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Study of Matrix Transformation of Uniformly Almost Surely Convergent Complex Uncertain Sequences

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Abstract. In this paper, we introduce the concept of convergence of complex uncertain series. We initiate matrix transformation of complex uncertain sequence and extend the study via linearity and boundedness. In this context, we prove Silverman-Toeplitz theorem and Kojima-Schur theorem considering complex uncertain sequences. Finally, we establish some results on co-regular matrices .

1. Introduction

The notion of uncertain sequences and their convergences is introduced by Liu [2] in the year 2007 and then the study was extended by You [3]. To describe the complex uncertain quantities, the notions of complex uncertain variable and complex uncertain distribution are presented by Peng [15]. Chen et. al [14] explored the work considering the sequence of complex uncertain variables due to Peng [15]. They reported five convergence concepts of sequence of complex uncertain variables, namely convergence in almost surely, convergence in measure, convergence in mean, convergence in distribution and convergence with respect to uniformly almost surely by establishing interrelationships among them. Since its initiation, the study of complex uncertain sequences got full attention from researchers. These convergence concept of complex uncertain sequence has also been generalised by Datta and Tripathy [4], Debnath and Tripathy [11], Saha et. al [13]. The study of sequence space through matrices are very much relevant in the current research flow [1, 5]. Interest in general matrix transformation was first stimulated in some extent by special results in summability theory which were obtained by Cesaro, Borel and others. However, it was Toeplitz who first made a detailed study on matrix transformation on sequence spaces and then mathematicians made progress enormously in this particular direction [6, 9, 10]. In this current treatise, we define the convergence of complex uncertain series to study the matrix transformation of complex uncertain sequences. Though convergence of uncertain series will be defined via five aspects, namely convergence in mean, in measure, in distribution, in almost surely and with respect to uniformly almost surely, our study will be limited to only on convergence of an uncertain series with respect to uniformly almost surely. In this context, we also establsih the famous Silverman-Toeplitz theorem and Kojima-Schur theorem via complex uncertain sequences.

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Before going to the main section we need some basic and preliminary ideas about the existing definitions and results which will play a major role in this study.

2. Preliminaries

In this section, we procure some fundamental concepts and results on uncertainty theory, those will be used throughout the paper.

2.1. Definition [2]

Let $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:
Axiom 1 (Normality Axiom). $\mathcal{M}\{\Gamma\}$=1;
Axiom 2 (Duality Axiom). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\}$=1, for any $\Lambda \in \mathcal{L}$;
Axiom 3 (Subadditivity Axiom). For every countable sequence of events $\{\Lambda_j\} \in \mathcal{L}$, we have

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}(\Lambda_j).$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathcal{L}$ is called an event. In order to obtain an uncertain measure of compound events, a product uncertain measure is defined as follows:
Axiom 4 (Product Axiom). Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces, for k = 1,2,3..... The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{j=1}^{\infty} \Lambda_j\right\} = \bigwedge_{j=1}^{\infty} \mathcal{M}(\Lambda_j),$$

where Λ_k are arbitrarily chosen events from Γ_k, for k=1,2,3,. . . ... respectively.

2.2. Definition [15]

A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set $\{\zeta \in B\}$=$\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.

2.3. Definition [2]

The expected value operator of an uncertain variable ζ is defined by

$$E[\zeta] = \int_0^{+\infty} \mathcal{M}\{\zeta \geq r\}dr - \int_{-\infty}^{0} M\{\zeta \leq r\}dr,$$

provided that at least one of the two integrals is finite.

2.4. Definition [14]

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent almost surely (a.s.) to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\}$=1 such that

$$\lim_{n\to\infty} ||\zeta_n(\gamma) - \zeta(\gamma)|| = 0, \text{ for every } \gamma \in \Lambda.$$

2.5. Definition [14]

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent in measure to ζ if for any $\varepsilon \geq 0$,

$$\lim_{n\to\infty} \mathcal{M}\{||\zeta_n - \zeta)|| \geq \varepsilon\} = 0.$$

2.6. Definition [14]

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent in mean to ζ if

$$\lim_{n\to\infty} E[||\zeta_n - \zeta||] = 0.$$

2.7. Definition [14]

Let $\Phi, \Phi_1, \Phi_2, \Phi_3, \ldots\ldots$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, \zeta_3\ldots..$ respectively. Then the complex uncertain sequence $\{\zeta_n\}$ is convergent in distribution to ζ if

$$\lim_{n\to\infty} \Phi_n(c) = \Phi(c),$$

for all $c \in \mathbb{C}$, at which Φ is continuous.

2.8. Definition [14]

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent uniformly almost surely (u.a.s.) to ζ if there exists a sequence of events $\{E_k\}$, with $\mathcal{M}\{E_k\} \to 0$ such that $\{\zeta_n\}$ converges uniformly to ζ in $\Gamma - \{E_k\}$, for any fixed $k \in \mathbb{N}$.

2.9. Lemma[7]

If $m \neq 0$, a multiplicative m matrix cannot sum all bounded sequences. The result does not hold if $m = 0$.

Throughout the article, the family of all convergent complex uncertain sequence in mean, in measure, in distribution, in almost surely and with respect to uniformly almost surely is denoted by $c(\Gamma_E), c(\Gamma_{\mathcal{M}}), c(\Gamma_{\mathcal{D}})$, $c(\Gamma_{a.s}), c(\Gamma_{u.a.s})$ respectively. Similarly, the collection of all null sequences in mean, in measure, in distribution, in almost surely and with respect to uniformly almost surely is denoted by $c_0(\Gamma_E), c_0(\Gamma_{\mathcal{M}}), c_0(\Gamma_{\mathcal{D}}), c_0(\Gamma_{a.s})$ and $c_0(\Gamma_{u.a.s})$ respectively.

3. Convergence of Complex Uncertain Series

Chen et. al [14] initiated five types of convergences of complex uncertain sequences. In this section, we introduce convergence of complex uncertain series which are going to very useful to study the matrix transformation of complex uncertain sequences.

