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Crossed and Quadratic Resolutions of Algebras

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Abstract. In this work, using crossed resolutions, we will give a construction of a free *reduced* quadratic resolutions of a commutative k-algebra and explain its 2-skeleton.

Introduction

To investigate homological properties of commutative algebras, André used simplicial methods in [1] and introduced 'step-by-step' construction of a resolution of a commutative algebra. This resolution is built up so that at each stage the next step is formed by adding in new simplicies to kill the homotopy modules of the previous step. Illusie [13], by using the simplicial resolution, constructed the cotangent complex of an algebra. Comparing this with results on crossed resolutions, in group cohomology theory, Porter [18] showed how this corresponds to a crossed resolution of algebras. Arvasi-Porter, [3], related André's construction to an obvious construction of a crossed resolution of an algebra by using a description of the passage from simplicial algebras to crossed complexes of algebras given by Carrasco and Cegarra [10] in the group case. This construction does give a 'step-by-step' construction of a crossed resolution given one of a simplicial resolution and its 1-and 2-skeleton.

As an algebraic model of homotopy connected 2-types, the notion of crossed module was introduced by Whitehead in [20] and these crossed modules are equivalent to the simplicial groups with Moore complex of length 1. The commutative algebra analogue of crossed modules has been studied by Porter in [18]. Conduché in [11] defined the notion of 2-crossed module as an algebraic model of homotopy connected 3-types and showed how to obtain a 2-crossed module from a simplicial group. The notion of 2-crossed module for commutative algebras was given in [12]. For detailed information about 2-crossed modules of commutative algebras see [2, 6]. Baues, defined quadratic modules of groups for homotopy connected 3-types and gave a construction of a quadratic module from a simplicial group in Appendix B to chapter IV of [8]. In [4], Arvasi and Ulualan gave the connections between quadratic modules, 2-crossed modules (cf. [11]) and simplicial groups. For the commutative algebra version see also their work, [5]. Reduced quadratic modules of commutative algebras are special kind of quadratic modules of algebras (cf. [5]), describing the 3-types of simply connected CW-complexes which are constructed with algebras of nilpotency degree 2. As a close relationship between crossed modules and reduced quadratic modules over groups, in [15], Muro defined the suspension functor from crossed modules to reduced quadratic modules which sends a

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2-type to the 3-type of its suspension. In [16], the notions of reduced quadratic complexes of commutative algebras was constructed, and the suspension functor from crossed modules to reduced quadratic modules of commutative algebras was given. Using these results, we extend this functor to crossed complexes and quadratic complexes and we give a construction *reduced* quadratic resolutions .

1. Preliminaries

In what follows 'algebras' will be commutative algebras over an unspecified commutative ring, **k**, but for convenience are not required to have a multiplicative identity.

1.1. Reduced Quadratic Modules

Crossed modules were initially defined by Whitehead as models for homotopy connected 2-types in [20]. The commutative algebra analogue of crossed modules has been studied by Porter in [18]. Throughout this paper we denote an action of $r \in R$ on $c \in C$ by $r \cdot c$.

Let R be a **k**-algebra with identity. A *pre-crossed module of commutative algebras* is an R-algebra C, together with an R-algebra morphism $\partial : C \to R$, such that for all $c \in C, r \in R$; $\partial(r \cdot c) = r\partial c$. This is a *crossed module* if in addition, for all $c, c' \in C$, $\partial c \cdot c' = cc'$. This condition is called the *Peiffer identity*. We denote such a crossed module by (C, R, ∂).

A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is pair of **k**-algebras morphisms, $\varphi : C \to C'$ and $\psi : R \to R'$ such that $\varphi(r \cdot c) = \psi(r) \cdot \varphi(c)$ and $\partial'\varphi(c) = \psi\partial(c)$.

Recall from [5] that *a nil(2)-module* is a pre-crossed module $\partial : C \to R$ with additional "nilpotency" condition. This condition is $P_3(\partial) = 0$ where $P_3(\partial)$ is generated by Peiffer elements $\langle x_1, x_2, x_3\rangle$ of length 3.

A Peiffer element in a pre-crossed module $\partial : C \to R$ is defined by

$$\langle x, y\rangle = xy - x \cdot \partial(y)$$

for $x, y \in C$.

For an algebra C, the C/C^2 is the quotient of the algebra C by its ideal of squares. Then, there is a functor from the category of **k**-algebras to the category of the **k**-modules. This functor C goes to C/C^2, plays the role of abelianization in the category of **k**-algebras. As modules are often called *singular* algebras.

$$\partial^{cr} : C^{cr} = C/P_2(\partial) \to R$$

is the crossed module associated to pre-crossed module $\partial : C \to R$, and

$$\partial^{nil} : C/P_3(\partial) \to R$$

is the *nil*(2)-module associated to pre-crossed module $\partial : C \to R$, where $P_2(\partial) = \langle C, C\rangle$ is the Peiffer ideal of C generated by the elements of the form

$$\langle x, y\rangle = xy - x \cdot \partial y,$$

for $x, y \in C$.

