Filomat 34:14 (2020), 4821–4827
https://doi.org/10.2298/FIL2014821C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Permutations That Preserve Asymptotically Null Sets and Statistical Convergence

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Abstract. The main result of this article is a characterization of the permutations $\theta : \mathbb{N} \to \mathbb{N}$ that map a set with zero asymptotic density into a set with zero asymptotic density; a permutation has this property if and only if the lower asymptotic density of C_p tends to 1 as $p \to \infty$ where p is an arbitrary natural number and $C_p = \left\{ l : \theta^{-1}(l) \leq lp \right\}$. We then show that a permutation has this property if and only if it maps statistically convergent sequences into statistically convergent sequences.

This main result of this note is a characterization of the permutations of the natural numbers that map sets with asymptotic density zero to sets with asymptotic density zero. Recall that, for $A \subseteq \mathbb{N}$, the upper and lower asymptotic density of A, denoted $\overline{\delta}(A)$ and $\underline{\delta}(A)$ respectively, are defined by

$$\overline{\delta}(A) = \limsup_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in A\}\right| \tag{1}$$

and

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in A\}\right| \tag{2}$$

In the case that $\overline{\delta}(A) = \underline{\delta}(A) = \gamma$, we say that A has density γ and write $\delta(A) = \gamma$. In this note we characterize the permutations $\theta : \mathbb{N} \to \mathbb{N}$ that have the property that $\delta(A) = 0$ implies that $\delta(\theta A) = 0$ where $\theta A = \{\theta(k) : k \in A\}$. We also show that this property also characterizes the permutations that map statistically convergent sequences to statistically convergent sequences.

It will be helpful to have the definition of statistical convergence available to us. Statistical convergence was introduced by Fast [3] and Steinhaus [11], and became an active area of research after the publication of Šalát [9] and Fridy's [4] oft-cited articles. A real-valued sequence $x = \left(x_j\right)$ is statistically convergent to L provided that $\delta(\{k : |x_k - L| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$. In this case we write $st - \lim x = L$. It is straightforward to verify, and well-known, for a bounded sequence $x = (x_k)$, that

$$\lim x = L \Rightarrow st - \lim x = L \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = L \tag{3}$$

2010 *Mathematics Subject Classification*. Primary 11B05, 40A35; Secondary 40G15
Keywords. permutations, asymptotically null sets, statistical convergence
Received: 20 February 2020; Revised: 12 March 2020; Accepted: 20 March 2020
Communicated by Eberhard Malkowsky
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and that none of the preceding implications can be reversed. Recall that a sequence $x = (x_k)$ is said to be Cesàro summable to L provided that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = L$. The scope of statistical convergence has been generalized extensively since Fridy and Šalát's articles first appeared, where the asymptotic density is typically replaced by finitely additive measures generated by regular summability matrices or, more generally, by ideals of subsets of $\mathbb{N}$ (cf. Kostryko et. al. [6], Connor [2], Kolk [5] and other related articles). Due to the computational nature of the proof of Theorem 1, this article is confined to the asymptotic density.

One class of permutations that preserve asymptotically null sets is the well-studied Lévy group. The Lévy group $\boldsymbol{G}$ [7] is the group of all permutations θ of $\mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{|\{k \le n : \theta(k) > n\}|}{n} = 0. \tag{4}$$

Any member $\theta \in \boldsymbol{G}$ has the property that if $A \subseteq \mathbb{N}$ with density $\delta(A) = \gamma$, then $\delta(\theta A) = \gamma$ (cf. [8], [7]). Note should be taken, though, that the Lévy group is a proper subgroup of the group of permutations that map Cesàro summable sequences to Cesàro summable sequences via the mapping $\theta x_k = x_{\theta^{-1}(k)}$ (Proposition 4.1 of [8]) and it is also noteworthy that there are permutations with the property that $\delta(A) = 0$ implies that $\delta(\theta A) = 0$, but for which there is a set B with non-zero density $\delta(B)$ such that $\delta(B) \ne \delta(\theta B)$ (Remark on pg. 667 of [1]). One can also characterize members of the Lévy group via statistical convergence [10], in particular a permutation $\theta : \mathbb{N} \to \mathbb{N}$ belongs to $\boldsymbol{G}$ if and only if

