



Some New Results on Six Types Mappings Between L -Convex Spaces

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Abstract. The aim of this paper is devoted to present some new results on six types of special mappings between L -convex spaces. For this purpose, we first make a summary about some novel elementary properties of L -CP mappings and L -CC mappings. Secondly, we propose the definitions of almost L -CC mappings, L -isomorphic mappings and L -embedding mappings and investigate their fundamental characterizations. Finally, we establish the connections between L -quotient mappings and L -quotient spaces. As a summary, we give a diagram to show the relationships among all the above-mentioned mappings.

1. Introduction

Since Zadeh [35] founded fuzzy set theory in 1965, it has been widely used in many mathematical structures in the past fifty-five years. Many fuzzy structures have been studied, such as fuzzy topological structures [22], fuzzy convergence structures [5], fuzzy neighborhood structures [34] and so on. As another important kind of fuzzy structures, the notion of fuzzy convex structures (or, fuzzy convexities) which was introduced by Rosa [15] is a natural generalization of the concept of abstract convex structures [21]. In Rosa's paper, the object of discussion was a classical structure composed of fuzzy convex sets and the research content was based on the real unit interval $[0, 1]$. Because of the limitation of $[0, 1]$ as fuzzy environment in theoretical research, in 2009, Maruyama [10] extended Rosa's notion of fuzzy convex structures to general lattice-value setting and investigated some properties of lattice-valued fuzzy convex sets in Euclidean spaces. Nowadays, both of fuzzy convex spaces in sense of Rosa and Maruyama are collectively called L -convex spaces (where is L a complete lattice). When $L = [0, 1]$, it degenerates to Rosa's fuzzy convex spaces.

Recently, L -convex spaces become a popular research direction. For example, Chen et.al. [1] generalized the concepts of arity, CUP, JHC and weakly JHC in classical convex spaces to the fuzzy case. Jin and Li [3] discussed the connections between classical convex spaces and stratified L -convex spaces from a categorical viewpoint. Li et.al. [6] presented some result on several special mappings in sense of degree based on a frame. Pang and Xiu [11] established the axiom of bases and subbases and gave their applications. Shen and Shi [17] provided some novel characterizations of L -convex structures based on way-below relations.

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Zhou and Shi [37] introduced some low-level separation axioms of L -convex spaces and researched their properties. More extensive studies on L -convex spaces can be found in [16, 23, 32, 33, 36].

From a different perspective, in 2014, Shi and Xiu [19] introduced a novel method to the fuzzification of convex structures and provided the concept of fuzzifying convex structures and fuzzifying convex hull operators, and proved that there is a one-to-one correspondence between them. At present, a nonempty set equipped with a fuzzifying convex structure is called an M -fuzzifying convex space (here M is a completely distributive lattice). Relevant results of M -fuzzifying convex spaces can be seen in [7, 9, 13, 18, 25–27, 30, 31]. Furthermore, a more generalized convex space was introduced in [20], which is called (L, M) -fuzzy convex spaces. In particular, the two kinds of convex spaces mentioned above are all special cases of (L, M) -fuzzy convex spaces under certain conditions. In [4, 8, 12, 14, 20, 24, 28, 29], the authors have given the relevant research contents of (L, M) -fuzzy convex spaces and obtained many good results.

As we all know, CP and CC mappings are two basic and important concepts of classical convexity theory. These two kinds of mappings play a key role in constructing and studying the properties of convex structures. For instance, the concepts of subspaces, product spaces and quotient spaces of convex spaces are defined based on them. In the monograph of convex spaces [21], the author made an in-depth study of CP mappings, CC mappings and other important concepts derived from them. As a generalization of classical convex structures, the study of L -convex structures is a great significance to the enrichment and development of fuzzy mathematics. It is meaningful to consider the properties of CP mappings, CC mappings and some special mappings related to them in the L -fuzzy case. Although the authors [6, 11, 17] have done some research work on special mappings in L -convex spaces, they have not discussed the differences and connections among them in detail. Therefore, it is necessary to discuss these special mappings systematically for the future study in L -convex spaces. Motivated by this, we make an inductive study on six kinds of mappings (L -CP mappings, L -CC mappings, almost L -CC mappings, L -isomorphic mappings, L -embedding mappings, L -quotient mappings) and investigate their relationships.

The structure of the paper is mainly expanded from the following parts. In Section 2, we recall some elementary knowledge that are required in the subsequent sections. In Section 3, we make a summary about some fresh results on L -CP mappings and L -CC mappings. In Section 4, we first introduce the definition of almost L -CC mappings, then study its related properties. In addition, we propose the concepts of L -isomorphic mappings and L -embedding mappings and give some of their characterizations. In Section 5, we study some properties of L -quotient mappings and L -quotient spaces. In the final, we use a diagram to show the relationships among the above-mentioned mappings.

2. Preliminaries

In this section, we recollect some elementary concepts and properties on fuzzy sets, lattices and L -convex structures. For concepts not defined in this paper, the reader can refer to [2, 11, 21, 22].

Assume L is a complete lattice. The two elements \perp and \top are smallest and largest elements of L , respectively. For $K \subseteq L$, we use $\bigvee K$ and $\bigwedge K$ to denote the supremum and infimum of K , respectively. We say that K is directed subset of L if it is nonempty and each finite subset of K has an upper bound in K . In order to facilitate the writing, we usually use $k = \bigvee^\uparrow K$ to express that the supremum of directed set K is k .

Definition 2.1 ([2]). Let L be a poset. For $a, b \in L$, a is way below b in L (in symbols, $a \ll b$) if for each directed subset $K \subseteq L$ such that $\bigvee^\uparrow K$ exists, $b \leq \bigvee^\uparrow K$ always implies the existence of some $k \in K$ with $a \leq k$.