Definition 1.1. *([5]) A reduced quadratic module* (ω, δ) *consists of the following diagram,*

$$\begin{array}{ccc} C \otimes C & & \\ {\scriptstyle \omega}\downarrow & \searrow^{w} & \\ C_2 & \xrightarrow[\delta]{} & C_1 \end{array}$$

of algebras such that the following axioms are satisfied.

RQM1- The algebra C_1 is a nil(2)-algebra and $C = C_1/(C_1)^2$ is the singularization of algebra C_1. The quotient map $q : C_1 \to C$ is given by $x \longrightarrow \{x\}$

RQM2- For the morphism δ and the quadratic map ω,

$$\delta\omega(\{x\} \otimes \{y\}) = w(\{x\} \otimes \{y\}) = xy$$

for $x, y \in C_1$.

RQM3- For $a \in C_2, x \in C_1$,

$$0 = \omega(\{\delta a\} \otimes \{x\} + \{x\} \otimes \{\delta a\}).$$

RQM4- For $a, b \in C_2$,

$$\omega(\{\delta a\} \otimes \{\delta b\}) = ab.$$

We denote the category of reduced quadratic modules by $\mathfrak{RQM}$.

Simplicial Commutative Algebras

Recall from [2] that a simplicial algebra **E** consists of a family of algebras E_n together with face and degeneracy maps $d_i^n : E_n \to E_{n-1}$, $0 \leq i \leq n$ $(n \neq 0)$ and $s_i^n : E_n \to E_{n+1}$, $0 \leq i \leq n$ satisfying the usual simplicial identities. In fact it can be completely described as a functor $\mathbf{E} : \Delta^{op} \to \mathfrak{Alg}$ where Δ is the category of finite ordinals. We obtain for each $k \geq 0$ a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of Δ with $j \leq k$. A k-truncated simplicial algebra is a functor from $\Delta^{op}_{\leq k}$ to the category of commutative algebras $\mathfrak{Alg}$. We denote the category of k-truncated simplicial algebra by $\mathfrak{Tr}_k\mathfrak{SimpAlg}$. A reduced simplicial algebra is a simplicial algebra in which the first component is trivial. We denote the category of reduced simplicial algebras by $\mathfrak{ReSimpAlg}$.

Given a simplicial algebra **E**, the Moore complex $(\mathbf{NE}, \partial)$ of **E**, is the chain complex defined by;

$$NE_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n : NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

Peiffer Pairings in the Moore Complex of a Simplicial Algebra

We recall briefly from [10] the construction of a family of **k**-linear morphisms. For details see [10] and [2]. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$ where $\alpha = (i_r, ...i_1), \beta = (j_s, ...j_1) \in S(n)$. The **k**-linear morphisms that we will need,

$$\{C_{\alpha,\beta} : NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \to NG_n : (\alpha, \beta) \in P(n), n \geq 0\}$$

are given as composites:

$$\begin{aligned} C_{\alpha,\beta}(x_\alpha \otimes y_\beta) &= p\mu(s_\alpha \otimes s_\beta)(x_\alpha \otimes y_\beta) \\ &= p(s_\alpha(x_\alpha)s_\beta(x_\beta)) \\ &= (1 - s_{n-1}d_{n-1})...(1 - s_0d_0)(s_\alpha(x_\alpha)s_\beta(x_\beta)), \end{aligned}$$

where

$$s_\alpha = s_{i_r}...s_{i_1} : NE_{n-\#\alpha} \to E_n, s_\beta = s_{j_s}...s_{j_1} : NE_{n-\#\beta} \to E_n,$$

$p : E_n \to NE_n$ is defined by composite projections $p = p_{n-1}...p_0$ with $p_j = 1 - s_jd_j$ for $j = 0, 1, ..., n-1$ and $\mu : E_n \otimes E_n \to E_n$ denotes multiplication.

We will now consider that the ideal I_n in E_n such that generated by all elements of the form;

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$$

where $x_\alpha \in NE_{n-\#\alpha}$ and $y_\beta \in NE_{n-\#\beta}$ and for all $(\alpha, \beta) \in P(n)$.

Proposition 1.2. ([2]) *Let* **E** *be simplicial algebra and $n > 0$, and D_n the ideal in E_n generated by degenerate elements. We suppose $E_n = D_n$, and let I_n be the ideal generated by elements of the form $C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$ with $(\alpha, \beta) \in P(n)$ where $x_\alpha \in NE_{n-\#\alpha}$, $y_\beta \in NE_{n-\#\beta}$ with $1 \leq r, s \leq n$. Then, $\partial_n(NE_n) = \partial_n(I_n)$.*

If $n = 2, 3$ or 4 , then the image of the Moore complex of the simplicial algebra **E** can be given in the form

$$\partial_n(NE_n) = \sum_{I,J} K_I K_J$$

where $I, J \subset [n-1]$, with $I \cup J = [n-1]$ and where $K_I = \bigcap_{i \in I} \ker d_i$ and $K_J = \bigcap_{j \in J} \ker d_j$ (cf. [2]).

