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Existence and Uniqueness Results for Nonlinear Implicit Riemann-Liouville Fractional Differential Equations with Nonlocal Conditions

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Abstract. In this paper, we use the fixed point theory to obtain the existence and uniqueness of solutions for nonlinear implicit Riemann-Liouville fractional differential equations with nonlocal conditions. An example is given to illustrate this work.

1. Introduction

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non integer order. Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[24], [26]–[30] and the references therein.

Recently, in [7], by using the lower and upper solutions method, the authors proved the existence of iterative solutions for a class of fractional initial value problem with non-monotone term

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f(t, x(t)), \ t \in (0, T], \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0 \neq 0, \ x_0 \in \mathbb{R}, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $f : (0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $0 < \alpha < 1$.

In [9], the authors discussed the existence and Ulam stability analysis of the following fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f\left(t, x(t), D_{0^+}^{\alpha} x(t)\right), \ t \in (0, T], \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0, \ x_0 \in \mathbb{R}, \end{cases}$$

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where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $f : (0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $0 < \alpha < 1$.

Inspired and motivated by the above works, we study the existence and uniqueness of solutions for the following fractional differential equation with nonlocal conditions

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f\left(t, x(t), D_{0^+}^{\alpha} x(t)\right), \ t \in (0, T], \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0 - g(x), \ x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $f : (0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : C((0, T], \mathbb{R}) \to \mathbb{R}$ are continuous nonlinear functions. To show the existence and uniqueness of solutions, we transform (1) into an integral equation and then use the Banach and Krasnoselskii fixed point theorems. Finally, we provide an example to illustrate our obtained results.

The rest of this paper is organized as follows. Some definitions from fractional calculus theory are recalled in Section 2. In Section 3, we prove the existence and uniqueness of solutions for (1). Finally, in Section 4, we give an example to illustrate the usefulness of our main results.

2. Preliminaries

In this section we present some basic definitions, notations and results of fractional calculus which are used throughout this paper.

Let $T > 0$, $J = [0, T]$. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|x\|_{\infty} = \sup\{|x(t)| : t \in J\}.$$

Let us set $AC(J)$ be the space of absolutely continuous valued functions on J, and set

$$AC^n(J) = \left\{x : J \to \mathbb{R} : x, x', x'', ..., x^{n-1} \in C(J, \mathbb{R}) \text{ and } x^{n-1} \in AC(J)\right\}.$$

In what follows $\gamma > 0$, we consider the weighted space of continuous functions

$$C_{\gamma}(J, \mathbb{R}) = \{x : (0, T] \to \mathbb{R} : t^{\gamma} x \in C(J, \mathbb{R})\},$$

with the norm

$$\|x\|_{C_{\gamma}} = \sup_{t \in J} |t^{\gamma} x(t)|.$$

Clearly $C_{\gamma}(J, \mathbb{R})$ is a Banach space.

Definition 2.1 ([16]). *The fractional integral of order $\alpha > 0$ of a function $x : J \to \mathbb{R}$ is given by*

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) \, ds,$$

provided the right side is pointwise defined on J. Where Γ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1} e^{-s} ds.$$

Definition 2.2 ([16]). *For a function $x \in AC^n(J)$, the Riemann-Liouville fractional order derivative of order α of x, is defined by*

$$D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(s) \, ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Lemma 2.3 ([16]). *The general solution of linear fractional differential equation*

$$D_{0^+}^{\alpha} x(t) = 0,$$

is given by

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n},\ c_i \in \mathbb{R},\ i = 1, 2, ..., n,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Lemma 2.4 ([16]). *We have*

$$I_{0^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1},\ \alpha \geq 0,\ \beta > 0.$$

Theorem 2.5 (Banach's fixed point theorem [25]). *Let Ω be a non-empty closed convex subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2.6 (Krasnoselskii's fixed point theorem [25]). *Let Ω be a non-empty closed bounded convex subset of a Banach space $(S, \|.\|)$. Suppose that F_1 and F_2 map Ω into S such that*
(i) $F_1 x + F_2 y \in \Omega$ for all $x, y \in \Omega$,
(ii) F_1 is continuous and compact,
(iii) F_2 is a contraction with constant $l < 1$.
Then there is a $z \in \Omega$ with $F_1 z + F_2 z = z$.