3.1. Definition

Suppose that $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An infinite complex uncertain series $\sum_{k=1}^{\infty} \zeta_k(\gamma)$ is said to be convergent in mean if the sequence of its partial sums $\{S_n(\gamma)\}$ is convergent in mean to S, where $S_n(\gamma) = \sum_{k=1}^{n} \zeta_k(\gamma)$, for any event $\gamma \in \Gamma$. That is,

$$\lim_{n\to\infty} E[||S_n(\gamma) - S(\gamma)||] = 0.$$

3.2. Definition

An infinite complex uncertain series $\sum_{k=1}^{\infty} \zeta_k(\gamma)$ is said to be convergent in measure if $\{S_n(\gamma)\}$, the sequence of its partial sums is convergent to S in measure, for all event $\gamma \in \Gamma$. Then, for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathcal{M}\{||S_n - S|| \geq \varepsilon\} = 0.$$

3.3. Definition

An infinite complex uncertain series $\sum_{k=1}^{\infty} \zeta_k(\gamma)$ is said to be convergent in distribution if $\{S_n(\gamma)\}$, the sequence of partial sums is convergent in distribution to S, for all event $\gamma \in \Gamma$. If Ψ_n and Ψ are the distribution functions of $||S_n||$ and $||S||$ then,

$$\lim_{n\to\infty} \Psi_n(c) = \Psi(c),$$

for all $c \in \mathbb{C}$, at which Ψ is continuous.

3.4. Definition

An infinite complex uncertain series $\sum_{k=1}^{\infty} \zeta_k(\gamma)$ is said to be convergent to S in almost surely if $\{S_n(\gamma)\}$, where $\gamma \in \Gamma$ is any event, is convergent in almost surely. In this case, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} \|S_n(\gamma) - S(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$.

3.5. Definition

An infinite complex uncertain series $\sum_{k=1}^{\infty} \zeta_k(\gamma)$ is said to be convergent with respect to uniformly almost surely to S if the sequence of partial sums is convergent with respect to uniformly almost surely. Then, there exists a sequence of events $\{E_k\}$, where $\mathcal{M}\{E_k\} \to 0$ such that $\{S_n\}$ converges uniformly to S in $\Gamma - \{E_k\}$, for any fixed $k \in \mathbb{N}$.

In this treatise, we will use the concept of convergence with respect to uniformly almost surely of a complex uncertain series whenever necessary.

4. Matrix Transformation of Complex Uncertain Sequences

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix}$ be a real infinite matrix and $\zeta = \{\zeta_n\}$ be a complex uncertain sequence which converges with respect to uniformly almost surely.
We apply A to ζ as an operator defined by simple matrix multiplication as follows:
Let us consider a collection of uncertain events $B = \{E_k\}$ with uncertain measure of each of the events tending to zero. Then,

$$A\zeta(\gamma) = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \zeta_1(\gamma) \\ \zeta_2(\gamma \\ \dots \end{pmatrix},$$
$$= \begin{pmatrix} a_{11}\zeta_1(\gamma) + a_{12}\zeta_2(\gamma) + \dots \\ a_{21}\zeta_1(\gamma) + a_{22}\zeta_2(\gamma) + \dots \\ \dots \quad \dots \quad \dots \end{pmatrix}. \ \forall\, \gamma \in \Gamma - \{E_k\} \text{ and uniformly for each } k.$$

We write $(A\zeta)_n \equiv A_n(\zeta)$ and it is given by $A_n(\zeta) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$, provided that the infinite series converges with respect to uniformly almost surely, for each n.
Thus A is said to be an operator which maps the complex uncertain sequence ζ into $A\zeta$.

We now find a necessary and sufficient condition under which an infinite matrix operator transforms a uniformly almost surely convergent complex uncertain sequence into another complex uncertain sequence of same type.

4.1. Theorem

Let $\lim_{n\to\infty} a_{nk} \to 0$ (uniformly for all $k \in \mathbb{N}$) and $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$ be finite. Then A is said to be a bounded linear operator on $c_0(\Gamma_{u.a.s})$ into itself and $\|A\| = M$.

Proof: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $\{\zeta_n(\gamma)\} \in c_0(\Gamma_{u.a.s})$. Then for any $\delta > 0$, there exists a sequence of uncertain events $\{E_l\}$ with $\mathcal{M}(E_l) < \delta$, for each l, such that $\{\zeta_n(\gamma)\}$ uniformly converges to $\zeta(\gamma) = 0$ in $\Gamma - \{E_l\}$. i.e., for any $\varepsilon > 0$, there exists $k > 0$ such that $\|\zeta_n(\gamma)\| < \varepsilon$, for all $\gamma \in \Gamma - \{E_l\}$ and $n \geq k$. At first we show that

$A\zeta(\gamma) \in c_0(\Gamma_{u.a.s})$, which implies that the complex uncertain series $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ is absolutely convergent with respect to uniformly almost surely for each n.
Now, for any $m \geq 1$ and $\gamma \in \Gamma - \{E_l\}$,

$$\begin{aligned} \|A_n(\zeta(\gamma))\| &= \sum_{k=1}^{\infty} \|a_{nk}\zeta_k(\gamma)\| \\ &\leq \sum_{k=1}^{m} \|a_{nk}\zeta_k(\gamma)\| + \sum_{k=m+1}^{\infty} \|a_{nk}\zeta_k(\gamma)\| \\ &\leq \|\zeta_k(\gamma)\| \sum_{k=1}^{m} |a_{nk}| + \max_{k\geq m+1} \|\zeta_k(\gamma)\| M. \end{aligned}$$

Take m and n so large that for any arbitrary small $\varepsilon > 0$, $\max\{\|\zeta_k(\gamma)\| : k \geq m+1, \gamma \in \Gamma - \{E_l\}\} < \varepsilon$ and $\sum_{k=1}^{m} |a_{nk}| < \varepsilon$, since $a_{nk} \to 0$ as $n \to \infty$ (k fixed).
Therefore, $A(\zeta(\gamma)) \in c_0(\Gamma_{u.a.s})$ and hence A defines an operator from $c_0(\Gamma_{u.a.s})$ into $c_0(\Gamma_{u.a.s})$.

To prove A is a linear operator, consider a scalar λ and then

$$\begin{aligned} A(\lambda\zeta^1(\gamma)+\zeta^2(\gamma)) &= \sum_{k=1}^{\infty} (a_{nk}(\lambda\zeta_k^1(\gamma)+\zeta_k^2(\gamma)), \quad \zeta^1(\gamma) \text{ and } \zeta^2(\gamma) \text{ are complex uncertain sequences}, n \in \mathbb{N}, \gamma \in \Gamma-\{E_l\} \\ &= \lambda \sum_{k=1}^{\infty} (a_{nk}\zeta_k^1(\gamma)) + \sum_{k=1}^{\infty} (a_{nk}\zeta_k^2(\gamma)), \\ &= \lambda A(\zeta^1(\gamma)) + A(\zeta^2(\gamma). \end{aligned}$$

Therefore, A is linear.
Also, for any uncertain event $\gamma \in \Gamma - \{E_l\}$,

$$\begin{aligned} \|A(\zeta(\gamma))\| &= \sup_n \| \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)\| \\ &\leq \|\zeta(\gamma)\| \sup_n \sum_{k=1}^{\infty} |a_{nk}| \\ &= M\|\zeta(\gamma)\|, \quad \text{for every } \zeta \in c_0(\Gamma_{u.a.s}). \end{aligned}$$

Hence, $\|A\| \leq M, \quad \forall\, \zeta \in c_0(\Gamma_{u.a.s})$ and so A is bounded.