1.2. From Reduced Simplicial Algebras to Reduced Quadratic Modules

By using the images of the $C_{\alpha,\beta}$ functions in the Moore complex of a simplicial commutative algebra given in [2], we can give a construction of a reduced quadratic module from a simplicial algebra.

Let **E** be a reduced simplicial algebra with Moore complex $(\mathbf{NE}, \partial)$ and $E_n = D_n$ for all $n \geq 0$. Let $M = NE_1/(NE_1)^3 = (NE_1)^{nil}$. Then the algebra M becomes a $nil(2)$-algebra. Let $q_1 : NE_1 \to M$ be the quotient map. Let P be the ideal of $(NE_2/\partial_3 NE_3)$ generated by elements of the form $s_1(xy)(s_1 z - s_0 z)$ or $s_1(x)(s_1(yz) - s_0(yz))$ for $x, y, z \in NE_1$. Let

$$L = (NE_2/\partial_3 NE_3)/P$$

be the quotient algebra and let

$$q_2 : NE_2/\partial_3 NE_3 \to L.$$

be the quotient morphism. Then, we have a commutative diagram

$$\begin{array}{ccc} NE_2/\partial_3(NE_3) & \xrightarrow{\partial_2} & NE_1 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle q_1} \\ L & \xrightarrow[\delta]{} & M. \end{array}$$

Since

$$\begin{aligned} \partial_2(s_1(xy)(s_1 z - s_0 z)) &= xy(z - s_0 d_1 z) \\ &= xyz - xy \cdot \partial_1(z) \\ &= xyz \in (NE_1)^3 \quad \text{(by reduced condition)} \end{aligned}$$

and

$$\begin{aligned} \partial_2(s_1(x)(s_1(yz) - s_0(yz))) &= x(yz - s_0 d_1(yz)) \\ &= xyz - x \cdot \partial_1(yz) \\ &= xyz \in (NE_1)^3 \quad \text{(by reduced condition)} \end{aligned}$$

we have $\partial_2(P) \subseteq (NE_1)^3$. Thus the map $\delta : L \to M$ given by $\delta(a + P) = \partial_2(a) + (NE_1)^3$ is a well-defined homomorphism. Indeed, if $a + P = b + P$ we have $a - b \in P$ and $\partial_2(a) - \partial_2(b) \in \partial_2(P)$ and since $\partial_2(P) \subseteq (NE_1)^3$ we have $\partial_2(a) - \partial_2(b) \in (NE_1)^3$ and then $\partial_2(a) + (NE_1)^3 = \partial_2(b) + (NE_1)^3$, that is we have $\delta(a + P) = \delta(b + P)$. Let

$$C = M/M^2$$

be the singularization of the algebra M. Thus we have the following commutative diagram

$$\begin{array}{ccc} C\otimes C & & \\ \downarrow{\scriptstyle\omega} & \searrow^{w} & \\ L & \xrightarrow[\delta]{} & M \\ \uparrow{\scriptstyle q_2} & & \uparrow{\scriptstyle q_1} \\ \dfrac{NE_2}{\partial_3(NE_3)} & \xrightarrow[\partial_2]{} & NE_1 \end{array}$$

where the map $w : C\otimes C \to M$ is given by $w(\{q_1(x)\}\otimes\{q_1(y)\}) = q_1(x)q_1(y)$ for $x, y \in NE_1$ and the quadratic map is defined by

$$\omega\{q_1x\}\otimes\{q_1y\} = q_2(s_1x(s_1y - s_0y) + \partial_(NE_3)$$

for $x, y \in NE_1$ and $q_1x, q_1y \in M$.

Thus, we have

Proposition 1.3. *The diagram*

$$\begin{array}{ccc} C\otimes C & & \\ \downarrow{\scriptstyle\omega} & \searrow^{w} & \\ L & \xrightarrow[\delta]{} & M \end{array}$$

is a reduced quadratic module.

Proof. The axioms of reduced quadratic module can be verified by using the images of the generate elements $C_{\alpha,\beta}$ in $\partial_3(NE_3) = \partial_3(I_3)$ similarly given in [2]. □

2. Crossed Complexes and Crossed Resolutions

A crossed complex of commutative algebras is a sequence of k-algebras

$$\mathfrak{C}: \qquad \cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1}\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which

(i) (C_1, C_0, ∂_1) is a crossed module,

(ii) for $i > 1$, C_i is an C_0- module on which $\partial_1 C_1$ operates trivially and each ∂_i is an C_0- module morphism,

(iii) for $i \geq 1$, $\partial_{i+1}\partial_i = 0$.

Morphisms of crossed complexes are defined in the obvious way.

The homology of a crossed complex $\mathfrak{C}$ can be defined by

$$H_n(\mathfrak{C}) = \ker\partial_n/\mathrm{Im}\partial_{n+1}.$$

A crossed complex $\mathfrak{C}$ is exact if for $n \geq 1$,

$$\ker(\partial_n) = \mathrm{Im}\partial_{n+1}.$$

A *crossed resolution* of a commutative k-algebra B is a crossed complex

$$\mathfrak{C}: \qquad \cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1}\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

of k-algebras, where ∂_1 is a crossed C_0-module, together with $f : C_0 \to B$ a morphism, such that the sequence

$$\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{f} B \longrightarrow 0$$

is exact.