3. Existence and uniqueness

Let us start by defining what we mean by a solution of the problem (1).

Definition 3.1. *A function $x \in C^1((0,T], \mathbb{R})$ is said to be a solution of (1) if x satisfies $D_{0^+}^{\alpha} x(t) = f\left(t, x(t), D_{0^+}^{\alpha} x(t)\right)$ for any $t \in (0,T]$ and $\left. t^{1-\alpha} x(t)\right|_{t=0} = x_0 - g(x)$.*

For the existence of solutions for the problem (1), we need the following auxiliary lemma.

Lemma 3.2. *The function x solves (1) if and only if it is a solution of the integral equation*

$$x(t) = t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), D_{0^+}^{\alpha} x(s)\right) ds,\ t \in (0,T]. \tag{2}$$

Proof. Suppose the function x satisfies the problem (1), then applying $I_{0^+}^{\alpha}$ to both sides of (1), we have

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = I_{0^+}^{\alpha} f\left(t, x(t), D_{0^+}^{\alpha} x(t)\right).$$

In view of Lemma 2.3, we get

$$x(t) = c_1 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), D_{0^+}^{\alpha} x(s)\right) ds. \tag{3}$$

The condition $\left. t^{1-\alpha} x(t)\right|_{t=0} = x_0 - g(x)$ implies that

$$c_1 = x_0 - g(x). \tag{4}$$

Substituting (4) in (3) we get the integral equation (2). The converse can be proven by direct computations. The proof is completed. □

In the following subsections we prove existence, as well as existence and uniqueness results, for the problem (1) by using a variety of fixed point theorems.

The following assumptions will be used in our main results.

(H1) There exist constants $k_1 > 0$ and $k_2 \in (0,1)$ such that

$$\left| f(t,u,v) - f(t,u^*,v^*) \right| \leq k_1 |u-u^*| + k_2 |v-v^*|,$$

for $t \in (0,T]$, $u,v,u^*,v^* \in \mathbb{R}$ and $f(.,0,0) \in C_{1-\alpha}(J,\mathbb{R})$.

(H2) There exist a constant $b \in (0,1)$ such that

$$\left| g(u) - g(u^*) \right| \leq b \|u-u^*\|_{C_{1-\alpha}},$$

for $u,u^* \in C_{1-\alpha}(J,\mathbb{R})$.

3.1. Existence and uniqueness results via Banach's fixed point theorem

Theorem 3.3. *Assume that the assumptions (H1) and (H2) are satisfied. If*

$$b + \frac{\Gamma(\alpha) k_1 T^{\alpha}}{\Gamma(2\alpha)(1-k_2)} < 1, \tag{5}$$

then there exists a unique solution for the problem (1) in the space $C_{1-\alpha}(J,\mathbb{R})$.

Proof. We define the operator $\Phi : C_{1-\alpha}(J,\mathbb{R}) \to C_{1-\alpha}(J,\mathbb{R})$ by

$$(\Phi x)(t) = t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds,\ t \in (0,T],$$

where $h : (0,T] \to \mathbb{R}$ be a function satisfying the functional equation

$$h(t) = f(t, x(t), h(t)).$$

By Lemma 3.2, the fixed points of operator Φ are solutions of (1). The operator Φ is well define, i.e. for every $x \in C_{1-\alpha}(J,\mathbb{R})$ and $t > 0$, the integral

