Non-Affine Nonlinear Stochastic Systems with Two ILC Update Laws under Random Data Dropouts

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Abstract. This article for non-affine nonlinear stochastic networked systems investigates the convergence analysis and the tracking performance verification of two iterative learning control (ILC) update laws. In first ILC update law, if the information isn't transferred, the update of the algorithm will stop. In second ILC update law, the update of the algorithm in each iteration will be continued using the newest accessible data even if no data is transferred in the current iteration. It is indicated that the input signals converge to the desired input, and no restrictive condition is imposed on the probabilities of the successful transfer of data. The convergence analysis of the two algorithms is based on concept almost sure. The comparisons of two presented ILC update laws are presented with a numerical example. Also, the tracking performance and effectiveness of the presented algorithms are shown.

1. Introduction

In most application industrial projects, the same task is iterated frequently at a fixed time to complete the industrial products. For such systems that consist of successive categories of the production project such that each batch follows the desired pattern at a fixed time and is reiterated repeatedly, operational data and experiences can be used for the next batch. This application of experience and information in the industry is the learning concept that is an incentive for providing and creating a branch of smart control, namely iterative learning control (ILC). ILC is a control process formed considering the process of human learning so that learning the iterative factors of the system based on information from prior done periods are performed to improve system tracking performance. In this control process, the dynamic information of the system is not much needed, and ILC is a control method based on the data. ILC can productively confront with different conventional control problems and have excellent performance despite all these control problems.

Besides, there are many results concerning systems with stochastic signals described by random variables. Systems involving stochastic signals such as system noise, measurement noise, and random data dropout are described with stochastic iterative learning control (SILC).

In this regard, many studies and researches have been done, so there is numerous literature in this branch of control in different fields of industrial and chemical applications, medical engineering, etc. Already, many

kinds of research have been done on different topics of ILC such as stability, robustness, frequency analysis, update law design, and applied research. The studies in this field can be seen in survey articles such as [1, 2, 7, 13, 14].

In [9], the status of the SILC according to basic techniques from three aspects was investigated.

Of Internet services, it is possible to network the control systems by helping techniques of network and communication due to low cost, high performance, robustness, flexibility, and facility. Control systems with these properties constitute network control systems (NCSs). In the implementation of networked control systems, data transfer suffers from the problem such as data dropouts which can impair tracking performance. Restrictions on the physical properties of wires or wireless net communication devices such as temporal oscillation of the net or the limit bandwidth enable delay in communication and data dropout. In NCSs, the network is in between the controller and the system, and the purpose is an efficient performance and a proper efficiency in difficult transfers i.e. loss of information. The network is modeled as a random switch. When the switch is opened, the data transmission becomes problematic, and the data is lost. Due to the random data dropouts, the direct methods of control and classical approximation are not used in NCSs. Several papers on the application of the ILC algorithm are available for NCSs concerning the main aspects, namely, the convergence analysis, data conditions, and the design of the compensation structure. One can observe that these articles listed in [8]. Indeed, [8] is a survey about ILC with incomplete information and the corresponding control systems.

In this paper, we present two ILC update laws (4), (5) and (6), (7) for non-affine nonlinear networked systems with measurement noise. The ILC update law (6), (7) implies that even with the data dropout event, the update of input continues with the newest accessible data. In the ILC update law (4), (5), update law with the data dropout event is stopped. In general, the data dropouts must be previously unknown, and no limiting condition is imposed on the data dropouts. We use the almost sure concept of convergence analysis for stochastic non-affine nonlinear systems with data dropouts considered with the Bernoulli model. That is why the randomness of data dropouts is included. Here, the system information is assumed to be unknown and the proposed update laws are data-driven. Also, there is no restrictive condition for the random data dropout probabilities in the convergence analysis of the input error. Let $\alpha_k(t)$ that has Bernoulli distribution be used for modeling the transfer of $y_k(t)$. It is proven that if $\left|1 - \rho_k C^+ D_k^2(t)\right| < 1$ then, input error converges to zero in the almost sure concept in the ILC update laws (4), (5) and (6), (7). ρ_k is the decreasing sequence with the (3) properties, and $D_k^2(t)$ is introduced in the requirement R4.

Notations: The real number field is indicated with $\mathbb{R}$. The probability of an event is shown by P. The mathematical expectation is indicated by E. The transpose of a vector or matrix is indicated by superscript T. $|.|$ is absolute value notation. "i.o." denotes "infinitely often", "a.s." shows "almost surely" and "w.p.1." indicates "with probability one". "i.i.d." is "independent and identically distributed".

In Section (2), we present the problem formulation. Convergence analysis is investigated in Section (3). In Section (4), for extra analyze of the convergence characteristics and the tracking performance of the proposed model in this paper, the numerical example is presented. In Section (5), the paper is concluded.

2. Problem formulation

The discrete-time non-affine nonlinear system with measurement noise is considered as below:

$$\begin{aligned}
x_k(t+1) &= f(t, x_k(t), u_k(t)), \\
y_k(t) &= C(t)x_k(t) + \zeta_k(t),
\end{aligned} \tag{1}$$

where $t = 0, 1, ..., N$ is the time index, the given positive integer N is the iteration length, and $k = 1, 2, ...$ indicates the iteration index. Vector input, vector output, and vector state are indicated by $u_k(t) \in \mathbb{R}$, $y_k(t) \in \mathbb{R}$, and $x_k(t) \in \mathbb{R}^n$, respectively. Measurement noise is indicated by the stochastic variable $\zeta_k(t)$. Nonlinear function $f(t, x_k(t), u_k(t))$ and time-varying vector $C(t)$ are unknown information on the system.

Some of the requirements are stated in the following.

- R1. $\forall t = 0, 1, ..., N$, $f(t, ., .)$ is a continuously differential function with respect to x and u.

- R2. For desired initial state $x_d(0)$ where, $y_d(0) = C(0)x_d(0)$ there exists a unique $u_d(t)$ for generating reference output $y_d(t)$, such that

$$x_d(t+1) = f(t, x_d(t), u_d(t)),$$
$$y_d(t) = C(t)x_d(t). \tag{2}$$

- R3. $\forall t$ the measurement noise sequence $\zeta_k(t), k = 0, 1, ...$ with $E\zeta_k(t) = 0$ and $\sup_k E|\zeta_k(t)|^2 < \infty$ is i.i.d. $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} \zeta_k(t) (\zeta_k(t))^T = \zeta_t$ a.s., so that ζ_t is an unknown matrix.