If, for $i \geq 0$, the C_i are free and ∂_1 is a free crossed module, then the resolution is called a free crossed resolution of the algebra B.

2.1. From Simplicial Resolution to Crossed Resolution

In this section, we recall the construction of the 2-skeleton of a free crossed resolution of a commutative algebra by using the free simplicial resolution. For more details see [3].

Step-by-Step Construction

This section is a brief résumé of how to construct simplicial resolutions. The work depends heavily on a variety of sources, mainly [1] and [3], [14]. The reader is referred to the book of André [1] and to the article of Arvasi and Porter [3] for full details and more references.

First, some notation and terminology. Let $[n] = \{0 < 1 < \cdots < n\}$ be an ordered set. We define the following maps.

First, the injective monotone map $\delta_i^n : [n-1] \to [n]$ is given by

$$\delta_i^n(x) = \begin{cases} x & \text{if } x < i, \\ x+1 & \text{if } x \geqslant i \end{cases}$$

for $0 \leqslant i \leqslant n \neq 0$. On the other hand, an increasing surjective monotone map $\sigma_i^n : [n+1] \to [n]$ is given by

$$\sigma_i^n(x) = \begin{cases} x & \text{if } x \leqslant i, \\ x-1 & \text{if } x > i \end{cases}$$

for $0 \leqslant i \leqslant n$. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$ (cf. [3]).

Killing Elements in Homotopy Modules

Let $\mathbf{E}$ be a simplicial algebra and $k \geqslant 1$ be fixed. Suppose we are given a set $\mathbf{\Omega}$ of elements

$$\mathbf{\Omega} = \{x_\lambda : \lambda \in \Lambda\},$$

$x_\lambda \in \pi_{k-1}(\mathbf{E})$; then we can choose a corresponding set of elements $w_\lambda \in NE_{k-1}$ so that

$$x_\lambda = w_\lambda + \partial_k(NE_k).$$

(If $k = 1$, then as $NE_0 = E_0$, the condition that $w_\lambda \in NE_0$ is empty). We want to define a simplicial algebra, $\mathbf{F} = \mathbf{E}[\mathbf{\Omega}]$ with a monomorphism $\mathbf{i} : \mathbf{E} \to \mathbf{F}$ such that

$$\pi_{k-1}(\mathbf{i}) : \pi_{k-1}(\mathbf{E}) \to \pi_{k-1}(\mathbf{F})$$

"kills off" the x_λ 's. We do this by adding new indeterminates into NE_k to enlarge it so as to make $\mathbf{i}(\mathbf{w}_\lambda) \in \partial(NF_k)$. More precisely,

(1) F_n is a free E_n-algebra,

$$F_n = E_n[y_{\lambda,t}] \qquad \text{with } \lambda \in \Lambda \text{ and } t \in \{n, k\}.$$

(2) For $0 \leqslant i \leqslant n$, the algebra homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : E_n \to E_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \qquad \text{with } u = t\sigma_i^n, t : [n] \to [k].$$

(3) For $0 \leqslant i \leqslant n \neq 0$, the algebra homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : E_n \to E_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map } u = t\delta_i^n \text{ is surjective} \\ t'(w_\lambda) & \text{if } t\delta_i^n = \delta_k^k t' \\ 0 & \text{if } t\delta_i^n = \delta_j^k t' \text{ with } j \neq k \end{cases}$$

by extending linearly.

Here $t' : [n-1] \to [k-1]$. It thus corresponds to a unique algebra $t' : E_{k-1} \to E_{n-1}$ (see André [1]).

Free Simplicial Algebras

Recall from [3] the definition of free simplicial algebra given by the step-by-step construction of André [1] according to the above statements.

Let $\mathbf{E}$ be a simplicial algebra and $k \geqslant 1$, k-skeletal be fixed. A simplicial algebra $\mathbf{F}$ is called a free if

(i) $F_n = E_n$ for $n < k$,

(ii) F_k = a free E_k-algebra over a set of non- degenerate indeterminates, all of whose faces are zero except the kth,

(iii) F_n is a free E_n-algebra over the degenerate elements for $n > k$.

A variant of the step-by-step construction gives: if $\mathbf{A}$ is a simplicial algebra, then there exists a free simplicial algebra $\mathbf{E}$ and an epimorphism $\mathbf{E} \to \mathbf{A}$ which induces isomorphisms on all homotopy modules. The details are omitted as they are well-known.

Now, we recall the 1- and 2-skeletons of a free simplicial algebra given as

$$\mathbf{E}^{(1)} : \cdots R[s_0X, s_1X] \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow} R[X] \substack{\longrightarrow\\\longrightarrow\\\longleftarrow} R \xrightarrow{f} R/I$$

$$\mathbf{E}^{(2)} : \cdots R[s_0X, s_1X][Y] \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow} R[X] \substack{\longrightarrow\\\longrightarrow\\\longleftarrow} R$$

with the simplicial structure defined as in Section 3 of [3]. Analysis of this 2-dimensional construction data shows that it consists of some 1-dimensional data, namely the function $\vartheta : X \to R$, that is used to induce $d_1 : R[X] \to R$, together with strictly 2-dimensional construction data consisting of the function $\psi : Y \to R^+[X]$ and this function is used to induce $d_2 : R[s_0X, s_1X][Y] \to R[X]$. We will denote this 2-dimensional construction data by (ϑ, ψ, R).