$$st - \lim_{n\to\infty} \frac{\theta(n)}{n} = 1. \tag{5}$$

Before starting, we establish some notation. For $C \subseteq \mathbb{N}$, we use $|C|$ to denote the cardinality of C and let χ_C denote the characteristic function of C. For $n \in \mathbb{N}$ and $C \subseteq \mathbb{N}$, we let $\delta_n(C) = (1/n)\,|\{k \le n : k \in C\}|$. It is also convenient to note that if the statistically convergent sequence $x = \left(x_j\right)$ is defined by $x_j = \chi_C(j)$ for some $C \subseteq \mathbb{N}$, and hence C has density 0 or 1, then $st - \lim x = \delta(C)$. Also note that for a permutation θ and a sequence $x = \chi_C$ such that θx is statistically convergent, we have

$$\theta x_j = \chi_C\left(\theta^{-1}(j)\right) = 1 \Leftrightarrow \theta^{-1}(j) \in C \Leftrightarrow j \in \theta C \tag{6}$$

and hence $st - \lim \theta x = \delta(\theta C)$.

We now turn to the main result:

Lemma 1. *Let A and B be subsets of $\mathbb{N}$ such that $\overline{\delta}(A) > z$ and $\underline{\delta}(B) > 1 - z/2$. Then $\overline{\delta}(A \bigcap B) > z/2$.*

Proof: The hypothesis yields that there are infinitely many n such that $\delta_n(A) > z$ and $\delta_n(B) > 1 - z/2$. For these n we have that

$$1 \ge \delta_n(A \cup B) = \delta_n(A) + \delta_n(B) - \delta_n(A \cap B) > 1 + z/2 - \delta_n(A \cap B) \tag{7}$$

and, as this occurs for infinitely many n, we have that $\overline{\delta}(A \cap B) > z/2$. ■

Theorem 1. *Let $\theta : \mathbb{N} \to \mathbb{N}$ be a permutation. Then the following are equivalent:*

(i) *If $A \subset \mathbb{N}$ and $\delta(A) = 0$, then $\delta(\theta A) = 0$ where $\theta A = \{\theta(k) : k \in A\}$.*

(ii) $\overline{\delta}\left(\left\{l : \theta^{-1}(l) > lp\right\}\right) \to 0$ *as $p \to \infty$, where p is a natural number.*

Proof: First we establish that (ii) implies (i). Suppose there is a set $A \subset \mathbb{N}$ such that $\overline{\delta}(\theta A) = z > 0$. We will show that if θ satisfies (ii), then $\overline{\delta}(A) > 0$. Hence $\delta(A) = 0$ implies $\delta(\theta A) = 0$.

Define $C_p = \left\{l : \theta^{-1}(l) \leq lp\right\}, \quad p \in \mathbb{N}$, and let $S = \left\{k \leq n : k \in \theta A \cap C_p\right\}$. Now $s \in S$ implies that $\theta^{-1}(s) \in A$ and that $\theta^{-1}(s) \leq sp$. Now $s \leq n$ yields $\theta^{-1}(s) \leq np$. Hence $s \in S$ implies that $\theta^{-1}(s) \in \{j \leq np : j \in A\}$ and we note that

$$\delta_n(S) = \frac{1}{n}\left|\left\{k \leq n : k \in \theta A \cap C_p\right\}\right| \leq p\left(\frac{1}{np}\left|\{j \leq np : j \in A\}\right|\right). \tag{8}$$

Now select $p \in \mathbb{N}$ such that $\underline{\delta}\left(C_p\right) \geq 1 - z/2$ and thus, by the lemma, $\overline{\delta}\left(\theta A \cap C_p\right) > z/2$. It follows that

$$0 < \frac{z}{2} < \overline{\delta}(\theta A \cap C_p) \leq p\overline{\delta}(A). \tag{9}$$

Hence (ii) yields $\overline{\delta}(\theta A) > z$ implies $\overline{\delta}(A) \geq z/(2p) > 0$.

Next we establish that (i) implies (ii). Set $E_p = \left\{l : \theta^{-1}(l) > lp\right\}$ and note that $E_p \supseteq E_{p+1}$ for all p, and hence $\overline{\delta}\left(E_p\right)$ is a nonincreasing sequence bounded below by 0. Suppose, for the sake of contradiction, $\overline{\delta}\left(E_p\right) \to \eta > 0$ as $p \to \infty$. We will construct a set F such that $\delta(F) = 0$ and $\overline{\delta}(\theta F) > 0$, which will establish the contrapositive of (i) implies (ii), and hence complete the proof of the theorem.