  Remark 2.1. *The condition applied to the measurement noise is based on the repetition axis, and because the repetitive and independent process is performed, it is no strict requirement.*

- R4. $\forall t = 0, 1, ..., N$, function $f(t, x_k(t), u_k(t))$ satisfies the global Lipschitz condition, that is, $\forall x_1, x_2 \in \mathbb{R}^n$ and $\forall u_1, u_2 \in \mathbb{R}, |f(t, x_1, u_1) - f(t, x_2, u_2)| \le l_1 |x_1 - x_2| + l_2 |u_1 - u_2|$, where $l_1 > 0$ and $l_2 > 0$ are the Lipschitz constants. We specify $D_k^1(t) = \frac{\partial f}{\partial x}\big|_{\tilde{x}_k(t)}$ and $D_k^2(t) = \frac{\partial f}{\partial u}\big|_{\tilde{u}_k(t)}$ where the vector $\tilde{u}_k(t)$ is between $u_d(t)$ and $u_k(t)$, and the vector $\tilde{x}_k(t)$ is between $x_d(t)$ and $x_k(t)$. Without loss of generality, we assume $D_k^2(t)$ is non-singular. Also, $\forall k, t, |D_k^1(t)| \le l_1$ and $|D_k^2(t)| \le l_2$.

- R5. It is assumed that unknown value $C(t+1)D_k^2(t)$ is nonzero, and its sign does not change during the learning process, in other words, it is considered $C(t+1)D_k^2(t) > 0$.

- R6. The i.i.d. initial state sequence is exactly resetting asymptotically in the concept that $x_k(0) \to x_d(0)$ when $k \to \infty$. Moreover, the initial state sequence and the measurement noise sequence are mutually independent.

  Remark 2.2. *It is noteworthy that the classically identical initial condition is a specific case of R6.*

In networked control systems on the ILC controller, the networks are employed to communicate the iterative learning controller and the operational plant. As it is observed in Figure 1, the system output of the current repetition is transferred through the communication network to the ILC controller. In this way, due to communication problems of wireless or wired, networks may have problems such as data dropouts or delays. Accordingly, conventionalized updating laws cannot be applied to control networked systems. This paper considers data dropouts on measurement side.

Remark 2.3. *If data dropout on both measurement and actuator sides are taken into account, a more detailed analysis is needed because the asynchronous update between the control signal fed to the plant and the control signal generated by the learning controller must be considered. Given that these are out of the scope of this article, for example, see [10] for more details.*

Similar to some articles on NCSs, in this paper, the Bernoulli random variables are used to express the packet dropout. For modeling the transfer of $y_k(t)$, $\alpha_k(t)$ that has Bernoulli distribution is used. For this purpose, if $y_k(t)$ is successfully transferred $\alpha_k(t) = 1$, and if $y_k(t)$ is not successfully transferred $\alpha_k(t) = 0$. It can be assumed the probability of successfully transferred output is $0 < r < 1$, that is, $P(\alpha_k(t) = 1) = r$ and $P(\alpha_k(t) = 0) = 1 - r$, $\forall k, t$. Therefore, we result in $E\alpha_k(t) = r$.

The purpose of the control in this paper is to design an ILC algorithm for updating and generating the inputs with due attention to minimize $\limsup\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} |y_d(t) - y_k(t)|^2$, $\forall t = 0, 1, ..., N$ in the data dropouts environment. It should also be borne in mind that given the unpredictable measurement noise, the convergence investigation of the suggested model in this paper is that the inputs of the system converge directly to the desired input.

A mechanism is required to assure the convergence of the input error to zero and overcome the influence of stochastic noises in the stochastic systems. Since the error includes two elements, the real tracking
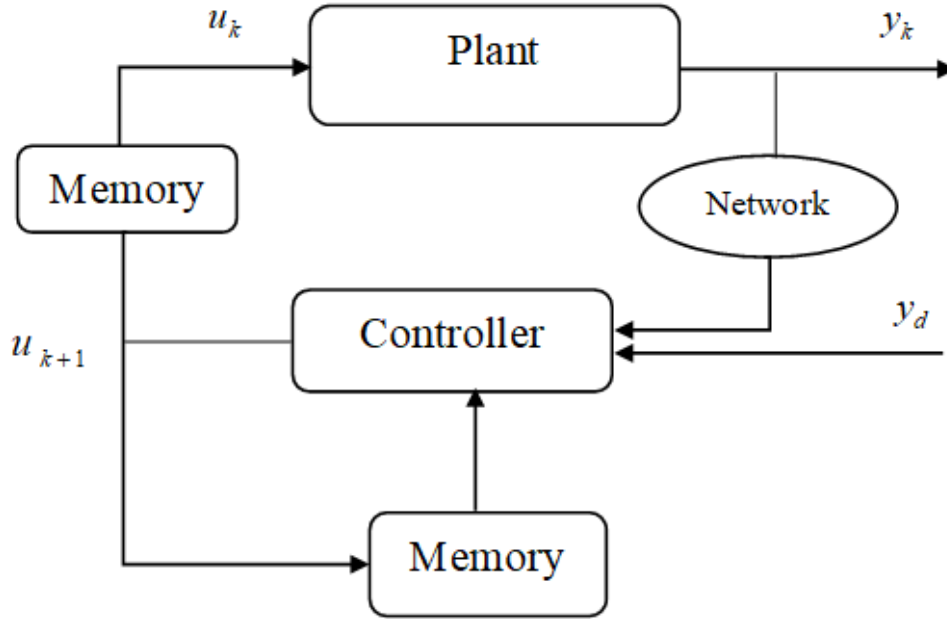
Figure 1: Diagram of a control system with a network at the measurement side

error, and the measurement noise, unlike the onset of the learning process where the actual output error is predominant, with increasing repetitions, the measurement noise in tracking error may be dominated. Thus, to counteract the effect of measured noise, avoiding unstable conditions, and ensuring the convergence of inputs, the decreasing sequence ρ_k with the following properties is considered for our update laws in this paper.

$$\rho_k \to 0,\ \rho_k > 0,\ \sum_{k=1}^{\infty} \rho_k^2 < \infty,\ \sum_{k=1}^{\infty} \rho_k = \infty,\ \forall k = 1,\ 2,\ ... \tag{3}$$

In this paper, the following ILC algorithms are proposed. In this way, the purpose of control is achieved under stochastic measurement noises for non-affine nonlinear systems. The first update law is az fallows:

$$u_{k+1}(t) = u_k(t) + \rho_k E_k(t + 1) \tag{4}$$

where,

$$E_k(t) = \begin{cases} e_k(t), & if\ \alpha_k(t) = 1 \\ 0, & if\ \alpha_k(t) = 0 \end{cases} \tag{5}$$

Also, $e_k(t) = y_d(t) - y_k(t)$ is the tracking error.

In ILC update law (4), (5), upon the successful transference of output, the algorithm updates its input. In this case, the update of the input signal will stop if the corresponding output is dropped out. Considering the inherent mechanism of (4), (5), the higher the probability of the data dropout, the slower the convergence of the algorithm. So, we look for faster convergence considering the high probability of the data dropout. Therefore, the following update law is considered.