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds, \tag{6}$$

belongs to $C_{1-\alpha}(J,\mathbb{R})$. Under the condition (H1),

$$|h(t)| = \left| f(t, x(t), h(t)) \right| \leq \frac{k_1}{1-k_2} |x(t)| + ct^{\alpha-1} \text{ for each } t \in (0,T], \tag{7}$$

where $c = \frac{\sup_{t\in J}\left|t^{1-\alpha} f(t,0,0)\right|}{1-k_2}$. For every $x \in C_{1-\alpha}(J,\mathbb{R})$, we have

$$\begin{aligned}
\left| \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds \right| &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s)|\,ds \\
&\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{k_1}{1-k_2} |x(s)| + cs^{\alpha-1} \right) ds \\
&\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left(\frac{k_1}{1-k_2} \left| s^{1-\alpha} x(s) \right| + c \right) ds \\
&\leq \left(\frac{k_1}{1-k_2} \|x\|_{C_{1-\alpha}} + c \right) t^{1-\alpha} I_{0^+}^{\alpha} \left(t^{\alpha-1} \right).
\end{aligned}$$

By Lemma 2.4, we have

$$\left|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds\right| \leq \left(\frac{k_1}{1-k_2}\|x\|_{C_{1-\alpha}} + c\right)\frac{\Gamma(\alpha)\,t^\alpha}{\Gamma(2\alpha)} \leq \left(\frac{k_1}{1-k_2}\|x\|_{C_{1-\alpha}} + c\right)\frac{\Gamma(\alpha)\,T^\alpha}{\Gamma(2\alpha)}.$$

That is to say that the integral exists and belongs to $C_{1-\alpha}(J,\mathbb{R})$.

Let $x, y \in C_{1-\alpha}(J,\mathbb{R})$. Then for $t \in (0,T]$, we have

$$\left|(\Phi x)(t) - (\Phi y)(t)\right| \leq t^{\alpha-1}\left|g(x) - g(y)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|h_x(s) - h_y(s)\right| ds$$

where $h_x, h_y \in C_{1-\alpha}(J,\mathbb{R})$ be such that

$$h_x(t) = f(t, x(t), h_x(t)),$$

and

$$h_y(t) = f\left(t, y(t), h_y(t)\right).$$

By (H1) we have

$$\left|h_x(t) - h_y(t)\right| = \left|f(t, x(t), h_x(t)) - f\left(t, y(t), h_y(t)\right)\right| \leq k_1\left|x(t) - y(t)\right| + k_2\left|h_x(t) - h_y(t)\right|.$$

Then

$$\left|h_x(t) - h_y(t)\right| \leq \frac{k_1}{1-k_2}\left|x(t) - y(t)\right|.$$

Therefore, for each $t \in (0,T]$

$$\begin{aligned}\left|(\Phi x)(t) - (\Phi y)(t)\right| &\leq b t^{\alpha-1}\left\|x-y\right\|_{C_{1-\alpha}} + \frac{k_1}{\Gamma(\alpha)(1-k_2)}\int_0^t (t-s)^{\alpha-1}\left|x(s) - y(s)\right| ds \\ &= t^{\alpha-1} b\left\|x-y\right\|_{C_{1-\alpha}} + \frac{k_1}{\Gamma(\alpha)(1-k_2)}\int_0^t (t-s)^{\alpha-1} s^{\alpha-1}\left|s^{1-\alpha}(x(s) - y(s))\right| ds \\ &\leq t^{\alpha-1} b\left\|x-y\right\|_{C_{1-\alpha}} + \frac{k_1}{1-k_2} I_{0^+}^{\alpha}\left(t^{\alpha-1}\right)\left\|x-y\right\|_{C_{1-\alpha}}.\end{aligned}$$