The set F will be constructed inductively. First, select n_1 such that $\delta_{n_1}(E_1) > \eta$ and set $I_1 = \left\{\theta^{-1}(l) : l \in E_1, l \leq n_1\right\}$. Now select $\beta(1)$ such that $\beta(1) \geq \max\{j : j \in I_1\}$ and $n_1/\beta(1) < 1/2$. Now select $n_2 > \beta(1)$ such that

$$\delta_{n_2}(E_2 \backslash \{1, 2, \ldots, \beta(1)\}) = \frac{1}{n_2}\left|\{\beta(1) < l \leq n_2 : \theta^{-1}(l) > 2l\}\right| > \eta \tag{10}$$

and set $I_2 = \left\{\theta^{-1}(l) : \beta(1) < l \leq n_2,\ l \in E_2\right\}$. Select $\beta(2)$ such that $\beta(2) \geq \max\{j : j \in I_2\}$ and $n_2/\beta(2) < 1/3$.

We pause to make a couple of observations. First, I_1 and I_2 are disjoint. This follows from observing that $\{j : j \in E_1, j \leq n_1\}$ and $E_2 \cap \{\beta(1), \ldots, n_2\}$ are disjoint and that θ^{-1} is one-to-one. Now let $\beta(1) < n \leq \beta(2)$ and note that

$$\delta_n(I_1) \leq \frac{|I_1|}{n} \leq \frac{n_1}{n} \leq \frac{n_1}{\beta(1)} < \frac{1}{2}. \tag{11}$$

Next we compute an estimate for $\delta_n(I_2)$. Suppose that $j \in I_2$ and $j = \theta^{-1}(l)$. Since $l \in E_2$, we have that

$$2\beta(1) \leq 2l \leq j = \theta^{-1}(l) \leq n. \tag{12}$$

This yields that $\beta(1) \leq l \leq n/2$ and, since θ is a one-to-one correspondence, $\left|\{j \leq n : n \in I_2\}\right| \leq n/2 - \beta(1)$. Thus

$$\delta_n(I_2) = \frac{1}{n}\left(\frac{n}{2} - \beta(1)\right) < \frac{1}{2} \tag{13}$$

and hence $\delta_n(I_1 \cup I_2) < 1$.

Now we continue with the construction of F. Select n_3 such that $n_3 > \beta(2)$ and

$$\delta_{n_3}(E_3 \backslash \{1, 2, \ldots, \beta(2)\}) = \frac{1}{n_3}\left|\left\{\beta(2) < l \leq n_3 : \theta^{-1}(l) > 3l\right\}\right| > \eta. \tag{14}$$

Set $I_3 = \{\theta^{-1}(l) : \beta(2) < l \leq n_3, l \in E_3\}$ and select $\beta(3)$ such that $\beta(3) \geq \max\{j : j \in I_3\}$ and $n_3/\beta(3) < 1/4$.

As before, I_1, I_2 and I_3 are the images of a one-to-one correspondence of a collection of disjoint sets, and hence disjoint. Thus $\delta_n(I_1 \cup I_2 \cup I_3) = \delta_n(I_1 \cup I_2) + \delta_n(I_3)$.

Now suppose that $\beta(2) < n \le \beta(3)$. Since $|I_1 \cup I_2| \le n_2$ and $\beta(2) < n$, it follows that

$$\delta_n(I_1 \cup I_2) < \frac{n_2}{\beta(2)} < \frac{1}{3}.$$

Next we estimate $\delta_n(I_3)$. Note that $j \in I_3$ implies $j = \theta^{-1}(l)$ where $l \in E_3$ and thus $n \ge j = \theta^{-1}(l) \ge 3l$. It follows that $\left|\{j \le n : n \in I_3\}\right| \le n/3 - \beta(2)$ and $\delta_n(I_3) < 1/3$. Hence

$$\delta_n(I_1 \cup I_2 \cup I_3) = \delta_n(I_1 \cup I_2) + \delta_n(I_3) < 1/3 + 1/3 = 2/3. \tag{15}$$

We proceed inductively. Suppose that n_p, I_p, and $\beta(p)$ have been selected such that:

1. $\delta_{n_p}\left(E_p \backslash \{1, 2, \ldots, \beta(p-1)\}\right) > \eta$;
2. $I_p = \left\{\theta^{-1}(l) : \beta(p-1) < l \le n_p, l \in E_p\right\}$;
3. $\beta(p) \ge \max\left\{j : j \in I_p\right\}$ and $n_p/\beta(p) < 1/(p+1)$;
4. $I_1, I_2, \cdots, I_p$ are disjoint; and
5. $\delta_n\left(\bigcup_{j=1}^{p} I_j\right) < 2/p$ for $\beta(p-1) < n \le \beta(p)$.