For the reverse inequality, there exists $n = m(\varepsilon)$ such that $\sum_{k=1}^{\infty} |a_{mk}| > M - \frac{\varepsilon}{2}$ and since $\sum_{k=1}^{\infty} |a_{mk}|$ is finite, there exists $p = p(\varepsilon)$ such that $\sum_{k>p} |a_{mk}| < \frac{\varepsilon}{2}$.
For all $\gamma \in \Gamma - \{E_l\}$, define the uncertain null sequence $\zeta = \{\zeta_k\}$ with respect to uniformly almost surely by

$$\zeta_k(\gamma) = \begin{cases} sgn\ a_{nk} & 1 \leq k \leq p; \\ 0 & k > p. \end{cases}$$

Then $\|\zeta(\gamma)\| = 1$ and

$$\begin{aligned} \|A(\zeta(\gamma))\|/\|\zeta_(\gamma)\| &= \sup_n \|A_n(\zeta(\gamma))\| \\ &\geq \|A_n(\zeta(\gamma))\| \\ &> M - \varepsilon. \end{aligned}$$

It implies that $M = \sup\{\|A(\zeta(\gamma))\|/\|\zeta(\gamma)\| : \zeta(\gamma) \neq 0\} = \|A\|$.

4.2. Theorem

Let $A : c_0(\Gamma_{u.a.s}) \to c_0(\Gamma_{u.a.s})$ be any bounded linear operator. Then A determines a matrix (a_{mn}) such that $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$, for every $\gamma \in \Gamma - \{E_l\}$ and $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$. Also, $\lim_{n\to\infty} a_{nk} = 0$, uniformly for all k.

Proof: Let $\zeta \in c_0(\Gamma_{u.a.s})$. Then $\zeta(\gamma) = \sum_{k=1}^{\infty} (\zeta_k(\gamma)e_k)$, where $\{e_n\}$ is a basis in $c_0(\Gamma_{u.a.s})$, which is given by

$$e_n(\gamma) = \begin{cases} 1 & if\ n = k; \\ 0 & otherwise; \end{cases}$$

Now $A\zeta(\gamma) = \sum_{k=1}^{\infty} \zeta_k(\gamma)Ae_k$

$= \sum_{k=1}^{\infty} \zeta_k(\gamma)(a_k^n),\ n \in \mathbf{N}$,

where Ae_k is a sequence $\{a_k^{(1)}, a_k^{(2)}, ...\} \in c_0(\Gamma_{u.a.s})$; $\quad k = 1, 2, 3, ...$

Then, $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_k^{(n)}\zeta_k(\gamma), \quad n = 1, 2, ...$

Since each $e_k \in c_0(\Gamma_{u.a.s})$, therefore $Ae_k \in c_0(\Gamma_{u.a.s})$ also, for $k = 1, 2, 3, ...$

That implies, $\lim_{n\to\infty} a_{nk} = 0$, keeping k fixed.

Thus, $\lim_{n\to\infty} A_n\zeta(\gamma) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = 0$.

We now prove that $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$.

For each n, we have, $\|A_n\zeta(\gamma)\| \le \|A\zeta(\gamma)\| \le \|A\|\|\zeta\|$, where $\gamma \in \Gamma - \{E_l\}$.

Since A is a bounded linear operator and $\zeta \in c_0(\Gamma_{u.a.s})$, then A_n is a bounded linear functional on $c_0(\Gamma_{u.a.s})$.

Thus we have the sequence $\{A_n\} \in c_0^*(\Gamma_{u.a.s})$ such that $\lim_{n\to\infty} A_n(\zeta(\gamma)) = 0$, where $c_0^*(\Gamma_{u.a.s})$ is the dual space.

Then, by Banach-Steinhaus theorem, for all n, $\|A_n\| \le H$, for some constant H.

By the table of dual spaces in page 110 of [8], $\|A_n\| = \sum_k |a_{nk}|$.

Then $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and by the above theorem $\|A\| = M$.

4.3. Definition

A complex uncertain sequence $\{\zeta_n\}$ in an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be bounded with respect to uniformly almost surely if for any $\varepsilon > 0$, there exists events E_k $(k \in \mathbb{N})$, each of which tending to zero uncertain measure such that

$$\sup_n \|\zeta_n(\gamma)\| < \infty, \text{ for all } \gamma \in \Gamma - \{E_k\}.$$

The set of all such types of sequences is denoted by $\ell_\infty(\Gamma_{u.a.s})$.

4.4. Theorem

The necessary and sufficient condition for an infinite bounded matrix operator $A = (a_{nk})$ from $\ell_\infty(\Gamma_{u.a.s})$ into $\ell_\infty(\Gamma_{u.a.s})$ is that

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty.$$

Proof: Let $\zeta = \{\zeta_k\} \in \ell_\infty(\Gamma_{u.a.s})$. Then, there exists some events E_l ($k \in \mathbb{N}$) with $\mathcal{M}(E_l) \to 0$ such that $\sup_n \|\zeta_n(\gamma)\| < \infty$, for all $\gamma \in \Gamma - \{E_l\}$.

Suppose $A = (a_{nk})$ be an infinite bounded matrix operator such that $\sup_n \sum_{k=1}^{\infty} |a_{nk}|$ is finite.

Since, $A = (a_{nk})$ is bounded uniformly for each n, $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ exists, for all $\gamma \in \Gamma - \{E_l\}$.

Then, $\sup_n \|(A\zeta)_n\| = \sup_n \|\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)\|$, uniformly for all n

$\leq \|\zeta(\gamma)\| \sup_n \sum_{k=1}^{\infty} |a_{nk}|$

$< \infty$, (since $\zeta \in \ell_\infty(\Gamma_{u.a.s})$).

Therefore, $A\zeta \in \ell_\infty(\Gamma_{u.a.s})$.

Hence, A defines a bounded linear operator from $\ell_\infty(\Gamma_{u.a.s})$ into $\ell_\infty(\Gamma_{u.a.s})$.