Throughout this paper, L denotes a continuous lattice, unless otherwise stated. For a nonempty set X , 2^X denotes the power set of X and L^X denotes the family of all L -sets on X . Obviously, L^X is also a continuous lattice under the pointwise order. For each $A \in 2^X$, its characteristic function $\chi_A \in L^X$ is defined as follows:

$$\chi_A(x) = \begin{cases} \top, & x \in A; \\ \perp, & x \notin A. \end{cases}$$

The two L -sets χ_\emptyset and χ_X are smallest and largest elements of L^X , respectively. We say that an L -set B is finite if its support set $\text{Supp}B = \{x \in X : B(x) \neq \perp\}$ is finite. We use L_{fin}^X denote the family of all finite L -sets on X .

An element $a \in L$ is called a co-prime element in L provided that $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ for all $b, c \in L$. We denote the family of all non-zero co-prime elements of L (resp., L^X) by $J(L)$ (resp., by $J(L^X)$). Obviously, $x_\lambda \in J(L^X)$ iff $x \in X$ and $\lambda \in J(L)$.

Let $f : X \rightarrow Y$ be a mapping. We define $f^\rightarrow : 2^X \rightarrow 2^Y$, $f^\leftarrow : 2^Y \rightarrow 2^X$, $f_L^\rightarrow : L^X \rightarrow L^Y$ and $f_L^\leftarrow : L^Y \rightarrow L^X$ as follows:

$$\forall U \in 2^X, f^\rightarrow(U) = \{f(x) : x \in U\}.$$

$$\forall V \in 2^Y, f^\leftarrow(V) = \{x \in X : f(x) \in V\}.$$

$$\forall B \in L^X, y \in Y, f_L^\rightarrow(B)(y) = \bigvee \{A(x) : f(x) = y\}.$$

$$\forall C \in L^Y, x \in X, f_L^\leftarrow(C)(x) = C(f(x)).$$

Obviously, we can obtain $B \leq f_L^\leftarrow(f_L^\rightarrow(B))$ and $f_L^\rightarrow(f_L^\leftarrow(C)) \leq C$.

For any $\mathfrak{A} \subseteq L^Y$, we denote $f_L^{-1}(\mathfrak{A}) = \{f_L^\leftarrow(B) : B \in \mathfrak{A}\}$.

In [11, 17], the authors extended the notion of convex structures to the L -fuzzy setting as follows.

Definition 2.2 ([11, 17]). A subset \mathfrak{C} of L^X is called an L -convex structure (or, L -convexity) on X if it fulfills the following conditions:

(LC1) $\chi_\emptyset, \chi_X \in \mathfrak{C}$.

(LC2) $\{B_i\}_{i \in I} \subseteq \mathfrak{C}$ implies $\bigwedge_{i \in I} B_i \in \mathfrak{C}$, where $I \neq \emptyset$.

(LC3) If $\{B_i\}_{i \in I} \subseteq \mathfrak{C}$ is directed, then $\bigvee_{i \in I}^\uparrow B_i \in \mathfrak{C}$.

We call the pair (X, \mathfrak{C}) an L -convex space if \mathfrak{C} is an L -convex structure on X . In this case, the elements in \mathfrak{C} are called L -convex sets on X .

Remark 2.3. For two L -convex structure \mathfrak{C} and \mathfrak{D} on X , we say that \mathfrak{C} is coarser than \mathfrak{D} or \mathfrak{D} is finer than \mathfrak{C} if $\mathfrak{C} \subseteq \mathfrak{D}$.

Proposition 2.4 ([20]). Let (X, \mathfrak{C}) be an L -convex space and $\emptyset \neq Y \in 2^X$. Then $\mathfrak{C}|_Y = \{B|_Y : B \in \mathfrak{C}\}$ is an L -convex structure on Y . In this case, we say that $(Y, \mathfrak{C}|_Y)$ is a subspace of (X, \mathfrak{C}) .

Definition 2.5. The pair $(Y, \mathfrak{C}|_Y)$ is called an convex subspace of the L -convex space (X, \mathfrak{C}) provided that χ_Y is an L -convex set on X .

Theorem 2.6 ([17]). Let (X, \mathfrak{C}) be an L -convex space. Let $co_{\mathfrak{C}} : L^X \rightarrow L^X$ be a mapping defined by

$$co_{\mathfrak{C}}(B) = \bigwedge \{C \in \mathfrak{C} : B \leq C\}$$

for all $B \in L^X$. Then $co_{\mathfrak{C}}$ has the following properties:

(CL1) $co_{\mathfrak{C}}(\chi_\emptyset) = \chi_\emptyset$.

(CL2) $B \leq co_{\mathfrak{C}}(B)$.

(CL3) $co_{\mathfrak{C}}(co_{\mathfrak{C}}(B)) = co_{\mathfrak{C}}(B)$.

(CL4) $co_{\mathfrak{C}}(B) = \bigvee \{co_{\mathfrak{C}}(F) : F \ll B\}$.

Conversely, let $co : L^X \rightarrow L^X$ be a mapping satisfying (CL1)-(CL4). Define a subset of L^X as follows:

$$\mathfrak{C}_{co} = \{B \in L^X : co(B) = B\}.$$

Then (X, \mathfrak{C}_{co}) is an L -convex space and $co_{\mathfrak{C}_{co}} = co$.

Remark 2.7. The operator $co_{\mathfrak{C}}$ defined in Theorem 2.6 is called the L -convex hull operator of the L -convex structure \mathfrak{C} . By (CL4), we know that $co_{\mathfrak{C}}$ is order preserving, that is, for each $B, C \in L^X$, $B \leq C$ implies $co_{\mathfrak{C}}(B) \leq co_{\mathfrak{C}}(C)$.