By Lemma 2.4, we have

$$\left|(\Phi x)(t) - (\Phi y)(t)\right| \leq t^{\alpha-1} b\left\|x-y\right\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha)\,k_1 t^{2\alpha-1}}{\Gamma(2\alpha)(1-k_2)}\left\|x-y\right\|_{C_{1-\alpha}},$$

which implies that

$$\begin{aligned}\left|t^{1-\alpha}\left((\Phi x)(t) - (\Phi y)(t)\right)\right| &\leq b\left\|x-y\right\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha)\,k_1 t^{\alpha}}{\Gamma(2\alpha)(1-k_2)}\left\|x-y\right\|_{C_{1-\alpha}} \\ &\leq b\left\|x-y\right\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha)\,k_1 T^{\alpha}}{\Gamma(2\alpha)(1-k_2)}\left\|x-y\right\|_{C_{1-\alpha}}.\end{aligned}$$

Thus

$$\left\|\Phi x - \Phi y\right\|_{C_{1-\alpha}} \leq \left(b + \frac{\Gamma(\alpha)\,k_1 T^{\alpha}}{\Gamma(2\alpha)(1-k_2)}\right)\left\|x-y\right\|_{C_{1-\alpha}}.$$

From (5), Φ is a contraction. As a consequence of Banach's fixed point theorem, we get that Φ has a unique fixed point which is a unique solution of the problem (1). □

3.2. Existence results via Krasnoselskii's fixed point theorem

Theorem 3.4. *Assume (H1), (H2) and the following hypothesis*

(H3) There exist $p_1 \in C_{1-\alpha}(J, \mathbb{R}^+)$, $p_2, p_3 \in C(J, \mathbb{R}^+)$ *with* $p_3^* = \sup_{t\in J} p_3(t) < 1$ *such that*

$$\left|f(t,u,v)\right| \leq p_1(t) + p_2(t)|u| + p_3(t)|v|,$$

for $t \in (0,T]$ *and each* $u, v \in \mathbb{R}$.

If

$$\lambda = b + \frac{p_2^*\Gamma(\alpha)T^\alpha}{\left(1-p_3^*\right)\Gamma(2\alpha)} < 1,$$

where $p_2^* = \sup_{t\in J} p_2(t)$. *Then the boundary value problem (1) has at least one solution in* Ω.

Proof. Set

$$R = \frac{1}{1-\lambda},\ \Lambda = |x_0| + Q + \frac{T^\alpha p_1^*\Gamma(\alpha)}{\left(1-p_3^*\right)\Gamma(2\alpha)},$$

where $p_1^* = \sup_{t\in J}\left\{t^{1-\alpha}p_1(t)\right\}$ and $Q = \left|g(0)\right|$. Let us fix

$$M \geq R\Lambda.$$

Consider the non-empty closed bounded convex subset $\Omega = \left\{x \in C_{1-\alpha}(J,\mathbb{R}) : \|x\|_{C_{1-\alpha}} \leq M\right\}$ and define two operators F_1 and F_2 on Ω, as follows

$$(F_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds,$$

and

$$(F_2x)(t) = t^{\alpha-1}\left(x_0 - g(x)\right),$$

where $h : (0,T] \to \mathbb{R}$ be a function satisfying the functional equation

$$h(t) = f(t, x(t), h(t)).$$

We shall use the Krasnoselskii fixed point theorem to prove there exists at least one fixed point of the operator $F_1 + F_2$ in Ω. The proof will be given in several steps.

Step 1. We prove that $F_1x + F_2y \in \Omega$ for all $x, y \in \Omega$.