Conversely, let $A \in (\ell_\infty(\Gamma_{u.a.s}), \ell_\infty(\Gamma_{u.a.s}))$. That is A transforms a complex uncertain sequence $\zeta \in \ell_\infty(\Gamma_{u.a.s})$ to another sequence $A\zeta \in \ell_\infty(\Gamma_{u.a.s})$.

This implies

$$\sup_n \|(A\zeta)_n\| = \sup_n \|\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)\| < \infty.$$

Then, by Banach-Steinhaus theorem, we have $\sup_n \|A_n\| = \sup_n |a_{nk}| < \infty$.

4.5. Corollary

$A : c_0(\Gamma_{u.a.s}) \to \ell_\infty(\Gamma_{u.a.s})$ is a bounded linear operator if $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

Proof: Let $\zeta = \{\zeta_k\} \in c_0(\Gamma_{u.a.s})$. Then, for any $\varepsilon > 0$, there exists a sequence of events $\{E_l\}$ with $\mathcal{M}(E_l) \to 0$ and $k \in \mathbb{N}$ such that $\|\zeta_n(\gamma) - 0\| < \varepsilon$, for all $n \geq k$ and $\gamma \in \Gamma - \{E_l\}$, that is $\|\zeta_n(\gamma)\| < \varepsilon$.

Let $H = \sup\{\|\zeta_1(\gamma)\|, \|\zeta_2(\gamma)\|, \ldots.., \|\zeta_k(\gamma)\|, \varepsilon\}$, for all $\gamma \in \Gamma - \{E_l\}$. Thus, $\|\zeta_n(\gamma)\| < H$, whenever $\gamma \in \Gamma - \{E_l\}$.

Therefore, $\zeta = \{\zeta_n\} \in \ell_\infty(\Gamma_{u.a.s})$. Hence,by theorem 4.4, we have $A\zeta \in \ell_\infty(\Gamma_{u.a.s})$.

4.6. Corollary

$A \in (c(\Gamma_{u.a.s}), \ell_\infty(\Gamma_{u.a.s}))$ is a bounded linear operator if $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

Proof: Let $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$. Since, every uniformly almost surely convergent complex uncertain sequence is bounded with respect to uniformly almost surely. Therefore, $\zeta \in c(\Gamma_{u.a.s}) \Rightarrow \zeta \in \ell_\infty(\Gamma_{u.a.s})$ and so $A\zeta \in \ell_\infty(\Gamma_{u.a.s})$, by theorem 4.4. Hence, $A \in (c(\Gamma_{u.a.s}), \ell_\infty(\Gamma_{u.a.s}))$.

4.7. Corollary

Theorem 4.2 is true for each bounded linear matrix operator between $(c_0(\Gamma_{u.a.s}), c(\Gamma_{u.a.s}))$, that is if $A : c_0(\Gamma_{u.a.s}) \to c(\Gamma_{u.a.s})$ is a bounded linear operator then $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

Proof: Let the matrix operator $A = (a_{nk}) \in (c_0(\Gamma_{u.a.s}), c(\Gamma_{u.a.s}))$. Then, for any complex uncertain sequence $\zeta \in c_0(\Gamma_{u.a.s})$, $A\zeta \in c(\Gamma_{u.a.s})$ and so $A\zeta \in \ell_\infty(\Gamma_{u.a.s})$. Moreover, $\zeta \in \ell_\infty(\Gamma_{u.a.s})$. Hence, $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ is the direct consequence of theorem 4.4.

4.8. Corollary

If $A \in (c(\Gamma_{u.a.s}), c(\Gamma_{u.a.s}))$, then also $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

Proof: Let $\zeta = \{\zeta_n\} \in c(\Gamma_{u.a.s})$ be a complex uncertain sequence which converges to ζ and $(A\zeta)_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ exists. Then, by the hypothesis $A\zeta \in c(\Gamma_{u.a.s})$ and let it converges to some finite limit. The existence of $A\zeta$ for each n and $\zeta \in c(\Gamma_{u.a.s})$ proves the boundedness of $\sum_{k=1}^{\infty} |a_{nk}|$, uniformly for all n and hence the result follows. Also, by Banach-Steinhaus theorem, $\|A_n\| = \sum_{k=1}^{\infty} |a_{nk}|$.

4.9. Theorem

Suppose $\lim_{n\to\infty} a_{nk} = 0$, uniformly for all $k \in \mathbb{N}$ and $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$. Then one may call A, a bounded linear operator on $c_0(\Gamma_E)$ into itself and $\|A\| = M$.

Proof: Consider an uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$ and $\{\zeta_n(\gamma)\} \in c_0(\Gamma_E)$. We first show that $A\zeta(\gamma) \in c_0(\Gamma_E)$, that is $A_n(\zeta(\gamma))$ converges to 0 in means, as $n \to \infty$.

This is true when the complex uncertain series $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ is absolutely convergent in mean for each n.

Now, for any $m \geq 1$, we have

$$\begin{aligned}
E\left[\|A_n(\zeta(\gamma))\|\right] &= E\left[\sum_{k=1}^{\infty} \|a_{nk}\zeta_k(\gamma)\|\right] \\
&\leq E\left[\sum_{k=1}^{m} \|a_{nk}\zeta_k(\gamma)\|\right] + E\left[\sum_{k=m+1}^{\infty} \|a_{nk}\zeta_k(\gamma)\|\right] \\
&\leq E\left[\|\zeta_k(\gamma)\| \sum_{k=1}^{m} |a_{nk}|\right] + E\left[\max_{k\geq m+1} \|\zeta_k(\gamma)\| M\right] \\
&= E\left[\|\zeta_k(\gamma)\|\right] \sum_{k=1}^{m} |a_{nk}| + M \max_{k\geq m+1} E\left[\|\zeta_k(\gamma)\|\right].
\end{aligned}$$

Take m and n so large that for any arbitrary small $\varepsilon > 0$, $\max\{E[\|\zeta_k(\gamma)\| : k \geq m+1]\} < \varepsilon$ and $\sum_{k=1}^{m} |a_{nk}| < \varepsilon$, since $a_{nk} \to 0$ as $n \to \infty$ (k fixed).

Therefore, $A(\zeta(\gamma)) \in c_0(\Gamma_E)$ and hence A defines an operator from $c_0(\Gamma_E)$ into $c_0(\Gamma_E)$.