Definition 2.8 ([11]). Let \mathfrak{C} be an L -convex structure on X and $\mathfrak{B}, \mathfrak{S} \subseteq \mathfrak{C}$. Then

- (1) \mathfrak{B} is called a base of (X, \mathfrak{C}) (or, \mathfrak{C}) provided that for any $C \in \mathfrak{C}$, there is a directed family $\mathfrak{B}_C \subseteq \mathfrak{B}$ such that $C = \bigvee^{\uparrow} \mathfrak{B}_C$.
- (2) \mathfrak{S} is called a subbase of (X, \mathfrak{C}) (or, \mathfrak{C}) provided that $\mathfrak{B}_{\mathfrak{S}}$ is a base of (X, \mathfrak{C}) , where $\mathfrak{B}_{\mathfrak{S}} = \{\bigwedge_{i \in I} B_i : \{B_i\}_{i \in I} \subseteq \mathfrak{S}\}$.

Definition 2.9 ([11, 17, 37]). Let $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ be a mapping between two L -convex spaces. Then

- (1) f is called L -convexity preserving (briefly, L -CP) provided that $D \in \mathfrak{D}$ implies $f_L^{\leftarrow}(D) \in \mathfrak{C}$.
- (2) f is called L -convex-to-convex (briefly, L -CC) provided that $C \in \mathfrak{C}$ implies $f_L^{\rightarrow}(C) \in \mathfrak{D}$.

3. Some new results on L -CP mappings and L -CC mappings

In this section, we first present some novel properties of L -CP mappings and L -CC mappings, then give their characterizations from different aspects.

Proposition 3.1. Let (X, \mathfrak{C}) , (Y, \mathfrak{D}) and (Z, \mathfrak{G}) be three L -convex spaces. Let $\emptyset \neq A \in 2^X$. Then we have the following statements:

- (1) The mapping $i|_A : (A, \mathfrak{C}|_A) \rightarrow (X, \mathfrak{C})$ is L -CP.
- (2) If the mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ and $g : (Y, \mathfrak{D}) \rightarrow (Z, \mathfrak{G})$ are L -CP, then $g \circ f$ is also L -CP.
- (3) If the mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -CP, then $f|_A : (A, \mathfrak{C}|_A) \rightarrow (Y, \mathfrak{D})$ is L -CP.
- (4) The mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -CP iff $f|^{f^{\rightarrow}(X)} : (X, \mathfrak{C}) \rightarrow (f^{\rightarrow}(X), \mathfrak{D}|_{f^{\rightarrow}(X)})$ is L -CP.
- (5) If the mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -CP, then $f|_A : (A, \mathfrak{C}|_A) \rightarrow (f^{\rightarrow}(A), \mathfrak{D}|_{f^{\rightarrow}(A)})$ is L -CP.

Proof. (1) By Proposition 2.4, we know that $(A, \mathfrak{C}|_A)$ is a subspace of (X, \mathfrak{C}) . For any $C \in \mathfrak{C}$, $C|_A \in \mathfrak{C}|_A$ and

$$(i|_A)_L^{\leftarrow}(C)(x) = C(i|_A(x)) = C(x) = (C|_A)(x)$$

for all $x \in A$. It means that $(i|_A)_L^{\leftarrow}(C) = C|_A \in \mathfrak{C}|_A$. This implies that $i|_A : (A, \mathfrak{C}|_A) \rightarrow (X, \mathfrak{C})$ is L -CP.

(2) The verification is straightforward.

(3) Note that $f|_A = f \circ (i|_A)$, so by (1) and (2), we can obtain that $f|_A : (A, \mathfrak{C}|_A) \rightarrow (Y, \mathfrak{D})$ is L -CP.

(4) *Necessity.* For any $B \in \mathfrak{D}|_{f^{\rightarrow}(X)}$, there exists $D \in \mathfrak{D}$ such that $B = D|_{f^{\rightarrow}(X)}$. Thus, we can obtain that

$$\begin{aligned} (f|^{f^{\rightarrow}(X)})_L^{\leftarrow}(B)(x) &= (f|^{f^{\rightarrow}(X)})_L^{\leftarrow}(D|_{f^{\rightarrow}(X)})(x) \\ &= (D|_{f^{\rightarrow}(X)})(f|^{f^{\rightarrow}(X)}(x)) \\ &= (D|_{f^{\rightarrow}(X)})(f(x)) \\ &= D(f(x)) \\ &= f_L^{\leftarrow}(D)(x) \end{aligned}$$

for all $x \in X$. This means $(f|^{f^{\rightarrow}(X)})_L^{\leftarrow}(B) = f_L^{\leftarrow}(D)$. Since $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -CP, we know that

$$(f|^{f^{\rightarrow}(X)})_L^{\leftarrow}(B) = f_L^{\leftarrow}(D) \in \mathfrak{C}.$$

It implies that $f|^{f^{\rightarrow}(X)} : (X, \mathfrak{C}) \rightarrow (f^{\rightarrow}(X), \mathfrak{D}|_{f^{\rightarrow}(X)})$ is L -CP.

Sufficiency. Since $f = i|_{f^{\rightarrow}(X)} \circ f|^{f^{\rightarrow}(X)}$ and $i|_{f^{\rightarrow}(X)} : (f^{\rightarrow}(X), \mathfrak{D}|_{f^{\rightarrow}(X)}) \rightarrow (Y, \mathfrak{D})$ is L -CP, it is easy to obtain the sufficiency by (2).