The second ILC update law is presented as follows:

$$u_{k+1}(t) = u_k(t) + \rho_k \bar{E}_k(t + 1) \tag{6}$$

where $\bar{E}_k(t) = y_d(t) - \bar{y}_k(t)$ and also, $\bar{y}_k(t)$ is the latest accessible output as fallows:

$$\bar{y}_k(t) = \begin{cases} y_k(t), & \text{if } \alpha_k(t) = 1 \\ \bar{y}_{k-1}(t), & \text{if } \alpha_k(t) = 0 \end{cases} \tag{7}$$

In ILC update law (6), (7), the update of input will always continue even if data dropout occurs. The update of input is performed using the transferred output of the end iteration. If the data dropout occurs, then the update of input is performed with the newest accessible output of previous iterations. Because of the permanent update ILC update law (6), (7), tracking the performance of ILC update law (6), (7) is better than ILC update law (4), (5). Also, the convergence speed of ILC update law (6), (7) is faster than ILC update law (4), (5).

Recall that in this paper, the convergence analysis of ILC update law (6), (7) and ILC update law (4), (5) is performed as well as their performance evaluation for stochastic non-affine nonlinear systems.

In the following for ease of writing, we set $f_k(t) = f(t, x_k(t), u_k(t))$, $f_d(t) = f(t, x_d(t), u_d(t))$, $\delta f_k(t) = f_d(t) - f_k(t)$, and $C^+ D_k^2(t) = C(t+1)D_k^2(t)$.

3. Convergence analysis

Here, the convergence argument, regarding the recommended laws (4), (5) and (6), (7) are performed. In this respect, the following Lemmas are needed to prove the convergence of the algorithms.

Lemma 3.1. *Consider assumptions R1-R6 for the system (1). If* $\lim_{k \to \infty} \delta u_k(m) = 0$, $m = 0, 1, ..., t$, *then* $\left| \delta f_k(t+1) \right| \xrightarrow[k \to \infty]{}$ 0 *and* $|\delta x_k(t+1)| \xrightarrow[k \to \infty]{} 0$ *at time t+1.*

Proof. Concerning (1) and (2) it results

$$\begin{aligned} \delta x_k(t+1) &= f(t, x_d(t), u_d(t)) - f(t, x_k(t), u_k(t)) \\ &= D_k^1(t)\delta x_k(t) + D_k^2(t)\delta u_k(t) \end{aligned} \tag{8}$$

The proof is provided by mathematical induction.

Initial step. Let $t = 0$, we have

$$\delta x_k(1) = D_k^1(0)\delta x_k(0) + D_k^2(0)\delta u_k(0) \tag{9}$$

With due attention to R6, $\delta x_k(0) \xrightarrow[k \to \infty]{} 0$, and because by concerning R4 $D_k^1(0)$ is bounded, we have $D_k^1(0)\delta x_k(0) \xrightarrow[k \to \infty]{} 0$. Considering assumptions of Lemma (3.1) $\delta u_k(0) \xrightarrow[k \to \infty]{} 0$, and since with respect to R4, $D_k^2(0)$ is bounded, it results in $D_k^2(0)\delta u_k(0) \xrightarrow[k \to \infty]{} 0$. Therefore, from (9), we result in $\delta x_k(1) \xrightarrow[k \to \infty]{} 0$. Now concerning R1, it results in $\delta f_k(1) \xrightarrow[k \to \infty]{} 0$.

Inductive step. We assume that conclusions of Lemma (3.1) hold for $i = 0, 1, ..., t$, we prove that results are true for $t + 1$. The process of proving here is similar to the initial step of induction. Thus, it is proven $\delta x_k(t+1) \xrightarrow[k \to \infty]{} 0$ and $\delta f_k(t+1) \xrightarrow[k \to \infty]{} 0$. $\square$

3.1. Convergence analysis of ILC update law (4), (5)

Here, we investigate the convergence of the ILC update law (4), (5). In ILC update law (4), (5), if the corresponding output is dropped out, the input is kept invariable.

In this regard, for convergence analysis, the below Theorem is established.

Theorem 3.2. *Let update law (4), (5) be used to the non-affine nonlinear stochastic networked system (1). If* $\left| 1 - \rho_k C^+ D_k^2(t) \right| < 1$ *then for* $u_k(t)$ *generated by update law (4), we conclude that* $u_k(t) \to u_d(t)$ *w.p.1, for all t when* $k \to \infty$.

Proof. To show convergence it must be proved $\delta u_k(t) = u_d(t) - u_k(t) \to 0$ for all $t = 0, 1, ..., N$ when $k \to \infty$. Note that we define the status error with $\delta x_k(t) = x_d(t) - x_k(t)$.

According to (1) and (2), from (4) and (5) we conclude that

$$
\begin{aligned}
\delta u_{k+1}(t) &= \delta u_k(t) - \rho_k E_k(t+1) \\
&= \delta u_k(t) - \rho_k \alpha_k(t+1)e_k(t+1) \\
&= \delta u_k(t) - \rho_k \alpha_k(t+1)C(t+1)(x_d(t+1) - x_k(t+1)) \\
&\quad + \rho_k \alpha_k(t+1)\zeta_k(t+1)
\end{aligned}
\tag{10}
$$

So, again from (1) and (2), we have that

$$
\begin{aligned}
\delta u_{k+1}(t) &= \delta u_k(t) - \rho_k \alpha_k(t+1)C(t+1)(f_d(t) - f_k(t)) \\
&\quad + \rho_k \alpha_k(t+1)\zeta_k(t+1) \\
&= \delta u_k(t) - \rho_k \alpha_k(t+1)C(t+1)(D_k^1(t)\delta x_k(t) \\
&\quad + D_k^2(t)\delta u_k(t)) \\
&\quad + \rho_k \alpha_k(t+1)\zeta_k(t+1)
\end{aligned}
\tag{11}
$$

Therefore, we have

$$
\begin{aligned}
\delta u_{k+1}(t) &= [1 - \rho_k C^+ D_k^2(t)]\delta u_k(t) \\
&\quad + \rho_k C^+ D_k^2(t)\delta u_k(t) \\
&\quad - \rho_k \alpha_k(t+1)C^+ D_k^2(t)\delta u_k(t) \\
&\quad - \rho_k \alpha_k(t+1)C(t+1)D_k^1(t)\delta x_k(t) \\
&\quad + \rho_k \alpha_k(t+1)\zeta_k(t+1)
\end{aligned}
\tag{12}
$$

Notice that $\alpha_k(t+1)$ is independent of $\zeta_k(t+1)$. We take the norm from both sides of (12), we have

$$
\begin{aligned}
|\delta u_{k+1}(t)| &\leq \left|1 - \rho_k C^+ D_k^2(t)\right| |\delta u_k(t)| \\
&\quad + \left|\rho_k\right| |C(t+1)| \left|D_k^2(t)\right| |\delta u_k(t)| \\
&\quad + \left|\rho_k\right| |\alpha_k(t+1)| |C(t+1)| \left|D_k^2(t)\right| |\delta u_k(t)| \\
&\quad + \left|\rho_k\right| |\alpha_k(t+1)| |C(t+1)| \left|D_k^1(t)\right| |\delta x_k(t)| \\
&\quad + \left|\rho_k\right| |\alpha_k(t+1)| |\zeta_k(t+1)| \qquad \text{a.s.}
\end{aligned}
\tag{13}
$$

It can be proven $\lim_{k\to\infty} \delta u_k(t) = 0$, $\forall t$ by using mathematical induction.