For any $x, y \in \Omega$ and $t \in (0,T]$, we have

$$\begin{aligned}\left|(F_1x)(t) + (F_2y)(t)\right| &\leq \left|t^{\alpha-1}\left(x_0 - g(x)\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds\right| \\ &\leq t^{\alpha-1}|x_0| + t^{\alpha-1}\left|g(x) - g(0)\right| + t^{\alpha-1}\left|g(0)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\left|s^{1-\alpha}h(s)\right|ds \\ &\leq t^{\alpha-1}|x_0| + t^{\alpha-1}b\|x\|_{C_{1-\alpha}} + t^{\alpha-1}Q + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\left|s^{1-\alpha}h(s)\right|ds \\ &\leq t^{\alpha-1}|x_0| + t^{\alpha-1}bM + t^{\alpha-1}Q + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\left|s^{1-\alpha}h(s)\right|ds. \qquad (8)\end{aligned}$$

By (H3), for each $t \in (0, T]$, we have

$$|h(t)| = \left|f(t, x(t), h(t))\right| \leq p_1(t) + p_2(t)|x(t)| + p_3(t)|h(t)|.$$

Hence, we get

$$\left|t^{1-\alpha}h(t)\right| \leq t^{1-\alpha}p_1(t) + p_2(t)\left|t^{1-\alpha}x(t)\right| + p_3(t)\left|t^{1-\alpha}h(t)\right| \leq p_1^* + p_2^*M + p_3^*\left|t^{1-\alpha}h(t)\right|,$$

then, we have

$$\left|t^{1-\alpha}h(t)\right| \leq \frac{p_1^* + p_2^*M}{1 - p_3^*}. \tag{9}$$

Replacing (9) in the inequality (8) and with Lemma 2.4, we get

$$\begin{aligned}\left|(F_1x)(t) + (F_2y)(t)\right| &\leq t^{\alpha-1}|x_0| + t^{\alpha-1}bM + t^{\alpha-1}Q + \left(\frac{p_1^* + p_2^*M}{1 - p_3^*}\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\ &\leq t^{\alpha-1}|x_0| + t^{\alpha-1}bM + t^{\alpha-1}Q + \left(\frac{p_1^* + p_2^*M}{1 - p_3^*}\right)\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}.\end{aligned}$$

Therefore

$$\left|t^{1-\alpha}((F_1x)(t) + (F_2x)(t))\right| \leq |x_0| + Q + \frac{T^\alpha p_1^*\Gamma(\alpha)}{\left(1 - p_3^*\right)\Gamma(2\alpha)} + \left(b + \frac{p_2^*\Gamma(\alpha)T^\alpha}{\left(1 - p_3^*\right)\Gamma(2\alpha)}\right)M.$$

Thus

$$\left\|F_1x + F_2x\right\|_{C_{1-\alpha}} \leq \Lambda + \lambda M \leq \frac{M}{R} + \left(1 - \frac{1}{R}\right)M = M.$$

Hence $F_1x + F_2y \in \Omega$ for all $x, y \in \Omega$.

Step 2. We show that F_1 is continuous.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in $C_{1-\alpha}(J, \mathbb{R})$, then for each $t \in (0, T]$, we have

$$\left|(F_1x_n)(t) - (F_1x)(t)\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h_n(s) - h(s)|ds, \tag{10}$$

where $h_n, h \in C_{1-\alpha}(J, \mathbb{R})$ be such that

$$h_n(t) = f(t, x_n(t), h_n(t)),$$

and

$$h(t) = f(t, x(t), h(t)).$$

By (H1) we have

$$|h_n(t) - h(t)| = \left|f(t, x_n(t), h_n(t)) - f(t, x(t), h(t))\right| \leq k_1|x_n(t) - x(t)| + k_2|h_n(t) - h(t)|.$$

Then

$$|h_n(t) - h(t)| \leq \frac{k_1}{1 - k_2}|x_n(t) - x(t)|. \tag{11}$$

By replacing (11) in inequality (10), we find

$$\begin{aligned}\left|(F_1x_n)(t)-(F_1x)(t)\right| &\leq \frac{k_1}{(1-k_2)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|x_n(t)-x(t)\right| ds\\ &= \frac{k_1}{(1-k_2)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\left|s^{1-\alpha}(x_n(t)-x(t))\right| ds\\ &\leq \frac{k_1}{1-k_2}I_{0^+}^{\alpha}\left(t^{\alpha-1}\right)\|x_n-x\|_{C_{1-\alpha}}\end{aligned}$$