Now select n_{p+1} such that

$$\delta_{n_{p+1}}\left(E_{p+1} \backslash \{1, 2, \cdots, \beta(p)\}\right) > \eta \tag{16}$$

and let

$$I_{p+1} = \left\{\theta^{-1}(l) : \beta(p) < l \le n_{p+1}, l \in E_{p+1}\right\}. \tag{17}$$

Select $\beta(p+1)$ such that $\beta(p+1) \ge \max\left\{j : j \in I_{p+1}\right\}$ and $n_{p+1}/\beta(p+1) < 1/(p+2)$. As in the preceding, the intervals $I_1, I_2, \cdots, I_{p+1}$ are disjoint and, if $\beta(p) < n \le \beta(p+1)$, then

$$\delta_n\left(\bigcup_{j=1}^{p} I_j\right) < \frac{n_p}{\beta(p)} < \frac{1}{p+1}. \tag{18}$$

Next we estimate $\delta_n\left(I_{p+1}\right)$. Note that $j \in I_{p+1}$ implies $j = \theta^{-1}(l)$ where $l \in E_{p+1}$ and $j \le n$, hence we have that $n \ge j = \theta^{-1}(l) \ge (p+1)l$ and consequently $n/(p+1) \ge l$. It follows that $\left|\left\{j \le n : n \in I_p\right\}\right| \le n/(p+1) - \beta(p)$ and $\delta_n\left(I_{p+1}\right) < 1/p+1$, hence

$$\delta_n\left(\bigcup_{j=1}^{p+1} I_j\right) < \frac{2}{p+1}. \tag{19}$$

Now let $F = \cup_{j=1}^{\infty} I_j$. Note that if $\beta(p) < n \le \beta(p+1)$ we have that, as the I_j 's are disjoint sets,

$$\frac{1}{n}\left|\{k \le n : k \in F\}\right| = \frac{1}{n}\left|\left\{k \le n : k \in \bigcup_{j=1}^{p+1} I_j\right\}\right| \le \frac{2}{p+1} \tag{20}$$

and thus $\delta(F) = 0$.

Next observe that

$$\delta_{n_{p+1}}\left(\theta F \backslash \{1, 2, \cdots, \beta(p)\}\right) = \frac{1}{n_p+1}\left|\left\{\beta(p) < l \le n_p : l \in E_p\right\}\right| > \eta \tag{21}$$

for all p, and hence $\overline{\delta}(\theta F) \geq \eta > 0$.

Thus $\overline{\delta}\left(E_p\right) \nrightarrow 0$ implies θ does not take null sets to null sets, and consequently θ takes null sets to null sets implies (ii), i.e., $\overline{\delta}\left(\left\{l : \theta^{-1}(l) > lp\right\}\right) \to 0$ as $p \to \infty$. ■

Example 1. *The following example shows that a permutation θ may have the property that $\delta(A) = 0$ implies $\delta(\theta A) = 0$ and fail to have the property $\delta(\theta A) = 0$ implies $\delta(A) = 0$. Define $\theta : \mathbb{N} \to \mathbb{N}$ by*

$$\theta(j) = \begin{cases} j^2 + 1 & j \quad \text{is even} \\ \min\{l : l \neq \theta(k), k < j\} & j \quad \text{is odd} \end{cases} \tag{22}$$

First we show that $\delta(A) = 0$ implies that $\delta(\theta A) = 0$. Suppose that $A \subseteq \mathbb{N}$ such that $\delta(A) = 0$. Let $A_O = \{k \in A : k \text{ odd }\}$ and $A_E = \{k \in A : k \text{ even }\}$. We will show that both θA_O and θA_E have asymptotic density zero and hence $\theta A = \theta A_O \cup \theta A_E$ also has asymptotic density zero.

First we show that θA_E is a null set. Suppose that $j \in \theta A_E$ and hence $j = \theta(l)$ where l is even. Now $\theta(l) = l^2 + 1$ and it follows that $j = \theta(l) \leq n$ implies that $l \leq \sqrt{n-1}$. Thus

$$\left|\{j \leq n : j \in \theta A_E\}\right| = \left|\{\theta(j) \leq n : j \in A_E\}\right| \leq \sqrt{n-1}. \tag{23}$$

As

$$\frac{1}{n}\left|\{j \leq n : j \in \theta A_E\}\right| \leq \frac{\sqrt{n-1}}{n} \to 0 \tag{24}$$

as $n \to \infty$, we have that $\delta(\theta A_E) = 0$.