(5) By (3) and (4), it is obvious. \square

Proposition 3.2. Let (X, \mathfrak{C}) be an L -convex space and $\emptyset \neq A \in 2^X$. Then

- (1) $\mathfrak{C}|_A$ is the coarsest L -convex structure on A such that $i|_A : (A, \mathfrak{C}|_A) \rightarrow (X, \mathfrak{C})$ is L -CP.
- (2) For each L -convex space (Z, \mathfrak{S}) and the mapping $f : (Z, \mathfrak{S}) \rightarrow (A, \mathfrak{C}|_A)$, f is L -CP iff $i|_A \circ f : (Z, \mathfrak{S}) \rightarrow (X, \mathfrak{C})$ is L -CP.

Proof. (1) By Proposition 3.1(1), we know that $i|_A : (A, \mathfrak{C}|_A) \rightarrow (X, \mathfrak{C})$ is L -CP. Suppose \mathfrak{D} is an L -convex structure on A such that $i|_A : (A, \mathfrak{D}) \rightarrow (X, \mathfrak{C})$ is L -CP. For each $C \in \mathfrak{C}|_A$, there exists $B \in \mathfrak{C}$ such that $C = B|_A$. Note that $i|_A : (A, \mathfrak{D}) \rightarrow (X, \mathfrak{C})$ is L -CP and $(i|_A)_L^{\leftarrow}(B) = B|_A$, so we have $C = (i|_A)_L^{\leftarrow}(B) \in \mathfrak{D}$. Hence $\mathfrak{C}|_A \subseteq \mathfrak{D}$.

(2) *Necessity.* It is trivial.

Sufficiency. For each $B \in \mathfrak{C}|_A$, there exists $C \in \mathfrak{C}$ such that $B = C|_A$. Since $i|_A \circ f : (Z, \mathfrak{S}) \rightarrow (X, \mathfrak{C})$ is L -CP, we obtain that

$$(i|_A \circ f)_L^{\leftarrow}(C) = f_L^{\leftarrow}((i|_A)_L^{\leftarrow}(C)) = f_L^{\leftarrow}(B) \in \mathfrak{S}.$$

This means that $f : (Z, \mathfrak{S}) \rightarrow (A, \mathfrak{C}|_A)$ is L -CP. \square

Next, we give a new characterization of L -CP mappings by means of the notion of remotehoods.

Definition 3.3 ([33]). Let (X, \mathfrak{C}) be an L -convex space, $x_\lambda \in J(L^X)$ and $B \in L^X$. The L -set B is called a remotehood of x_λ provided that there exists $C \in \mathfrak{C}$ such that $x_\lambda \not\leq C \geq B$. The family of all remotehoods of x_λ is denoted by \mathcal{R}_{x_λ} .

Definition 3.4. A mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L -convex spaces is called L -CP at x_λ provided that $B \in \mathcal{R}_{f_L^{\leftarrow}(x_\lambda)}$ implies $f_L^{\leftarrow}(B) \in \mathcal{R}_{x_\lambda}$ for all $x_\lambda \in J(L^X)$ and $B \in L^X$.

Proposition 3.5. If $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is a mapping between two L -convex spaces, then f is L -CP iff f is L -CP at x_λ for all $x_\lambda \in J(L^X)$.

Proof. *Necessity.* For any $x_\lambda \in J(L^X)$, if $B \in \mathcal{R}_{f_L^{\leftarrow}(x_\lambda)}$, then there exists $C \in \mathfrak{D}$ such that $f_L^{\leftarrow}(x_\lambda) \not\leq C \geq B$. It implies that $x_\lambda \not\leq f_L^{\leftarrow}(C) \geq f_L^{\leftarrow}(B)$. Since f is L -CP, we have $f_L^{\leftarrow}(C) \in \mathfrak{C}$. Hence $f_L^{\leftarrow}(B) \in \mathcal{R}_{x_\lambda}$. This implies that f is L -CP at x_λ .

Sufficiency. Suppose f is not L -CP. Then there exists $C \in \mathfrak{D}$ and $f_L^{\leftarrow}(C) \notin \mathfrak{C}$. This means $co_{\mathfrak{C}}(f_L^{\leftarrow}(C)) \not\leq f_L^{\leftarrow}(C)$. It implies that there exists $x_\lambda \in J(L^X)$ such that $x_\lambda \leq co_{\mathfrak{C}}(f_L^{\leftarrow}(C))$ and $x_\lambda \not\leq f_L^{\leftarrow}(C)$. Hence $f_L^{\leftarrow}(x_\lambda) \not\leq C \geq C$. By Definitions 3.3 and 3.4, we obtain $C \in \mathcal{R}_{f_L^{\leftarrow}(x_\lambda)}$ and $f_L^{\leftarrow}(C) \in \mathcal{R}_{x_\lambda}$. It implies that there exists $D \in \mathfrak{C}$ such that $x_\lambda \not\leq D \geq f_L^{\leftarrow}(C)$. Hence $D = co_{\mathfrak{C}}(D) \geq co_{\mathfrak{C}}(f_L^{\leftarrow}(C))$ and $x_\lambda \not\leq co_{\mathfrak{C}}(f_L^{\leftarrow}(C))$. This is a contradiction with $x_\lambda \leq co_{\mathfrak{C}}(f_L^{\leftarrow}(C))$ and thus $f_L^{\leftarrow}(C) \notin \mathcal{R}_{x_\lambda}$. This means that f is not L -CP at x_λ . Therefore, f is L -CP. \square

The following theorem present some characterizations of L -CP mappings.