By Lemma 2.4, we have

$$\left|(F_1x_n)(t)-(F_1x)(t)\right| \leq \frac{\Gamma(\alpha)k_1t^{2\alpha-1}}{(1-k_2)\Gamma(2\alpha)}\|x_n-x\|_{C_{1-\alpha}},$$

which implies that

$$\left|t^{1-\alpha}((F_1x_n)(t)-(F_1x)(t))\right| \leq \frac{\Gamma(\alpha)k_1t^{\alpha}}{(1-k_2)\Gamma(2\alpha)}\|x_n-x\|_{C_{1-\alpha}} \leq \frac{\Gamma(\alpha)k_1T^{\alpha}}{(1-k_2)\Gamma(2\alpha)}\|x_n-x\|_{C_{1-\alpha}}.$$

Thus

$$\left\|F_1x_n-F_1x\right\|_{C_{1-\alpha}} \leq \frac{\Gamma(\alpha)k_1T^{\alpha}}{(1-k_2)\Gamma(2\alpha)}\|x_n-x\|_{C_{1-\alpha}},$$

and hence

$$\left\|F_1x_n-F_1x\right\|_{C_{1-\alpha}} \to 0 \text{ as } n\to\infty.$$

Consequently, F_1 is continuous.

Step 3. We prove that F_1 is compact.

For all $x\in\Omega$ and $t\in(0,T]$, we have

$$\left|(F_1x)(t)\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\left|s^{1-\alpha}h(s)\right| ds. \tag{12}$$

Replacing (9) in the inequality (12) and with Lemma 2.4, we get

$$\left|(F_1x)(t)\right| \leq \left(\frac{p_1^*+p_2^*M}{1-p_3^*}\right)\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}.$$

Therefore

$$\left|t^{1-\alpha}(F_1x)(t)\right| \leq \left(\frac{p_1^*+p_2^*M}{1-p_3^*}\right)\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}.$$

Thus

$$\left\|F_1x\right\|_{C_{1-\alpha}} \leq \left(\frac{p_1^*+p_2^*M}{1-p_3^*}\right)\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}.$$

Hence $F_1(\Omega)$ is uniformly bounded.

It remains to show that $F_1(\Omega)$ is equicontinuous, let $0 \le t_1 < t_2 \le T$ and $x \in \Omega$. Then

$$\begin{aligned}
&\left|t_2^{1-\alpha}(F_1x)(t_2) - t_1^{1-\alpha}(F_1x)(t_1)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} t_2^{1-\alpha}(t_2-s)^{\alpha-1}h(s)\,ds + \int_{t_1}^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}h(s)\,ds - \int_0^{t_1} t_1^{1-\alpha}(t_1-s)^{\alpha-1}h(s)\,ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|t_2^{1-\alpha}(t_2-s)^{\alpha-1}s^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}s^{\alpha-1}\right|\left|s^{1-\alpha}h(s)\right|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}s^{\alpha-1}\left|s^{1-\alpha}h(s)\right|ds \\
&\le \frac{p_1^* + p_2^*M}{1-p_3^*}\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right|s^{\alpha-1}ds\right) \\
&+ \frac{p_1^* + p_2^*M}{1-p_3^*}\left(\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}s^{\alpha-1}ds\right).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. That is to say that $F_1(\Omega)$ is equicontinuous, then by Ascoli-Arzela theorem, we can conclude that the operator F_1 is compact.

Step 4. We prove that $F_2 : \Omega \to C_{1-\alpha}(J, \mathbb{R})$ is a contraction mapping.