Next we show that θA_O is a null set. Note that if $j \in \theta A_O$ then $j = \theta(l)$ where l is odd. Note that $\theta(l)$ is equal to the number of odds less than or equal to l plus the number of solutions to $m^2 + 1 \leq l$ where m is even, or

$$\theta(l) = \frac{l+1}{2} + \frac{1}{2}\left\lfloor \sqrt{l-1} \right\rfloor. \tag{25}$$

It follows that $\theta(l) \leq n$ implies that $\frac{l+1}{2} \leq n$ and consequently $l < 2n$. Now

$$\begin{aligned} \frac{1}{n}\left|\{j \leq n : j \in \theta A_O\}\right| &= \frac{1}{n}\left|\{\theta(k) \leq n : k \in A_O\}\right| \\ &\leq \frac{1}{n}\left|\{k \leq 2n : k \in A_O\}\right| \\ &\leq 2\left[\frac{1}{2n}\left|\{k \leq 2n : k \in A_O\}\right|\right] \end{aligned}$$

which tends to 0 as n tends to ∞. As $\delta(\theta A_O) = \delta(\theta A_E) = 0$, we have that $\delta(\theta A) = 0$.

Next we show there is a set B such that $\delta(\theta B) = 0$ but $\delta(B) \neq 0$. Let B denote the even integers and observe that $\theta B = \{\theta(2), \theta(4), \theta(6), \theta(8), \ldots\} = \{5, 17, 37, 65, \ldots\}$. Hence θ^{-1} does not map asymptotically zero sets to asymptotically zero sets. ■

The preceding theorem can be used to obtain a characterization of permutations such that θA is a null set if and only if A is a null set:

Corollary 1. *Let $\theta : \mathbb{N} \to \mathbb{N}$ be permutation and, for $p \in \mathbb{N}$, set*

$$D_p = \left\{l : \max\left(\theta(l), \theta^{-1}(l)\right) > lp\right\}.$$

Then the following are equivalent:

(i) *For $E \subseteq \mathbb{N}$, we have that $\delta(E) = 0$ if and only if $\delta(\theta E) = 0$.*

(ii) $\overline{\delta}\left(D_p\right) \to 0$ *as* $p \to \infty$.

Proof: First we establish (i) implies (ii) by establishing the contrapositive. Suppose that $\overline{\delta}\left(D_p\right) \nrightarrow 0$. Observe that

$$\left\{l : \max\left(\theta\left(l\right)\right), \theta^{-1}\left(l\right)\right) > lp\right\} = \left\{l : \theta\left(l\right) > lp\right\} \cup \left\{l : \theta^{-1}\left(l\right) > lp\right\} \tag{26}$$

and hence $\overline{\delta}\left(D_p\right) \leq \overline{\delta}\left(\left\{l : \theta\left(l\right) > lp\right\}\right) + \overline{\delta}\left(\left\{l : \theta^{-1}\left(l\right) > lp\right\}\right)$. As $\overline{\delta}\left(D_p\right) \nrightarrow 0$ as $p \to \infty$, at least one of $\overline{\delta}\left(\left\{l : \theta\left(l\right) > lp\right\}\right)$ or $\overline{\delta}\left(\left\{l : \theta^{-1}\left(l\right) > lp\right\}\right)$ does not tend to zero as p tends to infinity. Hence, by the theorem, there is an $A \subseteq \mathbb{N}$ such that $\delta\left(A\right) = 0$ and $\overline{\delta}(\theta A) > 0$ or such that $\delta\left(\theta A\right) = 0$ and $\overline{\delta}(A) > 0$.

Next we establish that (ii) implies (i). Observe that $\left\{l : \theta^{-1}\left(l\right) > lp\right\} \subseteq D_p$ and, as $\overline{\delta}\left(D_p\right) \to 0$ as $p \to \infty$, it follows that $\overline{\delta}\left(\left\{l : \theta^{-1}\left(l\right) > lp\right\}\right) \to 0$ as $p \to \infty$. Hence, by the theorem, $\delta(A) = 0$ implies that $\delta\left(\theta A\right) = 0$. Similarly $\overline{\delta}\left(\left\{l : \theta\left(l\right) > lp\right\}\right) \to 0$ as $p \to \infty$. Now by the theorem, $\delta\left(A\right) = 0$ implies $\delta\left(\theta^{-1}A\right) = 0$. Hence, if $\delta\left(\theta A\right) = 0$, we have that $\delta\left(\theta^{-1}\left(\theta A\right)\right) = \delta\left(A\right) = 0$. ■