Now, for any scalar λ, we have

$$\begin{aligned}
A(\lambda\zeta^1(\gamma) + \zeta^2(\gamma)) &= \sum_{k=1}^{\infty} (a_{nk}(\lambda\zeta_k^1(\gamma) + \zeta_k^2(\gamma)), \quad \text{where } \zeta^1(\gamma) \text{ and } \zeta^2(\gamma) \text{ are complex uncertain sequences, } n \in \mathbf{N} \\
&= \lambda \sum_{k=1}^{\infty} (a_{nk}\zeta_k^1(\gamma)) + \sum_{k=1}^{\infty} (a_{nk}\zeta_k^2(\gamma)), \\
&= \lambda A(\zeta^1(\gamma)) + A(\zeta^2(\gamma)).
\end{aligned}$$

Therefore, A is linear.

Again,

$$\begin{aligned}
\|A(\zeta(\gamma))\| &= \sup_n \| \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)\| \\
&\leq \|\zeta(\gamma)\| \sup_n \sum_{k=1}^{\infty} |a_{nk}| \\
&= M\|\zeta(\gamma)\|, \quad \text{for every } \zeta \in c_0(\Gamma_E).
\end{aligned}$$

Hence, $\|A\| \leq M$, $\forall\, \zeta \in c_0(\Gamma_E)$ and so A is bounded.

For the reverse inequality, there exists $n = n_0$ such that $\sum_{k=1}^{\infty} |a_{mk}| > M - \frac{\varepsilon}{2}$, for all $m > n_0$ and since $\sum_{k=1}^{\infty} |a_{mk}|$ is finite, there exists $p = p_0$ such that $\sum_{k>p} |a_{mk}| < \frac{\varepsilon}{2}$, for all $p > p_0$.

For all $\gamma \in \Gamma$, we define $\zeta = \zeta_k \in c_0(\Gamma_E)$ by

$$\zeta_k(\gamma) = \begin{cases} sgn\ a_{nk} & 1 \le k \le p; \\ 0 & k > p. \end{cases}$$

Then $\|\zeta(\gamma)\| = 1$ and

$$\begin{aligned} \|A(\zeta(\gamma))\|/\|\zeta(\gamma)\| &= \sup_n \|A_n(\zeta(\gamma))\| \\ &\ge \|A_n(\zeta(\gamma))\| \\ &> M - \varepsilon. \end{aligned}$$

This implies, $M = \sup\{\|A(\zeta(\gamma))\|/\|\zeta(\gamma)\| : \zeta(\gamma) \ne 0\} = \|A\|$.

4.10. Theorem

Let A be any bounded linear operator defined on $c_0(\Gamma_E)$ into itself. Then A determines a matrix (a_{mn}) such that $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$, for every $\gamma \in c_0(\Gamma_E)$ and $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.
Also, $a_{nk} \to 0$, as $n \to \infty$ (keeping k fixed).

Proof: Let $\zeta \in c_0(\Gamma_E)$. Then $\zeta(\gamma) = \sum_{k=1}^{\infty}(\zeta_k(\gamma)e_k)$, where $\{e_k\}$ is a basis in $c_0(\Gamma_E)$, which is given by

$$e_n(\gamma) = \begin{cases} 1 & n = k; \\ 0 & otherwise; \end{cases}$$

$$\begin{aligned} \text{Now, } A\zeta(\gamma) &= \sum_{k=1}^{\infty} \zeta_k(\gamma)Ae_k \\ &= \sum_{k=1}^{\infty} \zeta_k(\gamma)(a_k^n)_{n\in\mathbf{N}}, \text{ where } Ae_k \text{ is a sequence } \{a_k^{(1)}, a_k^{(2)}, ...\} \in c_0(\Gamma_E); k = 1, 2, 3, ... \end{aligned}$$

Then, we obtain, $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_k^{(n)}\zeta_k(\gamma), \ \ n = 1, 2, ...$
Since each $e_k \in c_0(\Gamma_E)$, therefore $Ae_k \in c_0(\Gamma_E)$ also, for $k = 1, 2, 3, ...$
That implies, $a_{nk} \to 0$ as $n \to 0$, keeping k fixed.
Thus, $\lim_{n\to\infty} A_n\zeta(\gamma) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = 0$.
We are to prove that $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$.
Now for each n, $\|A_n\zeta(\gamma)\| \le \|A\zeta(\gamma)\| \le \|A\|\|\zeta\|$.
Since A is a bounded linear operator and $\zeta \in c_0(\Gamma_E)$, then A_n is a bounded linear functional on $c_0(\Gamma_E)$. Thus we have the sequence $\{A_n\} \in c_0^*(\Gamma_E)$ such that $\lim_{n\to\infty} A_n(\zeta(\gamma)) = 0$.
Then, by Banach-Steinhaus theorem, for all n, $\|A_n\| \le H$, for some constant H.
From the table of dual spaces in page 110 of [8], $\|A_n\| = \sum_{k=1}^{\infty} |a_{nk}|$.
Then $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and by the above theorem $\|A\| = M$.

4.11. Theorem

Let $\lim_{n\to\infty} a_{nk} = 0$ (uniformly for all $k \in \mathbb{N}$) and $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$ to be finite. Then A is said to be a bounded linear operator on $c_0(\Gamma_{a.s})$ into itself and $\|A\| = M$.

Proof: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $\{\zeta_n(\gamma)\} \in c_0(\Gamma_{a.s})$. Then for any $\delta > 0$, there exists uncertain event Λ with $\mathcal{M}(\Lambda) < \delta$, such that $\{\zeta_n(\gamma)\}$ converges to $\zeta(\gamma) = 0$ in Λ.
i.e., for any $\varepsilon > 0$, there exists $k > 0$ such that $\|\zeta_n(\gamma)\| < \varepsilon$, for all $\gamma \in \Lambda$ and $n \ge k$. We now show that

$A\zeta(\gamma) \in c_0(\Gamma_{a.s})$, which implies that the complex uncertain series $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ is absolutely convergent with respect to almost surely for each n.
Now, for any $m \geq 1$ and $\gamma \in \Lambda$,

$$\begin{aligned}\|A_n(\zeta(\gamma))\| &= \sum_{k=1}^{\infty} \|a_{nk}\zeta_k(\gamma)\| \\ &\leq \sum_{k=1}^{m} \|a_{nk}\zeta_k(\gamma)\| + \sum_{k=m+1}^{\infty} \|a_{nk}\zeta_k(\gamma)\| \\ &\leq \|\zeta_k(\gamma)\| \sum_{k=1}^{m} |a_{nk}| + \max_{k\geq m+1} \|\zeta_k(\gamma)\| M\end{aligned}$$

Take m and n so large that for any arbitrary small $\varepsilon > 0$, $\max\{\|\zeta_k(\gamma)\| : k \geq m+1, \gamma \in \Lambda\} < \varepsilon$ and $\sum_{k=1}^{m} |a_{nk}| < \varepsilon$, since $a_{nk} \to 0$ as $n \to \infty$ (k fixed).
Therefore, $A(\zeta(\gamma)) \in c_0(\Gamma_{a.s})$ and hence A defines an operator from $c_0(\Gamma_{a.s})$ into $c_0(\Gamma_{a.s})$.