For all $x \in \Omega$ and from (H2), we have

$$\left|(F_2x)(t) - (F_2y)(t)\right| = \left|t^{\alpha-1}(g(x) - g(y))\right| \le t^{\alpha-1}b\left\|x-y\right\|_{C_{1-\alpha}}.$$

Therefore

$$\left|t^{1-\alpha}((F_2x)(t) - (F_2y)(t))\right| \le b\left\|x-y\right\|_{C_{1-\alpha}}.$$

Thus

$$\left\|F_2x - F_2y\right\|_{C_{1-\alpha}} \le b\left\|x-y\right\|_{C_{1-\alpha}}.$$

Hence, the operator F_2 is a contraction.

Clearly, all the hypotheses of the Krasnoselskii fixed point theorem (see [25]) are satisfied. Thus there a fixed point $x \in \Omega$ such that $x = F_1x + F_2x$, which is a solution of the problem (1). □

4. Example

We consider the following fractional initial value problem

$$\begin{cases} D_{0^+}^{\frac{2}{3}}x(t) = \dfrac{1}{4\exp(-t+2)\left(1+|x(t)|+\left|D_{0^+}^{\frac{2}{3}}x(t)\right|\right)} + \dfrac{1}{t^{\frac{1}{3}}},\ t \in (0,1], \\ t^{\frac{1}{3}}x(t)\Big|_{t=0} = \frac{1}{2} - \sum\limits_{i=1}^{n} c_i t_i^{\frac{1}{3}} y(t_i), \end{cases} \tag{13}$$

where $0 < t_1 < ... < t_n < 1$ and c_i, $i = 1, ..., n$ are positive constants with $\sum\limits_{i=1}^{n} c_i \le \frac{1}{4}$. Set

$$f(t,u,v) = \frac{1}{4\exp(-t+2)(1+|u|+|v|)} + \frac{1}{t^{\frac{1}{3}}},\ t \in (0,1],\ u,v \in \mathbb{R},$$

We have

$$C_{1-\alpha}([0,1],\mathbb{R}) = C_{\frac{1}{3}}([0,1],\mathbb{R}) = \left\{h : (0,1] \to \mathbb{R} : t^{\frac{1}{3}}h \in C([0,1],\mathbb{R})\right\},$$

with $\alpha = \frac{2}{3}$. Clearly the functions f and g are continuous, $f(.,0,0) \in C_{\frac{1}{3}}([0,1],\mathbb{R})$. For each $u, u^*, v, v^* \in \mathbb{R}$ and $t \in (0,1]$, we have

$$\left|f(t,u,v) - f(t,u^*,v^*)\right| = \left|\frac{1}{4\exp(-t+2)}\left(\frac{1}{(1+|u|+|v|)} - \frac{1}{(1+|u^*|+|v^*|)}\right)\right|$$
$$\leq \frac{|u-u^*|+|v-v^*|}{4\exp(-t+2)(1+|u|+|v|)(1+|u^*|+|v^*|)}$$
$$\leq \frac{1}{4e}(|u-u^*|+|v-v^*|),$$

and

$$\left|g(u) - g(u^*)\right| \leq \sum_{i=1}^{n} c_i t_i^{\frac{1}{3}} |u(t_i) - u^*(t_i)| \leq \sum_{i=1}^{n} c_i \|u-u^*\|_{C_{\frac{1}{3}}} \leq \frac{1}{4}\|u-u^*\|_{C_{\frac{1}{3}}}.$$

Hence, conditions (H1) and (H2) are satisfied with $k_1 = k_2 = \frac{1}{4e}$and $b = \frac{1}{4}$. The condition

$$b + \frac{\Gamma(\alpha)k_1T^\alpha}{\Gamma(2\alpha)(1-k_2)} = \frac{1}{4} + \frac{\frac{\Gamma\left(\frac{2}{3}\right)}{4e}}{\Gamma\left(\frac{4}{3}\right)\left(1-\frac{1}{4e}\right)} \simeq 0.4 < 1,$$

is satisfied with $T = 1$. It follows from Theorem 3.3 that the problem (13) has a unique solution in the space $C_{\frac{1}{3}}([0,1],\mathbb{R})$.

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