Theorem 3.6. For a mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L -convex spaces, the following statements are equivalent:

- (1) f is L -CP.
- (2) $f_L^{\leftarrow}(B) \in \mathfrak{C}$ for all $B \in \mathfrak{B}_{\mathfrak{D}}$, where $\mathfrak{B}_{\mathfrak{D}}$ is a base of (Y, \mathfrak{D}) .
- (3) $f_L^{\leftarrow}(B) \in \mathfrak{C}$ for all $B \in \mathfrak{S}_{\mathfrak{D}}$, where $B \in \mathfrak{S}_{\mathfrak{D}}$ is a subbase of (Y, \mathfrak{D}) .
- (4) $f_L^{\rightarrow}(co_{\mathfrak{C}}(F)) \leq co_{\mathfrak{D}}(f_L^{\rightarrow}(F))$ for all $F \in L_{fin}^X$.
- (5) $f_L^{\rightarrow}(co_{\mathfrak{C}}(C)) \leq co_{\mathfrak{D}}(f_L^{\rightarrow}(C))$ for all $C \in L^X$.
- (6) $co_{\mathfrak{C}}(f_L^{\leftarrow}(E)) \leq f_L^{\leftarrow}(co_{\mathfrak{D}}(E))$ for all $E \in L_{fin}^Y$.
- (7) $co_{\mathfrak{C}}(f_L^{\leftarrow}(B)) \leq f_L^{\leftarrow}(co_{\mathfrak{D}}(B))$ for all $B \in L^Y$.

(8) f is L -CP at x_λ for all $x_\lambda \in J(L^X)$.

Proof. The proofs of $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ were given in [11] and $(1) \Leftrightarrow (4) \Leftrightarrow (5)$ were presented in [17]. We need to show $(1) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8)$. By Proposition 3.5, we know that $(1) \Leftrightarrow (8)$. Next we prove $(5) \Leftrightarrow (7)$ and $(6) \Leftrightarrow (7)$.

$(5) \Rightarrow (7)$ For any $B \in L^Y, f_L^{\leftarrow}(B) \in L^X$. By (5), we obtain

$$f_L^{\rightarrow}(co_{\mathfrak{C}}(f_L^{\leftarrow}(B))) \leq co_{\mathfrak{D}}(f_L^{\rightarrow}(f_L^{\leftarrow}(B))) \leq co_{\mathfrak{D}}(B).$$

It implies that $f_L^{\leftarrow}(f_L^{\rightarrow}(co_{\mathfrak{C}}(f_L^{\leftarrow}(B)))) \leq f_L^{\leftarrow}(co_{\mathfrak{D}}(B))$.

Therefore, $co_{\mathfrak{C}}(f_L^{\leftarrow}(B)) \leq f_L^{\leftarrow}(co_{\mathfrak{D}}(B))$.

$(7) \Rightarrow (5)$ For any $C \in L^X, f_L^{\rightarrow}(C) \in L^Y$. By (7), we obtain

$$co_{\mathfrak{C}}(C) \leq co_{\mathfrak{C}}(f_L^{\leftarrow}(f_L^{\rightarrow}(C))) \leq f_L^{\leftarrow}(co_{\mathfrak{D}}(f_L^{\rightarrow}(C))).$$

It implies that $f_L^{\rightarrow}(co_{\mathfrak{C}}(C)) \leq co_{\mathfrak{D}}(f_L^{\rightarrow}(C))$.

$(6) \Rightarrow (7)$ For any $B \in L^Y$, by (CL4), we have

$$\begin{aligned} f_L^{\leftarrow}(co_{\mathfrak{D}}(B)) &= f_L^{\leftarrow}(\bigvee^{\uparrow}\{co_{\mathfrak{C}}(E) : E \ll B\}) \\ &= \bigvee^{\uparrow}\{f_L^{\leftarrow}(co_{\mathfrak{C}}(E)) : E \ll B\} \\ &\geq \bigvee^{\uparrow}\{co_{\mathfrak{C}}(f_L^{\leftarrow}(E)) : E \ll B\} \\ &= co_{\mathfrak{C}}(\bigvee^{\uparrow}\{f_L^{\leftarrow}(E) : E \ll B\}) \\ &= co_{\mathfrak{C}}(f_L^{\leftarrow}(\bigvee^{\uparrow}\{E : E \ll B\})) \\ &= co_{\mathfrak{C}}(f_L^{\leftarrow}(B)). \end{aligned}$$

$(7) \Rightarrow (6)$ It is straightforward. \square

We study some properties of L -CC mappings in following propositions.

Proposition 3.7. Let $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ be an L -CC mapping between two L -convex spaces. If $\emptyset \neq Z \in 2^Y$, then the mapping $f|_Z : (X, \mathfrak{C}) \rightarrow (Z, \mathfrak{D}|_Z)$ is L -CC.

Proof. Let $C \in \mathfrak{C}$. Since

$$\begin{aligned} (f|_Z)_L^{\rightarrow}(C)(y) &= \bigvee\{C(x) : x \in X, (f|_Z)(x) = y\} \\ &= \bigvee\{C(x) : x \in X, f(x) = y\} \\ &= f_L^{\rightarrow}(C)(y) \\ &= (f_L^{\rightarrow}(C)|_Z)(y) \end{aligned}$$

for all $y \in Z$, we obtain that $(f|_Z)_L^{\rightarrow}(C) = f_L^{\rightarrow}(C)|_Z$. Note that f is L -CC, so we have $f_L^{\rightarrow}(C) \in \mathfrak{D}$. Hence $(f|_Z)_L^{\rightarrow}(C) = f_L^{\rightarrow}(C)|_Z \in \mathfrak{D}|_Z$. This means that $f|_Z : (X, \mathfrak{C}) \rightarrow (Z, \mathfrak{D}|_Z)$ is L -CC. \square

Proposition 3.8. Let $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ be an L -CC mapping between two L -convex spaces. If $(A, \mathfrak{C}|_A)$ is a convex subspace of (X, \mathfrak{C}) , then the mapping $f|_A : (A, \mathfrak{C}|_A) \rightarrow (f^{\rightarrow}(A), \mathfrak{D}|_{f^{\rightarrow}(A)})$ is L -CC.