To prove A is a linear operator, consider a scalar c and then
$A(c\zeta^1(\gamma)+\zeta^2(\gamma)) = \sum_{k=1}^{\infty} (a_{nk}(c\zeta^1_k(\gamma)+\zeta^2_k(\gamma))$, where $\zeta^1(\gamma)$ and $\zeta^2(\gamma)$ are complex uncertain sequences, $n \in \mathbb{N}, \gamma \in \Lambda$
$= c\sum_{k=1}^{\infty} (a_{nk}(\zeta^1_k(\gamma)) + \sum_{k=1}^{\infty} (a_{nk}(\zeta^2_k(\gamma))$,
$= cA(\zeta^1(\gamma)) + A(\zeta^2(\gamma))$.
Therefore, A is linear.
Also, for any uncertain event $\gamma \in \Lambda$,

$$\|A(\zeta(\gamma))\| = \sup_n \|\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)\| \leq \|\zeta(\gamma)\| \sup_n \sum_{k=1}^{\infty} |a_{nk}| = M\|\zeta(\gamma)\|, \text{ for every } \zeta \in c_0(\Gamma_{a.s}).$$

Hence, $\|A\| \leq M$, $\forall\, \zeta \in c_0(\Gamma_{a.s})$ and so A is bounded.

For the reverse inequality, there exists $n = m(\varepsilon)$ such that $\sum_{k=1}^{\infty} |a_{mk}| > M - \frac{\varepsilon}{2}$ and since $\sum_{k=1}^{\infty} |a_{mk}|$ is finite, there exists $p = p(\varepsilon)$ such that $\sum_{k>p} |a_{mk}| < \frac{\varepsilon}{2}$.
For all $\gamma \in \Lambda$, define the uncertain null sequence $\zeta = \{\zeta_k\}$ with respect to almost surely by

$$\zeta_k(\gamma) = \begin{cases} sgn\ a_{nk} & 1 \leq k \leq p; \\ 0 & k > p. \end{cases}$$

Then $\|\zeta(\gamma)\| = 1$ and $\|A(\zeta(\gamma))\|/\|\zeta_(\gamma)\| = \sup_n \|A_n(\zeta(\gamma))\| \geq \|A_n(\zeta(\gamma))\| > M - \varepsilon$.

It implies that $M = \sup\{\|A(\zeta(\gamma))\|/\|\zeta(\gamma)\| : \zeta(\gamma) \neq 0\} = \|A\|$.
Hence the theorem.

4.12. Theorem

Let $A : c_0(\Gamma_{a.s}) \to c_0(\Gamma_{a.s})$ be any bounded linear operator. Then, $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and $\lim_{n\to\infty} a_{nk} = 0$, uniformly for all k.

Proof: Let $\zeta \in c_0(\Gamma_{a.s})$. Then, there exists events Λ with unit uncertain measure such that ζ converges to 0. Now, for all $\gamma \in \Lambda$, $\zeta(\gamma) = \sum_{k=1}^{\infty} (\zeta_k(\gamma)e_k)$, where $\{e_n\}$ is a basis in $c_0(\Gamma_{a.s})$ which is given by

$$e_n(\gamma) = \begin{cases} 1 & n = k; \\ 0 & otherwise; \end{cases}$$

Now $A\zeta(\gamma) = \sum_{k=1}^{\infty} \zeta_k(\gamma) Ae_k$

$= \sum_{k=1}^{\infty} \zeta_k(\gamma)(a_k^n),\ n \in \mathbf{N}$ and $\gamma \in \Lambda$,

where Ae_k is a sequence $\{a_k^{(1)}, a_k^{(2)}, ...\} \in c_0(\Gamma_{a.s})$; $\quad k = 1, 2, 3, ...$

Then, $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_k^{(n)} \zeta_k(\gamma),\quad n = 1, 2, ..., \gamma \in \Lambda$.

Since each $e_k \in c_0(\Gamma_{a.s})$, therefore $Ae_k \in c_0(\Gamma_{a.s})$ also, for $k = 1, 2, 3, ...$

That implies, $\lim_{n\to\infty} a_{nk} = 0$, keeping k fixed and thus $\lim_{n\to\infty} A_n\zeta(\gamma) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = 0$.

From theorem, 4.2, replacing the set $\Gamma - \{E_l\}$ by Λ, we can say $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$.

We now establish the following theorem due to Silverman-Toeplitz considering a complex uncertain sequence which is convergent with respect to uniformly almost surely.

4.13. Theorem

A bounded linear operator $A : c(\Gamma_{u.a.s}) \to c(\Gamma_{u.a.s})$ preserves the limit if and only if the following conditions are satisfied:

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$;

(ii) $a_{nk} \to 0$, as $n \to \infty$, while k is fixed ;

(iii) $\sum_{n=1}^{\infty} a_{nk} = 1$, for fixed k.

Proof: Consider an uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$ and $A : c(\Gamma_{u.a.s}) \to c(\Gamma_{u.a.s})$, a bounded linear operator which preserves limit.

Define the complex uncertain variables ζ_n as follows:

$$\zeta_n(\gamma) = \begin{cases} 1 & if\ n = k; \\ 0 & otherwise; \end{cases}$$

and let $\zeta(\gamma) = 0$, for all $\gamma \in \Gamma$.

Then, $\lim_{n\to\infty} \{\|\zeta_n - \zeta\|\} = 0$.

Hence, the complex uncertain sequence $\{\zeta_n\}$ is convergent with respect to uniformly almost surely and it converges to zero, for $k = 1, 2, 3......$

Thus, $\sum_{n=1}^{\infty} a_{nk}\zeta_k(\gamma) = 0$, $\quad$ (by our hypothesis),

which implies $\lim_{n\to\infty} a_{nk} = 0$, uniformly for all k. Thus, the condition (ii) is proved.

For the necessity of (iii), let $\{\zeta_n\}$ be a complex uncertain sequence such that $\zeta_n = 1,\ \forall\, n \in \mathbb{N}$ and let $\zeta = 1$.