Initial step. The case $t = 0$ is considered.

$$
\begin{aligned}
|\delta u_{k+1}(0)| &\leq \left|1 - \rho_k C^+ D_k^2(0)\right| |\delta u_k(0)| \\
&\quad + \left|\rho_k\right| |C(1)| \left|D_k^2(0)\right| |\delta u_k(0)| \\
&\quad + \left|\rho_k\right| |\alpha_k(1)| |C(1)| \left|D_k^2(0)\right| |\delta u_k(0)| \\
&\quad + \left|\rho_k\right| |\alpha_k(1)| |C(1)| \left|D_k^1(0)\right| |\delta x_k(0)| \\
&\quad + \left|\rho_k\right| |\alpha_k(1)| |\zeta_k(1)| \qquad \text{a.s.}
\end{aligned}
\tag{14}
$$

With respect to R5, for enough large k we conclude that $C^+ D_k^2(0) > \psi$, where ψ is an appropriate constant. Considering to R6, it concludes that $\delta x_k(0) \to 0$. Also, by R4, we conclude that $D_k^1(0)$ and $D_k^2(0)$ are bounded. Notice that $\alpha_k(1)$ is bounded. Thus, in (14), we conclude that $\left|\rho_k\right| |\alpha_k(1)| |C(1)| \left|D_k^1(0)\right| |\delta x_k(0)| \to 0$ w.p.1 when $k \to \infty$. In (14), concerning $\delta u_k(0)$ is input error vector therefore, its norm is bounded. Thus, concerning

$\lim\limits_{k\to\infty} \rho_k = 0$ we conclude that $\left|\rho_k\right|\left|C(1)\right|\left|D_k^2(0)\right|\left|\delta u_k(0)\right| \to 0$ and $\left|\rho_k\right|\left|\alpha_k(1)\right|\left|C(1)\right|\left|D_k^2(0)\right|\left|\delta u_k(0)\right| \to 0$ w.p.1, when $k \to \infty$. $\left|\zeta_k(1)\right|$ is bounded because of $\zeta_k(1)$ is continuous function white noise on $[0, N]$. Thus, concerning $\lim\limits_{k\to\infty} \rho_k = 0$ we conclude that $\left|\rho_k\right|\left|\alpha_k(1)\right|\left|\zeta_k(1)\right| \to 0$, w.p.1, when $k \to \infty$.

Let $\sigma_1 = \left|1 - \rho_k C^+ D_k^2(0)\right|$, $\sigma_i = 0$, $i = 2, 3, ..., e_k = \left|\delta u_k(0)\right|$, and $\varphi_k = 0$, given the assumption of Theorem (3.2) namely $\left|1 - \rho_k C^+ D_k^2(0)\right| < 1$ and Lemma (1) of paper [5] from inequality (14), it has resulted $\lim\limits_{k\to\infty} \left|\delta u_k(0)\right| = 0$, w.p.1.

Inductive step. We assume that $\delta u_k(m) \to 0$ is correct for $m = 0, 1, ..., t - 1$, then we prove $\delta u_k(m) \to 0$ for $m = t$. In (13), concerning the assumption of inductive and Lemma (3.1), we have $\delta x_k(t) \to 0$. Moreover, by R4, we conclude that $D_k^1(t)$ and $D_k^2(t)$ is bounded. Also, $\alpha_k(t + 1)$ is bounded. Therefore, in (13) we have $\left|\rho_k\right|\left|\alpha_k(t + 1)\right|\left|C(t + 1)\right|\left|D_k^1(t)\right|\left|\delta x_k(t)\right| \to 0$ w.p.1 when $k \to \infty$. Concerning $\delta u_k(t)$ is input error vector therefore, its norm is bounded. Therefore, concerning $\lim\limits_{k\to\infty} \rho_k = 0$ we conclude that $\left|\rho_k\right|\left|C(t + 1)\right|\left|D_k^2(t)\right|\left|\delta u_k(t)\right| \to 0$ and $\left|\rho_k\right|\left|\alpha_k(t + 1)\right|\left|C(t + 1)\right|\left|D_k^2(t)\right|\left|\delta u_k(t)\right| \to 0$ w.p.1, when $k \to \infty$.

$\zeta_k(t + 1)$ is continuous function white noise on $[0, N]$ therefore, $\left|\zeta_k(t + 1)\right|$ is bounded. With respect to $\lim\limits_{k\to\infty} \rho_k = 0$, we have $\left|\rho_k\right|\left|\alpha_k(t + 1)\right|\left|\zeta_k(t + 1)\right| \to 0$ w.p.1, when $k \to \infty$.

Set $\sigma_1 = \left|1 - \rho_k C^+ D_k^2(t)\right|$, $\sigma_i = 0$, $i = 2, 3, ..., e_k = \left|\delta u_k(t)\right|$, and $\varphi_k = 0$, with respect to $\left|1 - \rho_k C^+ D_k^2(t)\right| < 1$, Lemma (1) of paper [5], and (13), we have $\lim\limits_{k\to\infty} \left|\delta u_k(t)\right| = 0$, w.p.1. $\square$

Thus, in the ILC update law (4), (5), it was proven that input error converges to zero w.p.1 when the number of iterations tends to infinity.

3.2. Convergence analysis of ILC update law (6), (7)

In ILC update law (6), (7), the input of the algorithm is updated in each repetition. The error applied in ILC update law (6), (7) is probably unknown due to the lost output and the error data could be with different probabilities from each prior iteration. Therefore, similar to [11], we introduced $\left\{\tau_k^t\right\}$, $0 \le t \le N$, $k = 1, 2, ...$ to determine the random delay of iterations due to stochastic lost data. Thus, the updating law (6) is rewritten as below:

$$u_{k+1}(t) = u_{k+1}(t) + \rho_k e_{k-\tau_k^{t+1}}(t + 1) \tag{15}$$

It is to be mentioned that $\tau_k^{t+1} \le k$, and $e_i(t + 1)$ when $i > k - \tau_k^{t+1}$ is not available, namely just $e_{k-\tau_k^{t+1}}(t + 1)$ is accessible to update $u_{k+1}(t)$. Thus, in the ith-iteration, the updating law (15) successively updates the input $u_i(t)$ by the same error $e_{k-\tau_k^{t+1}}(t + 1)$ as $k - \tau_k^{t+1} < i \le k$.

Theorem 3.3. *Let update law (15) be used to the non-affine nonlinear stochastic networked system (1). If* $\left|1 - \rho_k C^+ D_k^2(t)\right| < 1$ *then for* $u_k(t)$ *generated by update law (15), we conclude that* $u_k(t) \to u_d(t)$ *w.p.1, for all t when* $k \to \infty$.