Proof. For any $B \in \mathfrak{C}|_A$, there exists $C \in \mathfrak{C}$ such that $B = C|_A$. For $x \in X$, we define an L -set C^* on X as follows:

$$C^*(x) = \begin{cases} C(x), & x \in A; \\ \perp, & x \notin A. \end{cases}$$

Then $C^* = C \wedge \chi_A$. Note that $(A, \mathfrak{C}|_A)$ is a convex subspace of (X, \mathfrak{C}) , so we obtain $\chi_A \in \mathfrak{C}$. Thus, $C^* \in \mathfrak{C}$. Since $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -CC, we know that $f_L^{\rightarrow}(C^*) \in \mathfrak{D}$ and $f_L^{\rightarrow}(C^*)|_{f^{\rightarrow}(A)} \in \mathfrak{D}|_{f^{\rightarrow}(A)}$. By

$$\begin{aligned} (f|_A)_L^{\rightarrow}(B)(y) &= \bigvee\{B(x) : x \in A, (f|_A)(x) = y\} \\ &= \bigvee\{(C|_A)(x) : x \in A, f(x) = y\} \\ &= \bigvee\{C(x) : x \in A, f(x) = y\} \end{aligned}$$

and

$$\begin{aligned} f_L^{-\rightarrow}(C^*)|_{f^{-\rightarrow}(A)}(y) &= f_L^{-\rightarrow}(C^*)(y) \\ &= \bigvee \{C^*(x) : x \in X, f(x) = y\} \\ &= \bigvee \{C^*(x) : x \in A, f(x) = y\} \\ &= \bigvee \{C(x) : x \in A, f(x) = y\} \end{aligned}$$

for all $y \in f^{-\rightarrow}(A)$, it follows that $(f|A)_L^{-\rightarrow}(B) = f_L^{-\rightarrow}(C^*)|_{f^{-\rightarrow}(A)} \in \mathfrak{D}|_{f^{-\rightarrow}(A)}$. Therefore, $f|A : (A, \mathfrak{C}|_A) \rightarrow (f^{-\rightarrow}(A), \mathfrak{D}|_{f^{-\rightarrow}(A)})$ is L-CC. \square

Remark 3.9. Clearly, if $L = \{\perp, \top\}$, then the above proposition can be regarded as a generalization of the classical result as follows:

Let $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ be a CC mapping between two convex spaces. If $A \in \mathfrak{C}$, then the mapping $f|A : (A, \mathfrak{C}|_A) \rightarrow (f^{-\rightarrow}(A), \mathfrak{D}|_{f^{-\rightarrow}(A)})$ is CC.

The following proposition is trivial and the proof of it omitted here.

Proposition 3.10. Let $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ and $g : (Y, \mathfrak{D}) \rightarrow (Z, \mathfrak{E})$ be two mappings between L-convex spaces.

- (1) If f and g are L-CC, then $g \circ f$ is L-CC.
- (2) If $g \circ f$ is L-CC, f is surjective and L-CP, then g is L-CC.
- (3) If $g \circ f$ is L-CC, g is injective and L-CP, then f is L-CC.

Theorem 3.11. For a mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L-convex spaces, the following statements are equivalent:

- (1) f is L-CC;
- (2) $f_L^{-\rightarrow}(B) \in \mathfrak{D}$ for all $B \in \mathfrak{B}_{\mathfrak{C}}$, where $\mathfrak{B}_{\mathfrak{C}}$ is a base of (X, \mathfrak{C}) ;
- (3) for any $F \in L_{fin}^X$, $f_L^{-\rightarrow}(co_{\mathfrak{C}}(F)) \geq co_{\mathfrak{D}}(f_L^{-\rightarrow}(F))$;
- (4) for any $C \in L^X$, $f_L^{-\rightarrow}(co_{\mathfrak{C}}(C)) \geq co_{\mathfrak{D}}(f_L^{-\rightarrow}(C))$.

Proof. The proof of (1) \Leftrightarrow (2) was given in [11]. We need to proof (1) \Leftrightarrow (3) \Leftrightarrow (4).

(1) \Rightarrow (3) For each $F \in L_{fin}^X$, we obtain $F \leq co_{\mathfrak{C}}(F) \in \mathfrak{C}$. Since f is L-CC and $f_L^{-\rightarrow}(F) \leq f_L^{-\rightarrow}(co_{\mathfrak{C}}(F))$, it follows that $f_L^{-\rightarrow}(co_{\mathfrak{C}}(F)) \in \mathfrak{D}$. Hence $f_L^{-\rightarrow}(co_{\mathfrak{C}}(F)) \geq co_{\mathfrak{D}}(f_L^{-\rightarrow}(F))$.