The permutations that take asymptotically null sets to asymptotically null sets are also the same permutations that rearrange statistically convergent sequences into statistically convergent sequences. Recall that a sequence $x = \left(x_j\right)$ is statistically convergent to L if and only if $\delta(\{j : \left|x_j - L\right| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$. A permutation θ of $\mathbb{N}$ can be used to rearrange a sequence, denoted θx, by defining $\theta x_j = x_{\theta^{-1}(j)}$. The reader should be aware that, depending upon an author's preference, θx is sometimes defined by $\theta x_j = x_{\theta(j)}$; in this note we are following the convention used by Obata [8]. If one is using the definition $\theta x_j = x_{\theta(j)}$, one should replace θA by $\theta^{-1}A$ in the second assertion of Theorem 1.

Theorem 2. *Let θ be a permutation from $\mathbb{N}$ onto itself. The following are equivalent:*

(i) *If x is statistically convergent, then θx is statistically convergent.*

(ii) *If $A \subseteq \mathbb{N}$ and A has asymptotic density zero , then θA has asymptotic density zero, i.e., if $\delta(A) = 0$, then $\delta\left(\theta A\right) = 0$.*

(iii) *If x is statistically convergent, then θx is statistically convergent and $st - \lim x = st - \lim \theta x$.*

Proof: First we establish (i) implies (ii). Let $A \subseteq \mathbb{N}$ such that $\delta\left(A\right) = 0$ and let $x = \chi_A$. Since $\delta\left(A\right) = 0$, the sequence x is statistically null and hence, by hypothesis, θx is statistically convergent. Observe that, since θx is a sequence of 0 's and 1 's, we have that $st - \lim \theta x = 0$ or $st - \lim \theta x = 1$.

We suppose that $st - \lim \theta x = \delta\left(\theta A\right) = 1$ and arrive at a contradiction. Since $\delta\left(\theta A\right) = 1$, then if we let B denote the even elements of θA, then B has density $1/2$ and hence is not statistically convergent. Now set $C = \theta^{-1}\left(B\right)$ and note, since $C \subseteq A$, we have that $\delta\left(C\right) = 0$. Thus $y = \chi_C$ is statistically convergent but $\theta y = \chi_{\theta C} = \chi_B$ is not statistically convergent. Hence it must be the case that $st - \lim \theta x = \delta\left(\theta A\right) = 0$.

Next we establish (ii) implies (iii). First we will establish that if $st - \lim x = 0$, then $st - \lim \theta x = 0$. Let $\varepsilon > 0$ and set $A = \left\{j : \left|x_j\right| \geq \varepsilon\right\}$. By definition, $\delta\left(A\right) = 0$ and hence $\delta\left(\theta A\right) = 0$. Now recall that $\left\{j : \left|\theta x_j\right| \geq \varepsilon\right\} = \theta A$ and consequently θx is statistically convergent to 0.

Now suppose that x is statistically convergent and let $e = \chi_{\mathbb{N}}$. Then there is an L such that $x - Le$ is statistically null and hence $\theta\left(x - Le\right)$ is statistically null. Now observe that

$$\theta\left(x - Le\right)_j = \left(x - Le\right)_{\theta^{-1}(j)} = x_{\theta^{-1}(j)} - Le_{\theta^{-1}(j)} = x_{\theta^{-1}(j)} - L \tag{27}$$

as $e_j = e_{\theta^{-1}(j)} = 1$ for all j. As $\theta x - Le$ is statistically null, $st - \lim \theta x = L$.

Finally, the statement of (iii) immediately yields the assertion of (i). ■

Note that Example 1 provides an example of a permutation with the property that if x is statistically convergent then θx is statistically convergent but for which there is a statistically convergent sequence x

such that $\theta^{-1}x$ is not statistically convergent. Observe that if one sets $x = \chi_{\theta B}$ in the last part of Example 3, then $\theta^{-1}x = \chi_B$ is not statistically convergent even though $st-\lim x = 0$. We also note that the preceding work can be used to establish the analogous result that, given a permutation θ, one can show that the statistical convergence of θx implies the statistical convergence of x if and only if $\delta(F) = 0$ implies $\delta(\theta^{-1}F) = 0$ as well as a result similar to Theorem 1.

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