Proof. First, we specify that $k - \tau_k^t \xrightarrow[k\to\infty]{} \infty$ a.s., $\forall t$. Given that the data dropouts have the Bernoulli distribution, τ_k^t has a geometric distribution, namely $\tau_k^t \sim G(r)$. For each $T \sim G(r)$ with geometric distribution, we have $E(T)=\frac{1}{r}$, $\text{var}(T) = \frac{1-r}{r^2}$, and $E(T)^2 = \frac{1}{r}$. Also, we conclude that $\sum\limits_{m=1}^{\infty} P\left\{T \ge m^{\frac{1}{2}}\right\} = \sum\limits_{m=1}^{\infty} P\left\{T^2 \ge m\right\} = \sum\limits_{m=1}^{\infty}\sum\limits_{i=m}^{\infty} P\left\{i \le T^2 \le i + 1\right\} = \sum\limits_{i=1}^{\infty} iP\left\{i \le T^2 \le i + 1\right\}$

$\le E(T^2) < \infty$. Considering the Borel-Cantelli Lemma, we have $P\left\{T \ge m^{\frac{1}{2}} \ i.o.\right\} = 0$. Therefore, it concludes that $\lim\limits_{k\to\infty} \frac{\tau_k^t}{k} = 0$, $\forall t$, *a.s.*, namely $\lim\limits_{k\to\infty} \frac{k-\tau_k^t}{k} = 1$ and $\lim\limits_{k\to\infty} k - \tau_k^t = \infty$ a.s., $\forall t$.

According to (1) and (2), from (15) we conclude that

$$
\begin{aligned}
\delta u_{k+1}(t) &= \delta u_k(t) - \rho_k e_{k-\tau_k^{t+1}}(t+1) \\
&= \delta u_k(t) - \rho_k C(t+1)(x_d(t+1) - x_{k-\tau_k^{t+1}}(t+1)) \\
&\quad + \rho_k \zeta_{k-\tau_k^{t+1}}(t+1)
\end{aligned}
\tag{16}
$$

So, again from (1) and (2), we have that

$$
\begin{aligned}
\delta u_{k+1}(t) &= \delta u_k(t) - \rho_k C(t+1)(f_d(t) - f_{k-\tau_k^{t+1}}(t)) \\
&\quad + \rho_k \zeta_{k-\tau_k^{t+1}}(t+1) \\
&= \delta u_k(t) - \rho_k C(t+1)(D^1_{k-\tau_k^{t+1}}(t)\delta x_{k-\tau_k^{t+1}}(t) \\
&\quad + D^2_{k-\tau_k^{t+1}}(t)\delta u_{k-\tau_k^{t+1}}(t)) \\
&\quad + \rho_k \zeta_{k-\tau_k^{t+1}}(t+1)
\end{aligned}
\tag{17}
$$

Therefore, we have

$$
\begin{aligned}
\delta u_{k+1}(t) &= [1 - \rho_k C^+ D_k^2(t)]\delta u_k(t) \\
&\quad + \rho_k C^+ D_k^2(t)\delta u_k(t) \\
&\quad - \rho_k C(t+1)D^1_{k-\tau_k^{t+1}}(t)\delta x_{k-\tau_k^{t+1}}(t) \\
&\quad - \rho_k C^+ D^2_{k-\tau_k^{t+1}}(t)\delta u_{k-\tau_k^{t+1}}(t) \\
&\quad + \rho_k \zeta_{k-\tau_k^{t+1}}(t+1)
\end{aligned}
\tag{18}
$$

We take the norm from both sides of (18), we have

$$
\begin{aligned}
|\delta u_{k+1}(t)| &\le \left|1 - \rho_k C^+ D_k^2(t)\right| |\delta u_k(t)| \\
&\quad + \left|\rho_k\right| |C(t+1)| \left|D_k^2(t)\right| |\delta u_k(t)| \\
&\quad + \left|\rho_k\right| |C(t+1)| \left|D^2_{k-\tau_k^{t+1}}(t)\right| \left|\delta u_{k-\tau_k^{t+1}}(t)\right| \\
&\quad + \left|\rho_k\right| |C(t+1)| \left|D^1_{k-\tau_k^{t+1}}(t)\right| \left|\delta x_{k-\tau_k^{t+1}}(t)\right| \\
&\quad + \left|\rho_k\right| \left|\zeta_{k-\tau_k^{t+1}}(t+1)\right| \qquad \text{a.s.}
\end{aligned}
\tag{19}
$$

By using mathematical induction, it can prove $\lim_{k\to\infty} |\delta u_k(t)| = 0$.

Initial step. In (19), let $t = 0$

$$
\begin{aligned}
|\delta u_{k+1}(0)| &\le \left|1 - \rho_k C^+ D_k^2(0)\right| |\delta u_k(0)| \\
&\quad + \left|\rho_k\right| |C(1)| \left|D_k^2(0)\right| |\delta u_k(0)| \\
&\quad + \left|\rho_k\right| |C(1)| \left|D^2_{k-\tau_k^{t+1}}(0)\right| \left|\delta u_{k-\tau_k^1}(0)\right| \\
&\quad + \left|\rho_k\right| |C(1)| \left|D^1_{k-\tau_k^1}(0)\right| \left|\delta x_{k-\tau_k^1}(0)\right| \\
&\quad + \left|\rho_k\right| \left|\zeta_{k-\tau_k^1}(1)\right| \qquad \text{a.s.}
\end{aligned}
\tag{20}
$$

With respect to R5, for enough large k we conclude that $C^+ D_k^2(0) > \psi$, where ψ is an appropriate constant. Considering to R6, it concludes that $\left|\delta x_{k-\tau_k^1}(0)\right| \to 0$ when $k \to \infty$. Also, by R4, we conclude that $D^1_{k-\tau_k^1}(0)$

is bounded. Thus, in (20), concerning $\lim\limits_{k\to\infty} \rho_k = 0$ we conclude that $\left|\rho_k\right| |C(1)| \left|D^1_{k-\tau^1_k}(0)\right| \left|\delta x_{k-\tau^1_k}(0)\right| \to 0$ w.p.1, when $k \to \infty$. By R4, we conclude that $D^2_k(0)$ and $D^2_{k-\tau^1_k}(0)$ are bounded. In (20), concerning $\delta u_k(0)$ and $\delta u_{k-\tau^1_k}(0)$ are input error vectors therefore, their norm are bounded. Thus, concerning $\lim\limits_{k\to\infty} \rho_k = 0$ we conclude that $\left|\rho_k\right| |C(1)| \left|D^2_k(0)\right| |\delta u_k(0)| \to 0$ and $\left|\rho_k\right| |C(1)| \left|D^2_{k-\tau^{t+1}_k}(0)\right| \left|\delta u_{k-\tau^1_k}(0)\right| \to 0$ w.p.1, when $k \to \infty$. $\left|\zeta_{k-\tau^1_k}(1)\right|$ is bounded because of $\zeta_{k-\tau^1_k}(1)$ is continuous function white noise on $[0, N]$. Thus, concerning $\lim\limits_{k\to\infty} \rho_k = 0$ it results, $\left|\rho_k\right| \left|\zeta_{k-\tau^1_k}(1)\right| \to 0$, w.p.1, when $k \to \infty$.