Then, $\|\zeta_n - \zeta\| = 0$, for all $n \in \mathbb{N}$.

Thus the sequence $\{\zeta_n\}$ converges to ζ with respect to uniformly almost surely.

Consequently, $\sum_{n=1}^{\infty} a_{nk}\zeta_k(\gamma) \longrightarrow 0$, as $n \longrightarrow \infty$ (for fixed k), which implies

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1.$$

Now, $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ exists for each n and tends to ζ, whenever $\{\zeta_k\}$ converges to ζ with respect to uniformly almost surely.

Then by corollary 4.8, we can say that $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

For sufficiency, let the three conditions holds true and the complex uncertain sequence $\{\zeta_n\}$ converges with respect to uniformly almost surely to ζ.

Now, $\sum_{n=1}^{\infty} a_{nk}\zeta_k(\gamma) = \sum_{n=1}^{\infty} a_{nk}(\zeta_k(\gamma) - \zeta(\gamma)) + \zeta(\gamma)\sum_{n=1}^{\infty} a_{nk}$.

Using condition (i) and the fact that $\zeta_n \to \zeta$ with respect to uniformly almost surely, we have the first term of the right hand side of the above equation is zero.

Again by condition (iii), the second term of the right hand side tends to ζ.

Therefore, $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = \zeta$.

Hence $A \in (c(\Gamma_{u.a.s}), c(\Gamma_{u.a.s}))$ and it keeps the limit preserved.

Now, we prove the Kojima-Schur theorem related to matrix transformation of sequences in uncertain environment via convergence with respect uniformly almost surely.

4.14. Theorem

$A : c(\Gamma_{u.a.s}) \to c(\Gamma_{u.a.s})$ is a bounded linear operator if and only if the following conditions are satisfied.

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}|$ is finite;

(ii) for each $p \in \mathbb{N}$, there exists $a_p = \lim_n \sum_{k=p}^{\infty} a_{nk}$, (eliminating n from the notation since it is fixed).

Proof: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space and $A : c(\Gamma_{u.a.s}) \to c(\Gamma_{u.a.s})$ be a bounded linear operator. Suppose the complex uncertain sequence $\{\zeta_n\} \in c(\Gamma_{u.a.s})$ converges to ζ.

Then, by the corollary 4.8, $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and thus (i) is proved.

Consider the complex uncertain sequence $\{\zeta_n\}$ in such a way that

$$\zeta_n(\gamma) = \begin{cases} 0 & if\ n < p; \\ 1 & otherwise; \end{cases}$$

for some finite p and $\zeta(\gamma) = 1,\ \forall \gamma \in \Gamma$.

Then, $\lim_{n\to\infty} \|\zeta_n - \zeta\| = 0$ and so $\{\zeta_n\} \in c(\Gamma_{u.a.s})$, which converges to $\zeta = 1$ with respect to uniformly almost surely.

Thus $\lim_{n\to\infty} A\zeta_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\zeta_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = a_p$.

Conversely, let conditions (i) and (ii) holds true and $\{\zeta_n\} \in c(\Gamma_{u.a.s})$ converges to ζ.

$$\text{Then, } \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = \sum_{k=1}^{\infty} (\zeta_k(\gamma) - \zeta(\gamma))a_{nk} + \zeta(\gamma)\sum_{k=1}^{\infty} a_{nk}$$
$$= S_{\Gamma_n} + \zeta \sum_{k=1}^{\infty} a_{nk}, \text{ where } S_{\Gamma_n} = \sum_{k=1}^{\infty} (\zeta_k(\gamma) - \zeta(\gamma))a_{nk}.$$

Now, by condition (i), $\zeta(\gamma)\sum_{k=1}^{\infty} a_{nk}$ tends to $\zeta(\gamma)a_1$.

$$\text{Suppose, } b_k = \lim_{n\to\infty} a_{nk}$$
$$= \lim_{n\to\infty} \left(\sum_{j=k}^{\infty} a_{nj} - \sum_{j=k+1}^{\infty} a_{nj} \right)$$
$$= a_k - a_{k+1}, \text{ for each } k.$$

$$\text{So, } \sum_{k\to 1}^{\infty} |b_k| = \sum_{k\to 1}^{\infty} |\lim_{n\to\infty} a_{nk}|$$
$$\leq \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty. \quad \text{(by condition (i)).}$$

Again, $\sum_{k=1}^{\infty} (a_{nk} - b_k)\|\zeta_k(\gamma) - \zeta(\gamma)\|$

$$= \sum_{k=1}^{\infty} (a_{nk} - b_k)\|\zeta_k(\gamma) - \zeta(\gamma)\|.$$

Since $\{\zeta_n(\gamma)\}$ converges to ζ uniformly almost surely, so S_{Γ_n} tends to $\sum_{k=1}^{\infty} b_k(\zeta_k(\gamma) - \zeta(\gamma))$.
Hence $A \in (c(\Gamma_{u.a.s}), c(\Gamma_{u.a.s.}))$.

4.15. Definition

A complex uncertain sequence $\{\zeta_n\}$ is said to be uniformly almost surely A-summable to ζ if there exists a sequence of events $\{E_k\}$ with uncertain measure of each events tending to zero such that the A-limit of the complex uncertain sequence $\{\zeta_n\}$ is ζ. That is

$$\lim_{n\to\infty} (A\zeta)_n = \lim_{n\to\infty} A(\zeta_n(\gamma)) = \zeta(\gamma),$$

for all $\gamma \in \Gamma - \{E_k\}$.

We now show the existence of complex uncertain sequence which is uniformly almost surely A-summable to some limit. For that we need to define the C_1-summability due to Cesaro considering a complex uncertain sequence.

4.16. Definition

Let $\{\zeta_n\}$ be a complex uncertain sequence. Then the C_1 transformation of the sequence $\{\zeta_n\}$ is defined by

$$(C_1\zeta)_n = \frac{1}{n+1} \sum_{k=0}^{n} \zeta_k(\gamma), \ \forall\, k \in \mathbb{N}.$$

4.17. Example

Let E_k, where $k \in \mathbb{N}$, be some events in an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with uncertainty measure tending to zero.
Define the complex uncertain sequence $\{\zeta_n\}$ as follows:

$$\zeta_n(\gamma) = \begin{cases} -i & if \ \ n \text{ is even;} \\ i & if \ \ n \text{ is odd;} \\ 0 & otherwise. \end{cases}$$

for all $\gamma \in \Gamma$.
Let $\alpha \in \Gamma - \{E_k\}$. Then

$$(C_1\zeta(\alpha))_n = \begin{cases} \frac{i}{n+1} & if \ \ n \text{ is odd;} \\ \\ 0 & if \ \ n \text{ is even;} \end{cases}$$

Therefore, $\lim_{n\to\infty} (C_1\zeta)_n = \lim_{n\to\infty} (C_1\zeta(\alpha))_n = 0, \ \forall \alpha \in \Gamma - \{E_k\}$.
Hence, the complex uncertain sequence $\{\zeta_n\}$ is uniformly almost surely A-summable to $\zeta \equiv 0$.