Proposition 2.1. ([3]) *Given a presentation $P = (R; x_1, ..., x_n)$ of an R-algebra B and $\mathbf{E}^{(1)}$ the 1-skeleton of the free simplicial algebra generated by this presentation, then*

$$\delta : NE_1^{(1)}/\partial_2(NE_2^{(1)}) \to NE_0^{(1)}$$

is the free crossed module on $(x_1, ..., x_n) \to R$.

Proposition 2.2. ([3])*Let $\mathbf{E}$ be a simplicial algebra; then defining*

$$C_n(\mathbf{E}) = \frac{NE_n}{NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})}$$

with

$$\partial_n(\overline{z}) = \overline{d_n(z)}$$

yields a crossed complex $C(\mathbf{E})$ of algebras.

By using the 1- and 2-skeletons of the free simplicial resolution of algebra $R/(x_1, ..., x_n)$ and the image of the Peiffer elements in the Moore complex of this simplicial resolution (cf. [2]) and by using the functor from simplicial algebras to crossed complexes analogously to that given by Carrasco and Cegarra (cf. [10]), Arvasi and Porter constructed the 2-skeleton of a free crossed resolution of the commutative algebra $B = R/(x_1, ..., x_n)$ in section 4 of [3] as given in the following proposition.

Proposition 2.3. ([3]) *Let $\mathbf{E}^{(2)}$ be the 2-skeleton of a free simplicial algebra in 2-dimensional construction data. Then*

$$\mathfrak{C}^{(2)}: \qquad \frac{(R[s_0(X),s_1(X)])^+[Y]}{[Q_2+P_2]} \xrightarrow{\partial_2} R^+[X]/P_1 \xrightarrow{\partial_1} R$$

is the 2-skeleton of a free crossed resolution of $R/(x_1,...,x_n)$, where ∂_2 and ∂_1 are given respectively by $\partial_2(Y_i+(Q_2+P_2)) = d_2(Y_i)+P_1$ and $\partial_1(X_i+P_1) = d_1(X_i)$ for $Y_i \in (R[s_0(X),s_1(X)])^+[Y]$ and $X_i \in R[X]^+$ and where

$$\frac{(R[s_0X,s_1X])^+[Y]+(s_0X-s_1X)}{Q_2+P_2} = C_2(\mathbf{E}^{(2)})$$

and

$$R^+[X]/P_1 = C_1(\mathbf{E}^{(1)}).$$

Note that $Q_2 = NE_2^{(2)} \cap D_2$ is the ideal of $(R[s_0X,s_1X])^+[Y]+(s_0X-s_1X)$ generated by elements of the form

$$s_1(X_i)(s_0(X_j)-s_1(X_j))$$

for $X_i, X_j \in R[X]$.

On the other hand $P_2 = \partial_3(NE_3^{(2)})$ is the ideal of $(R[s_0X,s_1X])^+[Y]+(s_0X-s_1X)$ generated by elements of the form

$$\begin{array}{rr}
(s_1s_0d_1(X_i)-s_0(X_i))Y_j & (i)\\
Y_i(s_1d_2Y_j-Y_j) & (ii)\\
(s_0X_i-s_1X_i)(s_1d_2Y_j-Y_j) & (iii)\\
Y_i(Y_j+s_0d_2Y_j-s_1d_2Y_j) & (iv)\\
s_1X_i(s_0d_2Y_j-s_1d_2Y_j+Y_j) & (v)\\
(s_0d_2Y_i-s_1d_2Y_i+Y_i)(s_1d_2Y_j-Y_j) & (vi)
\end{array}$$

for $X_i, X_j \in R^+[X]$, $Y_i, Y_j \in (R[s_0X,s_1X])^+[Y]+(s_0X-s_1X)$ and P_1 is the Peiffer ideal of $R^+[X]$.

3. Free Reduced Quadratic Resolution of a Commutative Algebra

Muro gave in [15] the suspension functor by using central push-out from crossed modules to reduced quadratic modules, and showed that this functor preserves the free crossed modules of groups. In [16], Odabas and Ulualan gave this functor for crossed modules of commutative algebras. We recall briefly this functor from [16].

Let $\partial: L \to M$ be a crossed module of commutative algebras. Let $I = \{1,2,3\}$ be index set with partially ordered $1<2$ and $1<3$. We know that the direct system, $F: I \to C$, is the following diagram;

$$\begin{array}{ccc}
F_1 & \xrightarrow{\varphi_3^1} & F_3\\
\big\downarrow{\scriptstyle \varphi_2^1} & & \\
F_2 & &
\end{array}$$

We will construct a functor from I to the category of commutative algebras, using the crossed module $\partial: L \to M$.