(3) \Rightarrow (4) For any $C \in L^X$, by (CL4), we obtain

$$\begin{aligned} f_L^{-\rightarrow}(co_{\mathfrak{C}}(C)) &= f_L^{-\rightarrow}(\bigvee^{\uparrow} \{co_{\mathfrak{C}}(F) : F \ll C\}) \\ &= \bigvee^{\uparrow} \{f_L^{-\rightarrow}(co_{\mathfrak{C}}(F)) : F \ll C\} \\ &\geq \bigvee^{\uparrow} \{co_{\mathfrak{D}}(f_L^{-\rightarrow}(F)) : F \ll C\} \\ &= co_{\mathfrak{D}}(\bigvee^{\uparrow} \{f_L^{-\rightarrow}(F) : F \ll C\}) \\ &= co_{\mathfrak{D}}(f_L^{-\rightarrow}(\bigvee^{\uparrow} \{F : F \ll C\})) \\ &= co_{\mathfrak{D}}(f_L^{-\rightarrow}(C)). \end{aligned}$$

(4) \Rightarrow (1) For any $C \in \mathfrak{C}$, we have $co_{\mathfrak{D}}(f_L^{-\rightarrow}(C)) \leq f_L^{-\rightarrow}(co_{\mathfrak{C}}(C)) = f_L^{-\rightarrow}(C)$. It implies that $f_L^{-\rightarrow}(C) \in \mathfrak{D}$. Hence f is L-CC. \square

4. L-isomorphic mappings and L-embedding mappings

In this section, we first propose the concept of almost convex-to-convex mappings and then extend it to the L-fuzzy case. Moreover, we study related properties of almost L-CC mappings.

Definition 4.1. A mapping $f : (X, \mathcal{C}) \rightarrow (Y, \mathcal{D})$ between two convex spaces is called almost convex-to-convex (briefly, almost CC) provided that for each $U \in \mathcal{C}$, there exists $V \in \mathcal{D}$ such that $U = f^{\leftarrow}(V)$.

Remark 4.2. Comparing the definitions of CC mappings and almost CC mappings in crisp case, we can see that an injective CC mapping is an almost CC mapping and a surjective almost CC mapping is a CC mapping. Obviously, a bijective mapping f is almost CC iff f is CC.

We generalize the definition of almost convex-to-convex mappings to L-fuzzy case.

Definition 4.3. A mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L-convex spaces is called almost L-convex-to-convex (briefly, almost L-CC) provided that for each $C \in \mathfrak{C}$, there exists $D \in \mathfrak{D}$ such that $C = f_L^{\leftarrow}(D)$.

The following proposition is trivial and the proof of it omitted here.

Proposition 4.4. Let $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ and $g : (Y, \mathfrak{D}) \rightarrow (Z, \mathfrak{S})$ be two mappings between L-convex spaces.

- (1) If f is surjective and almost L-CC, then f is L-CC.
- (2) If f is injective and L-CC, then f is almost L-CC.
- (3) If f and g are almost L-CC, then $g \circ f$ is almost L-CC.
- (4) If $g \circ f$ is an almost L-CC, f is surjective and L-CP, then g is almost L-CC.
- (5) If $g \circ f$ is an almost L-CC and g is L-CP, then f is almost L-CC.

According to Proposition 4.4, we can easily obtain the following conclusion.

Proposition 4.5. A bijective mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L-convex spaces is almost L-CC iff f is L-CC.

In the following, we put forward the notion of L-isomorphic mappings and present several characterizations of it. Furthermore, we introduce the definition of L-embedding mappings by means of L-isomorphic mappings and discuss some properties of it.

Definition 4.6. An L-CP mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L-convex spaces is called L-isomorphic (briefly, L-isomorphism) provided that there is an L-CP mapping $g : (Y, \mathfrak{D}) \rightarrow (X, \mathfrak{C})$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.

If there is an L-isomorphism between (X, \mathfrak{C}) and (Y, \mathfrak{D}) , then we say that (X, \mathfrak{C}) and (Y, \mathfrak{D}) are L-isomorphic.

Remark 4.7. For each L-convex space (X, \mathfrak{C}) , the identity mapping $id_X : (X, \mathfrak{C}) \rightarrow (X, \mathfrak{C})$ is an L-isomorphism. One can readily verify that if f is an L-isomorphism, then the inverse mapping f^{-1} is an L-isomorphism as well and that the composition $g \circ f$ of two L-isomorphisms f and g is again an L-isomorphism. Thus, the relation “ (X, \mathfrak{C}) and (Y, \mathfrak{D}) are L-isomorphic” is an equivalence relation.

Theorem 4.8. For a mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L-convex spaces. Let $co_{\mathfrak{C}}$ and $co_{\mathfrak{D}}$ be L-convex hull operators of (X, \mathfrak{C}) and (Y, \mathfrak{D}) , respectively. The following statements are equivalent:

- (1) f is an L-isomorphism.
- (2) f is bijective and $f^{-1} : (Y, \mathfrak{D}) \rightarrow (X, \mathfrak{C})$ is L-CP.
- (3) f is bijective and f is L-CC.

- (4) f is bijective and f is almost L -CC.
- (5) f is bijective and $co_{\mathfrak{D}}(f_L^{\rightarrow}(F)) \leq f_L^{\rightarrow}(co_{\mathfrak{C}}(F))$ for all $F \in L_{fin}^X$.
- (6) f is bijective and $co_{\mathfrak{D}}(f_L^{\rightarrow}(C)) \leq f_L^{\rightarrow}(co_{\mathfrak{C}}(C))$ for all $C \in L^X$.
- (7) f is bijective and $f_L^{\rightarrow}(B) \in \mathfrak{D}$ for all $B \in \mathfrak{B}_{\mathfrak{C}}$, where $\mathfrak{B}_{\mathfrak{C}}$ is a base of (X, \mathfrak{C}) .

Definition 4.9. A mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L -convex spaces is called L -embedding if $f|^{f^{-1}(X)} : (X, \mathfrak{C}) \rightarrow (f^{-1}(X), \mathfrak{D}|_{f^{-1}(X)})$ is an L -isomorphism.

If there is an L -embedding mapping between (X, \mathfrak{C}) and (Y, \mathfrak{D}) , then we say that (X, \mathfrak{C}) can be embedded in (Y, \mathfrak{D}) .

Theorem 4.10. A mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L -convex spaces is L -embedding iff f is injective, L -CP and almost L -CC.