With respect to Lemma (1) of paper [5], let $\sigma_1 = \left|1 - \rho_k C^+ D^2_k(0)\right|$, $\sigma_i = 0$, $i = 2, 3, ...$, $e_k = |\delta u_k(0)|$, and $\varphi_k = 0$, concerning $\left|1 - \rho_k C^+ D^2_k(0)\right| < 1$ and Lemma (1) of paper [5] from inequality (20), it has resulted $\lim\limits_{k\to\infty} |\delta u_k(0)| = 0$, w.p.1.

Inductive step. We assume that $\delta u_k(m) \to 0$ is correct for $m = 0, 1, ..., t - 1$, then we prove $\delta u_k(t) \to 0$ for $m = t$. In (19), concerning the assumption of inductive and Lemma (3.1), we have $\left|\delta x_{k-\tau^{t+1}_k}(t)\right| \to 0$ when $k \to \infty$. Also, by R4, we conclude that $D^1_{k-\tau^{t+1}_k}(t)$ is bounded. Thus, we conclude that $\left|\rho_k\right| |C(t + 1)| \left|D^1_{k-\tau^1_k}(t)\right| \left|\delta x_{k-\tau^{t+1}_k}(t)\right| \to 0$ w.p.1, when $k \to \infty$. Concerning $\delta u_k(t)$ and $\delta u_{k-\tau^1_k}(t)$ are input error vectors therefore, their norm are bounded. Similar to initial step, we can conclude that $\left|\rho_k\right| |C(t + 1)| \left|D^2_k(t)\right| |\delta u_k(t)| \to 0$, $\left|\rho_k\right| |C(t + 1)| \left|D^2_{k-\tau^{t+1}_k}(t)\right| \left|\delta u_{k-\tau^{t+1}_k}(t)\right| \to 0$, and $\left|\rho_k\right| \left|\zeta_{k-\tau^{t+1}_k}(t + 1)\right| \to 0$ w.p.1, when $k \to \infty$.

Concerning Lemma (1) of paper [5], let $\sigma_1 = \left|1 - \rho_k C^+ D^2_k(t)\right|$, $\sigma_i = 0$, $i = 2, 3, ...$, $e_k = |\delta u_k(t)|$, and $\varphi_k = 0$, by considering the assumption of Theorem (3.3) namely $\left|1 - \rho_k C^+ D^2_k(t)\right| < 1$, Lemma (1) of paper [5], and (19), we conclude that $\lim\limits_{k\to\infty} |\delta u_k(t)| = 0$, w.p.1. $\square$

4. Numerical example

A stochastic non-affine nonlinear networked system is considered as follows for indicating the desired convergence of the recommended models (4), (5) and (6), (7).

$$x^1_k(t + 1) = -0.5 \cos(t) \cos(x^1_k(t)) + x^1_k(t) \sin(x^2_k(t))$$
$$+ \frac{1}{3} \sin(\frac{x^1_k(t) + u_k(t)}{3}) u_k(t)$$
$$x^2_k(t + 1) = -0.75 \sin(t) \sin(x^2_k(t)) + 0.3 \cos(t) \cos(x^1_k(t)) \tag{21}$$
$$+ (0.5 + \frac{\cos(u_k(t))}{5}) u_k(t)$$
$$y_k(t) = 0.3 x^1_k(t) + 0.25 t^{0.25} x^2_k(t) + \zeta_k(t)$$

Where $\begin{bmatrix} x^1_k(t) \\ x^2_k(t) \end{bmatrix}$ is the state vector, $u_k(t)$ is the input, and $y_k(t)$ is the output of the system (21). $\zeta_k(t)$ is the measurement noise of the system (20) with normal distribution $N(0, 0.01^2)$. The desired output is demonstrated as $y_d(t) = 0.75 \sin(\frac{\pi}{15} t) + 0.5 \sin(\frac{\pi}{20} t)$, and the time interval is $[0, 60]$. $x^1_k(0) = x^2_k(0) = 0$ is the initial state. The initial iteration input signal is $u_1(t) = 0$. For evaluation of the tracking performance, the algorithm is implemented for 500 iterations. The average absolute tracking error of outputs in the kth iteration is $|e_k| = \left(\sum\limits_{l=1}^{N} |y_d(t) - y_k(t)|\right) / N$. In the following, convergence attributes of the recommended models and tracking performances concerning the different probabilities of the loss of data are investigated. To simplify, ILC update law (4), (5) is denoted by model I and ILC update law (6), (7) is denoted by model II.
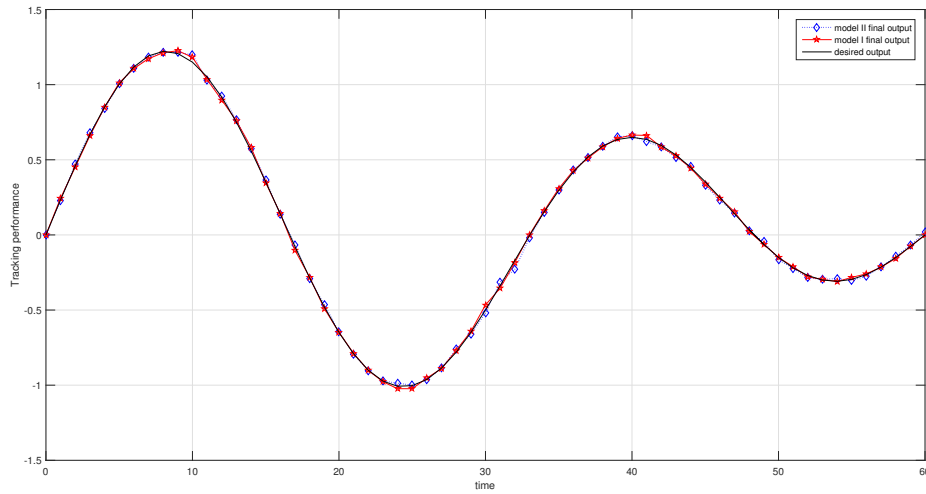
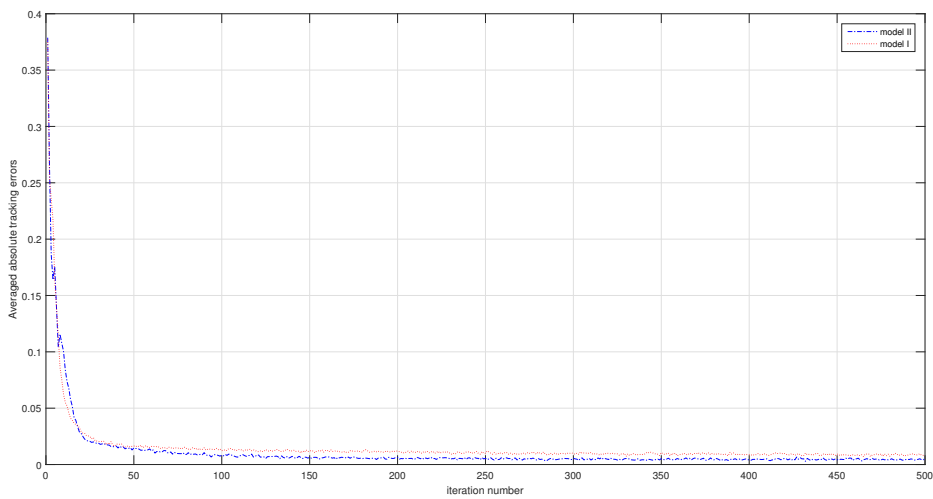Figure 2: Tracking performances of model I and model II for $r = 0.85$.