Suppose that $F_1 = L\otimes M$, $F_2 = L$ and $F_3 = \left(M/M^2 \otimes M/M^2\right)/K$. We can define the morphisms between them

$$\begin{array}{rccc}
\varphi_2^1: & F_1 & \longrightarrow & F_2\\
 & (l\otimes m) & \longmapsto & l\cdot m
\end{array}$$

This morphism satisfies the following;

$$\begin{aligned} \varphi_2^1(id \otimes \partial)(l \otimes l') &= \varphi_2^1(l \otimes \partial l') \\ &= l \cdot \partial l' \\ &= ll' \\ &= w'(l \otimes l') \end{aligned}$$

that is,

$$\varphi_2^1(id \otimes \partial) = w' : L \otimes L \to L$$

where w' is the multiplication map and

$$\begin{aligned} \partial\varphi_2^1(l \otimes m) &= \partial(l \cdot m) \\ &= (\partial l)m \\ &= w'(\partial l \otimes m) \\ &= w'(\partial \otimes id)(l \otimes m) \end{aligned}$$

thus, we have

$$\partial\varphi_2^1 = w'(\partial \otimes id) : L \otimes M \to M.$$

We now define the morphism

$$\varphi_3^1 : F_1 = L \otimes M \to (M/M^2 \otimes M/M^2)/K = F_3$$

by composition of the following maps

$$L \otimes M \xrightarrow{q \otimes q} L/L^2 \otimes M/M^2 \xrightarrow{\partial^2 \otimes id} (M/M^2 \otimes M/M^2)/K,$$

where $q : M \to M/M^2$ is the quotient map and K is the image of

$$\partial^2 \otimes id + id \otimes \partial^2 : L/L^2 \otimes M/M^2 \longrightarrow M/M^2 \otimes M/M^2.$$

That is, φ_3^1 is given by

$$\varphi_3^1(l \otimes m) = \partial^2 q(l) \otimes q(m) + K.$$

Therefore, we have the following diagrams

$$\begin{array}{ccc} L \otimes M & \xrightarrow{\varphi_3^1} & (M/M^2 \otimes M/M^2)/K \\ {\scriptstyle \varphi_2^1}\downarrow & & \\ L & & \end{array}$$

and

$$\begin{array}{ccc} L \otimes M & \xrightarrow{\varphi_3^1} & (M/M^2 \otimes M/M^2)/K \\ {\scriptstyle \varphi_2^1}\downarrow & & \downarrow{\scriptstyle \omega} \\ L & \xrightarrow[r]{} & L^{\Sigma} \end{array}$$

and this diagram is a push-out and where

$$L^{\Sigma} = \frac{L \times (M/M^2 \otimes M/M^2)/K}{W},$$

$$W = \{(l \cdot m, \partial^2 q(l) \otimes qm + K) : l \in L, m \in M\}.$$

There is a morphism w given by

$$\begin{array}{rcl} w : (M/M^2 \otimes M/M^2)/K & \longrightarrow & M^{nil} \\ (qm \otimes qm') + K & \longmapsto & mm'. \end{array}$$

Furthermore, there is also a morphism

$$L \xrightarrow{\partial^{nil}\overline{q}} M^{nil}$$

given by composition of the following maps

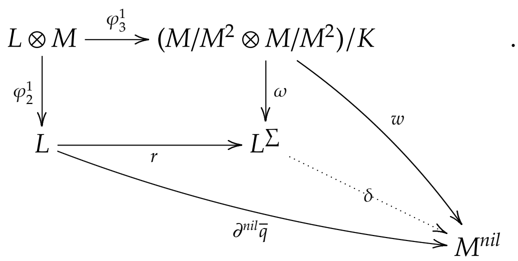
$$L \xrightarrow{\overline{q}} L^{nil} \xrightarrow{\partial^{nil}} M^{nil}$$

where, $\overline{q} : L \to L^{nil}$ is the quotient map.

Obviously, according to above descriptions, we have

$$w(\partial^2 \otimes id)(q \otimes q) = \partial^{nil}\overline{q}\varphi_2^1.$$

Thus, we have the following diagram



There is a unique morphism

$$\delta : L^{\Sigma} \to M^{nil}$$

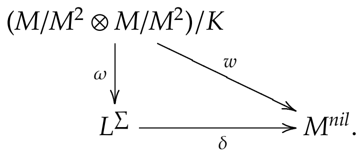
satisfying the following equalities

$$\delta\omega = w$$

and

$$\delta r = \partial^{nil}\overline{q}.$$

Thus the following diagram



is a reduced quadratic module (cf. [16]).

3.1. Free crossed and reduced quadratic modules

Let (C, R, ∂) be a crossed module, let Y be a set, and let $\nu : Y \to C$ be a function, then (C, R, ∂) is said to be a free crossed module with basis ν if for any crossed module (C', R, ∂') and a function $\nu' : Y \to C'$ such that $\partial'\nu' = \partial\nu$, there is a unique morphism $\Phi : C \to C'$ such that $\Phi\nu = \nu'$.

The free crossed module (C, R, ∂) is totally free if R is a free algebra. On replacing "crossed" by "pre-crossed" in the above definition of a (totally) free crossed module, we obtain the definition of a (totally) free pre-crossed module.

