma 2.2. Since $\pm 1 \in \sigma(W_1)$ and $\dim H_2 \geq 2$, it follows that $\sigma(W_2) = \{-1, 1\}$. Let $w_1 \in \ker(W_2 - I)$ and $w_2 \in \ker(W_2^* - I)$. Then

$$w_1 = c_1 e_1 + c_2 u_1 \quad \text{and} \quad w_2 = d_1 e_6 + d_2 u_2$$

for some constants c_1, c_2, d_1, d_2. Since $w_1, w_2 \perp H_1$, if we assume that $c_2 \neq 0$ and $d_2 \neq 0$, then

$$(1+i)(\bar{a}b/c - 2) = |a|^2 \quad \text{and} \quad (1-i)(b\bar{c}/a + 2) = -|c|^2.$$

But we can check that these imply a contradiction. Thus $c_2 = 0$ or $d_2 = 0$. That is, either e_1 or e_6 belongs to H_2, which contradicts to the fact that $H_2 \perp H_1$.

Case 3: $c = 0$ and $ab \neq 0$. Then

$$\ker(W_s - iI) = \operatorname{Span}\left\{\Big(\frac{(1+i)a}{-\sqrt{2}}, 1, 0, 0, 0, 0\Big)\right\}, \quad \ker(W_s^* - iI) = \operatorname{Span}\{e_5\},$$

$$\ker(W_s + iI) = \operatorname{Span}\left\{\Big(\frac{(1+i)ab}{2\sqrt{2}}, \frac{-b}{2}, \frac{(1+i)a}{-2}, 0, 1, 0\Big)\right\},$$

$$\ker(W_s^* + iI) = \operatorname{Span}\left\{(0, 1, 0, \frac{(1-i)\bar{a}}{\sqrt{2}}, \frac{\bar{b}}{2}, 0)\right\}.$$

Assume that H_1 contains $\ker(W_s^* - iI)$. By using Lemma 2.2, we can show that

$$\ker(W_s - iI) + \ker(W_s + iI) + \ker(W_s^* + iI) + \ker(W_s^* - iI) \subseteq H_1$$

Also, $W_s e_5 \in H_1$. It follows that $\dim H_1 \geq 5$, which is a contradiction.

Case 4: $a = 0$ and $bc \neq 0$. By the same argument as in Case 3, we can show that $\dim H_1 \geq 5$, which is a contradiction.

Therefore W_s is irreducible, and the proof is complete. □

In Theorem 1.5(i), we assumed that $\sigma(A)$ is not equal to $\{\lambda, \lambda\omega, \lambda\omega^2\}$. We pretty sure that the theorem is true for this case.

References

[1] C. Gu, J. Park, C. Peak, and J. Rowley, Decomposition of the Kronecker sums of matrices into a direct sum of irreducible matrices, Bull. Korean Math. Soc., 58 (2021), no. 3, 637–657.
[2] C. Gu, A. Mendes, and J. Park, Reducing subspaces of tensor products of operators and representation of permutation group, in preparation.
[3] P. R. Halmos, Irreducible operators, Michigan Math J. 15 (1968) 215–223.
[4] C. S. Kubrusly, Regular lattices of tensor products, Linear Algebra Appl. 438 (2013) 428–435.
[5] W. Tung, Group Theory in Physics, World Scientific, Singapore, 1985.
[6] D. Voiculescu, A non-commutative Weyl–von Neumann theorem, Rev. Roumaine Math. Pures Appl. 21 (1976) 97-113.