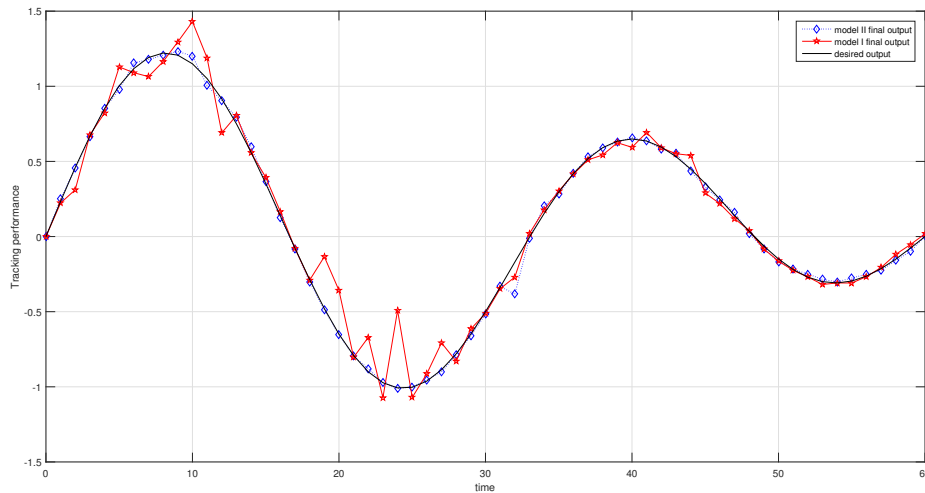Figure 3: Average absolute tracking errors of model I and model II for $r = 0.85$.

Figure 4: Tracking performances of model I and model II for $r = 0.3$.

Let first $r = 0.85$, in this mode, the probability of successful transmission of the outputs are 85%, and only 15% of the data are dropped out. In Figure 2 tracking performance of the system (20) for $r = 0.85$ is shown for both model I and model II. Figure 3 displays the average absolute tracking error of outputs for both model I and model II for $r = 0.85$. Results in Figure 2 and Figure 3 indicate that the model I and model II have good tracking performances and are effective. Moreover, both algorithms have almost similar performance, when the probability of successfully transmitting the data is high.

The other mode that we consider is $r = 0.30$. This means that 70% of the data are lost, and the probability of successful transmission of data is 30%. The final outputs of both model I and model II considering $r = 0.30$ are shown in Figure 4. The average absolute tracking error of outputs for both model I and model II considering $r = 0.30$ is plotted in Figure 5. As it is observed, the probability of successful transmission of outputs in algorithms in this mode is low. In this situation, the model II works better than the model I because more updates are performed in model II than model I over the same number of iterations. The reason for this is that in the model I if the corresponding data is dropped out, then the update of the control signal is not carried out, while in model II the update of the control signal is done, even if the corresponding data is lost, and the update is never stopped. Unlike Figure 3, in Figure 5, the model II has more updates due to the high probability of data dropout and larger learning gain so, the great increase of average absolute tracking error in the initial repetitions are generated. In the following, we set the different probabilities of the successful transmission of outputs, i.e, $r = 0.9, 0.7, 0.5$, and 0.3, for scrutiny of the impact of different probabilities of data dropout. As observed in Figure 6 and Figure 8, in the model II, almost a similar performance is preserved even if the probability of successful transmission of outputs decreases as iterations increase. Whereas in the model I, when the probability of successful transmission of outputs decreases, the tracking performance gets worse at the identical iterations, as observed in Figure 7 and Figure 9.

5. Conclusion

In this paper, the analysis of convergence, and the tracking performance scrutiny of two update laws for stochastic non-affine nonlinear networked systems were investigated. For these stochastic non-affine nonlinear networked systems in this paper, the random packet dropout formulated with random Bernoulli variables was considered in the measurement side. In ILC update law (6), (7), the algorithm updates by using the newest accessible output even if no output is transferred in the current iteration. In ILC update law
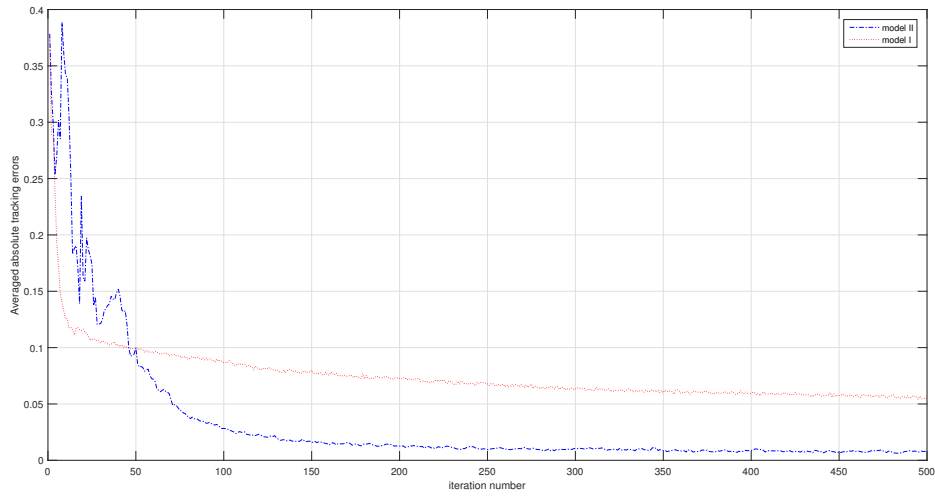
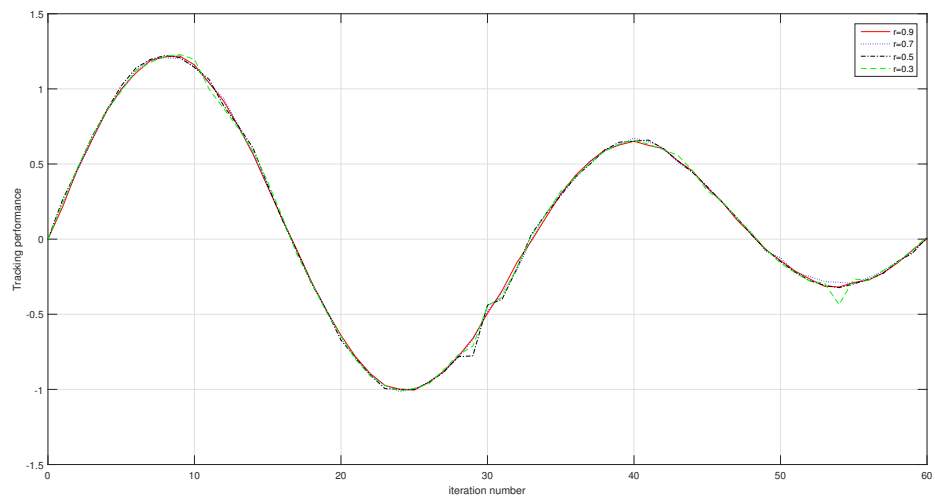Figure 5: Average absolute tracking errors of model I and model II for *r* = 0.3.