Theorem 3.1. ([3]) *A free crossed module (C, R, ∂) exists on any function $f : Y \to R$ with codomain R.*

Definition 3.2. *Let*

$$C \otimes C \xrightarrow{\ \omega\ } L \xrightarrow{\ \delta\ } M$$

be a reduced quadratic module, let Y be a set and let $\nu : Y \to L$ be a function and M is free nil(2)-algebra, then this reduced quadratic module is called the totally free reduced quadratic module with basis $\nu : Y \to L$, or alternatively on the function $\delta\nu : Y \to M$, if for any reduced quadratic module $(L', M, \delta', \omega')$ and a function $\nu' : Y \to L'$ such that $\delta'\nu' = \delta\nu$, there is a unique morphism $\Phi : L \to L'$ such that $\Phi\nu = \nu'$.

Let R be a free algebra and let Y be a set and $f : Y \to R$ be a function with codomain R. Let $E = R^+[Y]$, the positively graded part of the polynomial ring on Y so that R acts on E by multiplication. The function f induces a morphism of R-algebras $\theta : R^+[Y] \to R$ given by $\theta(y) = f(y)$. Let P_2 be Peiffer ideal of $R^+[Y]$, then take $C = R^+[Y]/P_2$. We have functions: $\varphi : C \otimes R \to C$ given by $\varphi(y \otimes r) = y \cdot r$ and $\varphi' : C \otimes R \to (R/R^2 \otimes R/R^2)/K$ given by $\varphi'(y \otimes r) = \theta^2 q_1(y) \otimes q_2(r) + K$, where $q_1 : C \to C/C^2$ and $q_2 : R \to R/R^2$ are the quotient maps and K is image of the function

$$\theta^2 \otimes id + id \otimes \theta^2 : C/C^2 \otimes R/R^2 \to R/R^2 \otimes R/R^2.$$

Thus the diagram

$$\begin{array}{ccc} C \otimes R & \xrightarrow{\ \theta^2(q_1 \otimes q_2)\ } & (R/R^2 \otimes R/R^2)/K \\ \varphi \downarrow & & \downarrow \omega \\ C & \xrightarrow[\ r\ ]{} & C^{\Sigma} \end{array}$$

is a push-out, where

$$C^{\Sigma} = \frac{R^+[Y]/P_2 \times (R/R^2 \otimes R/R^2)/K}{W}$$

and

$$W = \{(y \cdot r, \theta^2(q_1 \otimes q_2)(y, r)) : y \in R^+[Y]/P_2, r \in R\}.$$

Proposition 3.3. *([16]) The diagram*

$$\begin{array}{ccc} (R/R^2 \otimes R/R^2)/K & & \\ \omega \downarrow & \searrow^{w} & \\ C^{\Sigma} & \xrightarrow[\ \delta\ ]{} & R^{nil} \end{array}$$

is a totally free reduced quadratic module on the function $f^{nil} : Y \to R^{nil}$, where $\delta = \theta^{nil}\overline{q}$ and $\overline{q} : C \to C^{nil}$ is the quotient map.

Now, we define the notion of free reduced quadratic resolution of a commutative k-algebra and we give its 2-skeleton by using the suspension functor from crossed to reduced quadratic modules.

A *reduced quadratic complex* of commutative k-algebras is a sequence of k-algebras

$$\begin{array}{ccccccccc} & & & & & & C\otimes C & & \\ & & & & & & \downarrow{\scriptstyle\omega} & \searrow^{w} & \\ \sigma: & \cdots\longrightarrow & \sigma_n & \xrightarrow{\partial_n} & \sigma_{n-1}\cdots\longrightarrow & \sigma_2 \xrightarrow{\partial_2} & \sigma_1 & \xrightarrow[\partial_1]{} & \sigma_0 \end{array}$$

in which

(*i*)

$$\begin{array}{ccc} C\otimes C & & \\ {\scriptstyle\omega}\downarrow & \searrow^{w} & \\ \sigma_1 & \xrightarrow[\partial_1]{} & \sigma_0 \end{array}$$

is a reduced quadratic module,

(*ii*) for $i > 1$, σ_i is an σ_0-module on which $\partial_1\sigma_1$ operates trivially and each ∂_i is an σ_0-module morphism,

(*iii*) for $i \geq 1$, $\partial_{i+1}\partial_i = 0$.