4.18. Theorem

Every uniformly almost surely convergent complex uncertain sequence $\{\zeta_n\}$ is uniformly almost surely C_1 summable. Moreover, it preserves the limit.
Proof: Let the complex uncertain sequence $\{\zeta_n\}$ converges to ζ with respect to uniformly almost surely. Then there exists a sequence of events E_l with $\mathcal{M}(E_l) \to 0$ such that

$$\lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0, \ \forall \ \gamma \in \Gamma - \{E_l\}.$$

Therefore, for any given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\|\zeta_{k_0}(\gamma) - \zeta(\gamma)\| < \varepsilon, \ \forall \ n > k_0 \text{ and } \gamma \in \Gamma - \{E_l\}.$$

Also, there is a constant M such that

$$\|\zeta_k(\gamma)\| \leq M, \ \forall \ k \in \mathbb{N} \text{ and } \gamma \in \Gamma - \{E_l\}.$$

Now take $K > 0$ such that $K > 2M\frac{k_0+1}{\varepsilon}$ i.e., $\frac{2M(k_0+1)}{K} < \varepsilon$.
Then, for all $n \geq k$
$\|C_1(\zeta_n(\gamma)) - \zeta(\gamma)\|$
$= \|\frac{(\zeta_1(\gamma)-\zeta(\gamma))+(\zeta_2(\gamma)-\zeta(\gamma))+......+(\zeta_{n+1}(\gamma)-\zeta(\gamma))}{n+1}\|$
$\leq \|\frac{(\zeta_1(\gamma)-\zeta(\gamma))+(\zeta_2(\gamma)-\zeta(\gamma))+......+(\zeta_{k_0}(\gamma)-\zeta(\gamma))}{n+1}\| + \|\frac{(\zeta_{k_0+1}(\gamma)-\zeta(\gamma))+(\zeta_{k_0+2}(\gamma)-\zeta(\gamma))+......+(\zeta_{n+1}(\gamma)-\zeta(\gamma))}{n+1}\|$
$\leq \frac{2M(k_0+1)}{n+1} + \frac{(n-k_0)\varepsilon}{n+1}$
$< \varepsilon + \varepsilon = 2\varepsilon.$
Therefore, $\lim_{n\to\infty}(C_1\zeta_n(\gamma)) = \zeta(\gamma)$.
Hence the complex uncertain sequence is uniformly almost surely A-summable and it also preserves the same limit.

5. Co-Regular Matrix

Let $A = (a_{nk}) \in (c(\Gamma_{u.a.s}), c(\Gamma_{u.a.s}))$, that is A is a conservative matrix (or convergence preserving matrix). The characteristic of A is defined by

$$\chi(A) = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} - \sum_{k=1}^{\infty} (\lim_{n\to\infty} a_{nk})$$

The numbers $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}$ and $\sum_{k=1}^{\infty} (\lim_{n\to\infty} a_{nk})$ are referred to as the characteristic numbers of A, for a details of co-regular matrices over sequences of real or complex terms, one may refer to Maddox [8].

5.1. Definition

Let $A : c(\Gamma_{u.a.s}) \to c(\Gamma_{u.a.s})$ be a conservative (limit preserving) bounded linear matrix operator. The matrix A in an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is called co-regular if and only if $\chi(A) \neq 0$ and co-null matrix if $\chi(A) = 0$.

5.2. Remark

If A is a regular matrix, then from the above results it is clear that $\chi(A) = 1 - 0 = 1$. This shows that every regular matrix is co-regular but not necessarily conversely. This is clear from the following example.

5.3. Example

Consider the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} 1 & for\ \ n; \\ 0 & otherwise. \end{cases}$$

Let $\{\zeta_n(\gamma)\}$ be convergent uniformly almost surely to $\zeta(\gamma)$. Then $\{A(\zeta_n(\gamma))\}$ will converge to $2\zeta(\gamma)$. We have $\chi(A) = 2$. Hence A is a co-regular matrix, but not a regular matrix.

5.4. Theorem

In an uncertain space, a co-regular matrix cannot sum every uniformly almost surely bounded complex uncertain sequences.
Proof: Let $A = (a_{nk})$ be a conservative matrix and $B = (b_{nk})$ be another matrix defined by $b_{nk} = a_{nk} - \lim\limits_{n\to\infty} a_{nk}$. Then, by Mazur's theorem [12]

$$A\zeta(\gamma) = \chi(A)I\zeta(\gamma) + \sum_{k=1}^{\infty}(\lim_{n\to\infty} a_{nk})\zeta_k(\gamma),$$

for all $\zeta = \{\zeta_k\} \in (I_{u.a.s}) = c(\Gamma_{u.a.s})$ and $\gamma \in \Gamma - \{E_l\}$, where E_l are some events with $\mathcal{M}(E_l) \to 0$, I is the infinite identity matrix operator and by $(I_{u.a.s})$ we represent those complex uncertain sequences which are transformed into uniformly almost surely convergent complex uncertain sequences by I.
Then, $B\zeta(\gamma) = \chi(A)I\zeta(\gamma)$ is multiplicative $\chi(A)$.
Also, since A is conservative, $\|A\| < \infty$ and thus $\sum\limits_{k=1}^{\infty} a_k < \infty$, where $a_k = \lim\limits_{n\to\infty} a_{nk}$.
Therefore, $B\zeta$ sums exactly those uniformly almost surely bounded complex uncertain sequences that the infinite matrix A does. Thus, if $\chi \neq 0$, $\chi^{-1}B$ is a regular matrix and by lemma 2.9, the result follows, due to Steinhaus [7].

5.5. Remark

The above definitions and results for co-regular matrices will hold for the cases of almost sure convergence, convergence in mean, convergence in measure and convergence in distribution.

6. Conclusion

In this article, five types of convergence concepts of complex uncertain series is introduced. Initial study of matrix transformation of complex uncertain sequences is made considering a convergent complex uncertain sequence with respect to uniformly almost surely. Summability of a complex uncertain sequence is also been introduced and some results are established. These concepts can be generalized and applied for further studies.

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