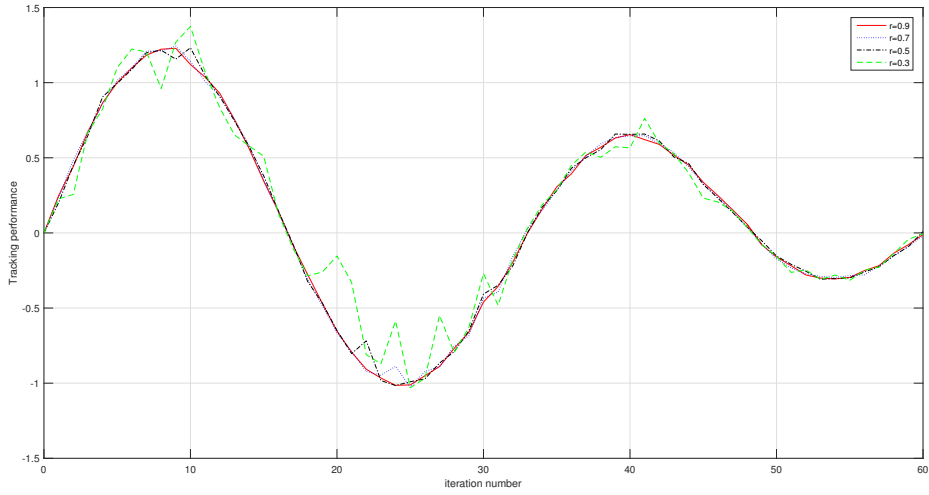Figure 6: Final outputs of model II for *r* = 0.9, *r* = 0.7, *r* = 0.5, *r* = 0.3.

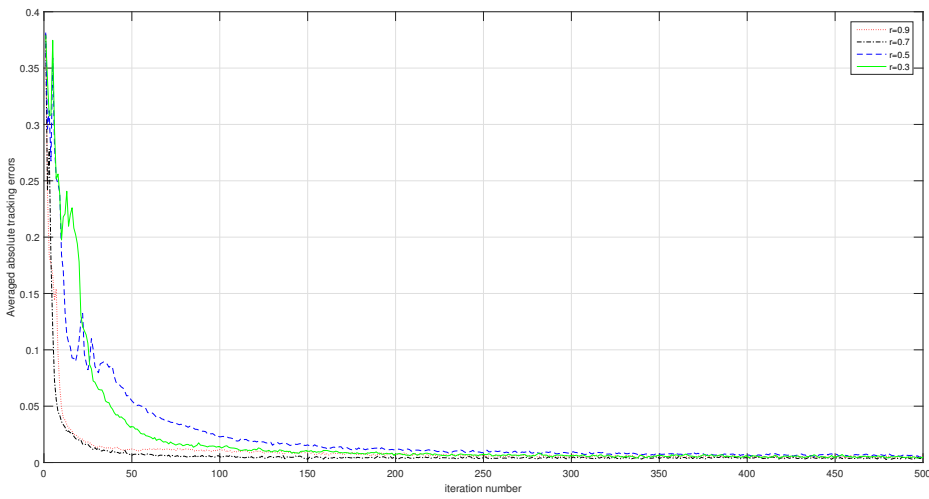Figure 7: Final outputs of model I for $r = 0.9, r = 0.7, r = 0.5, r = 0.3$.

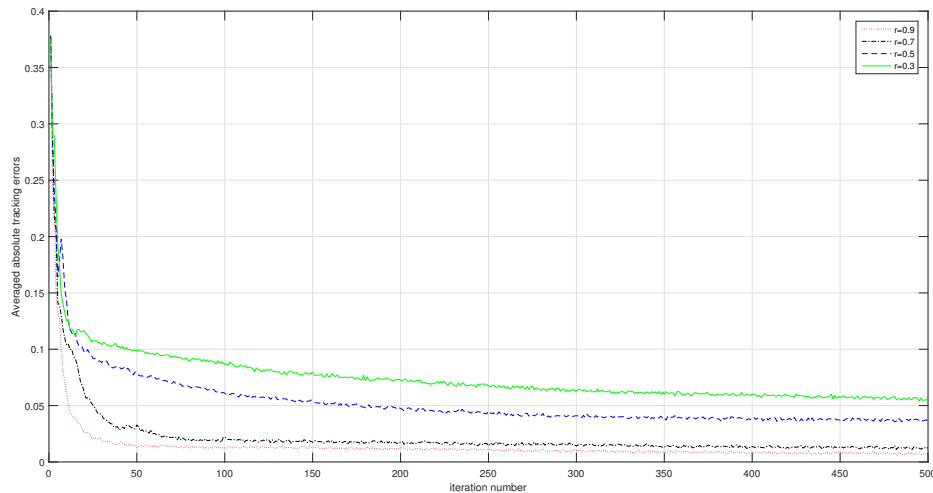Figure 8: Average absolute tracking error of model II for $r = 0.9, r = 0.7, r = 0.5, r = 0.3$.

Figure 9: Average absolute tracking error of model I for $r = 0.9, r = 0.7, r = 0.5, r = 0.3$.

(4), (5), if the data were not received, the update of the algorithm would stop. In the presence of stochastic measurement noise, the convergence analysis of the input error instead of the convergence analysis of the output error for ILC update law (4), (5) and ILC update law (6), (7) was investigated. As noted, there is no restrictive condition for the random data dropout probabilities in the convergence analysis of the input error. It was indicated that if $\left|1 - \rho_k C^+ D_k^2(t)\right| < 1$ then input error converges to zero in the almost sure concept in the ILC update laws (4), (5) and (6), (7). By a numerical example, the theoretical results were more testified. The advantage of ILC update law (6), (7) is keeping similar performance with different probabilities of data dropouts as iterations increase, while, in ILC update law (4), (5) by decreasing the probability of the successful transfer of outputs, the performance gets worse at the same iterations.

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