The homology of a reduced quadratic complex σ can be defined by

$$H_n(\sigma) = \ker\partial_n / \mathrm{Im}\partial_{n+1}.$$

A reduced quadratic complex σ is exact if for $n \geq 1$,

$$\ker(\partial_n) = \mathrm{Im}\partial_{n+1}.$$

A *reduced quadratic resolution* of a commutative k-algebra B is a reduced quadratic complex

$$\begin{array}{ccccccccc} & & & & & & C\otimes C & & \\ & & & & & & \downarrow{\scriptstyle\omega} & \searrow^{w} & \\ \sigma: & \cdots\longrightarrow & \sigma_n & \xrightarrow{\partial_n} & \sigma_{n-1}\cdots\longrightarrow & \sigma_2 \xrightarrow{\partial_2} & \sigma_1 & \xrightarrow[\partial_1]{} & \sigma_0 \end{array}$$

of k-algebras, where

$$\begin{array}{ccc} C\otimes C & & \\ {\scriptstyle\omega}\downarrow & \searrow^{w} & \\ \sigma_1 & \xrightarrow[\partial_1]{} & \sigma_0 \end{array}$$

is a reduced quadratic module, together with $f : \sigma_0 \to B$ a morphism, such that the sequence

$$\cdots \longrightarrow \sigma_2 \xrightarrow{\partial_2} \sigma_1 \xrightarrow{\partial_1} \sigma_0 \xrightarrow{f} B \longrightarrow 0$$

is exact.

If, for $i \geq 0$, the σ_i are free and

$$\begin{array}{ccc} C\otimes C & & \\ {\scriptstyle\omega}\downarrow & \searrow^{w} & \\ \sigma_1 & \xrightarrow[\partial_1]{} & \sigma_0 \end{array}$$

is a free reduced quadratic module, then the resolution is called a free reduced quadratic resolution of the algebra B. Note that if

$$\begin{array}{ccccccccccc} & & & & & & & & C\otimes C & & \\ & & & & & & & & \downarrow{\scriptstyle\omega} & \searrow{\scriptstyle w} & \\ \sigma: & \cdots \longrightarrow & \sigma_n & \xrightarrow{\partial_n} & \sigma_{n-1}\cdots & \longrightarrow & \sigma_2 & \xrightarrow{\partial_2} & \sigma_1 & \xrightarrow[\partial_1]{} & \sigma_0 \end{array}$$

is a reduced quadratic complex, then the sequence

$$\cdots \longrightarrow \sigma_n \xrightarrow{\partial_n} \sigma_{n-1}\cdots \longrightarrow \sigma_2 \xrightarrow{\overline{\partial_2}} \frac{\sigma_1}{\omega(C\otimes C)} \xrightarrow{\overline{\partial_1}} \frac{\sigma_0}{w(C\otimes C)}$$

becomes a chain complex of commutative algebras, where $\overline{\partial_2} = q_2\partial_2$ and $\overline{\partial_2} = q_1\partial_1$, and where

$$q_2 : \sigma_1 \to \frac{\sigma_1}{\omega(C\otimes C)}, q_1 : \sigma_0 \to \frac{\sigma_0}{w(C\otimes C)}$$

are the quotient maps. Since $\partial_1(\omega(C\otimes C)) = w(C\otimes C)$, $\overline{\partial_1}$ is a well defined homomorphism.

Now, consider the crossed complex

$$\mathfrak{C}: \qquad \cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1}\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which $\partial_1 : C_1 \to C_0$ is a crossed module. If we apply the suspension functor to this crossed module, we have a reduced quadratic module

$$\begin{array}{ccc} (C_0/(C_0)^2\otimes C_0/(C_0)^2)/K & & \\ \downarrow{\scriptstyle\omega} & \searrow{\scriptstyle w} & \\ C_1^{\Sigma} & \xrightarrow[\delta]{} & (C_0)^{nil} \end{array}$$

as explained in section **??**. Suppose that for $n \geq 2$, $\sigma_n = C_n$, we have a reduced quadratic complex,

$$\begin{array}{ccccccccccc} & & & & & & & & (C_0/(C_0)^2\otimes C_0/(C_0)^2)/K & & \\ & & & & & & & & \downarrow{\scriptstyle\omega} & \searrow{\scriptstyle w} & \\ \sigma: & \cdots \longrightarrow & \sigma_n & \xrightarrow{\partial_n} & \sigma_{n-1}\cdots & \longrightarrow & \sigma_2 & \xrightarrow{\partial_2} & C_1^{\Sigma} & \xrightarrow[\delta]{} & (C_0)^{nil}. \end{array}$$

Now, we recall the 2-skeleton of a free crossed resolution of $R/(x_1, ..., x_n)$ from [3];

$$\mathfrak{C}^{(2)}: \qquad \cdots \longrightarrow C_2(\mathbf{E}^{(2)}) \xrightarrow{\partial_2} C_1(\mathbf{E}^{(1)}) \xrightarrow{\partial_1} R$$

as briefly explained in section 2.1.

Thus, we have that the following diagram

$$\begin{array}{ccccc} & & ((R/R^2)\otimes(R/R^2))/K & & \\ & & \downarrow{\scriptstyle\omega} & \searrow{\scriptstyle w} & \\ \sigma: \quad \dfrac{(R[s_0X, s_1X])^+[Y] + (s_0X - s_1X)}{Q_2 + P_2} & \xrightarrow[\partial_2]{} & (\dfrac{R^+[X]}{P_1})^{\Sigma} & \xrightarrow[\delta]{} & (R)^{nil} \end{array}$$

is the 2-skeleton of a free reduced quadratic resolution of the commutative algebra $B = R/(x_1, ..., x_n)$.

This follows immediately from the construction of the suspension functor and simplicial resolution and from the results of [3] and section 2 of this paper.

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