Proof. Necessity. Since $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -embedding, we can know that $f|^{f^{-1}(X)} : (X, \mathfrak{C}) \rightarrow (f^{-1}(X), \mathfrak{D}|_{f^{-1}(X)})$ is an L -isomorphism. This shows that f is injective and $f|^{f^{-1}(X)}$ is L -CP, L -CC. By Proposition 3.1(4), we can get that f is L -CP. Note that $f|^{f^{-1}(X)}$ is L -CC, so we have $f_L^{\rightarrow}(C) \in \mathfrak{D}|_{f^{-1}(X)}$ for all $C \in \mathfrak{C}$. It implies that there exists $D \in \mathfrak{D}$ such that $f_L^{\rightarrow}(C) = D|_{f^{-1}(X)}$. Thus, $C = f_L^{\leftarrow}(f_L^{\rightarrow}(C)) = f_L^{\leftarrow}(D|_{f^{-1}(X)}) = f_L^{\leftarrow}(D)$. Hence f is almost L -CC.

Sufficiency. Since $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is injective, L -CP and almost L -CC, we can obtain that $f|^{f^{-1}(X)} : (X, \mathfrak{C}) \rightarrow (f^{-1}(X), \mathfrak{D}|_{f^{-1}(X)})$ is bijective and L -CP. We need to show that $f|^{f^{-1}(X)}$ is L -CC. Since f is almost L -CC, it follows that there exists $D \in \mathfrak{D}$ such that $C = f_L^{\leftarrow}(D)$ for all $C \in \mathfrak{C}$. For any $y \in f^{-1}(X)$,

$$\begin{aligned} (f|^{f^{-1}(X)})_L^{\rightarrow}(f_L^{\leftarrow}(D))(y) &= \bigvee \{f_L^{\leftarrow}(D)(x) : x \in X, f|^{f^{-1}(X)}(x) = y\} \\ &= \bigvee \{D(f(x)) : x \in X, f(x) = y\} \\ &= D(y) \\ &= D|_{f^{-1}(X)}(y). \end{aligned}$$

It implies that $(f|^{f^{-1}(X)})_L^{\rightarrow}(f_L^{\leftarrow}(D)) = D|_{f^{-1}(X)}$. Note that $C = f_L^{\leftarrow}(D)$, so we obtain

$$(f|^{f^{-1}(X)})_L^{\rightarrow}(C) = D|_{f^{-1}(X)} \in \mathfrak{D}|_{f^{-1}(X)}.$$

This shows that $f|^{f^{-1}(X)}$ is L -CC. Hence f is L -embedding. \square

Lemma 4.11. Let \mathfrak{C} be an L -convex structure on X and $\emptyset \neq Y \in 2^X$.

- (1) If $\mathfrak{B}_{\mathfrak{C}}$ is a base of (X, \mathfrak{C}) , then $\mathfrak{B}_{\mathfrak{C}|_Y}$ is base of $(Y, \mathfrak{C}|_Y)$.
- (2) If $\mathfrak{S}_{\mathfrak{C}}$ is a subbase of (X, \mathfrak{C}) , then $\mathfrak{S}_{\mathfrak{C}|_Y}$ is subbase of $(Y, \mathfrak{C}|_Y)$.

Proof. (1) For any $D \in \mathfrak{C}|_Y$, there exists $C \in \mathfrak{C}$ such that $D = C|_Y$. Since $\mathfrak{B}_{\mathfrak{C}}$ is a base of (X, \mathfrak{C}) , there is a directed family $\{B_i\}_{i \in I} \subseteq \mathfrak{B}_{\mathfrak{C}}$ such that $C = \bigvee_{i \in I} B_i$. Thus, we have

$$D = (\bigvee_{i \in I} B_i)|_Y = \bigvee_{i \in I} (B_i|_Y).$$

One can readily verify that $\{B_i|_Y\}_{i \in I} \subseteq \mathfrak{B}_{\mathfrak{C}|_Y}$ is also directed. This means that $\mathfrak{B}_{\mathfrak{C}|_Y}$ is base of $(Y, \mathfrak{C}|_Y)$.

- (2) Since $\mathfrak{S}_{\mathfrak{C}}$ is a subbase of (X, \mathfrak{C}) , $\mathfrak{B}_{\mathfrak{S}_{\mathfrak{C}}}$ defined by

$$\mathfrak{B}_{\mathfrak{S}_{\mathfrak{C}}} = \{\bigwedge_{i \in I} B_i : \{B_i\}_{i \in I} \subseteq \mathfrak{S}_{\mathfrak{C}}\}$$

is a base of (X, \mathfrak{C}) . By (1), we know that

$$\begin{aligned} \mathfrak{B}_{\mathfrak{C}|Y} &= \{(\bigwedge_{i \in I} B_i)|_Y : \{B_i\}_{i \in I} \subseteq \mathfrak{C}\} \\ &= \{\bigwedge_{i \in I} (B_i|_Y) : \{B_i\}_{i \in I} \subseteq \mathfrak{C}\} \\ &= \{\bigwedge_{i \in I} C_i : \{C_i\}_{i \in I} \subseteq \mathfrak{C}|_Y\} \end{aligned}$$

is a base of $(Y, \mathfrak{C}|_Y)$. This implies that $\mathfrak{C}|_Y$ is subbase of $(Y, \mathfrak{C}|_Y)$. \square

Theorem 4.12. Let (X, \mathfrak{C}) and (Y, \mathfrak{D}) be two L -convex spaces, $\mathfrak{B}_{\mathfrak{D}}$ a base of (Y, \mathfrak{D}) . Then the mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -embedding iff it is injective and $f_L^{-1}(\mathfrak{B}_{\mathfrak{D}})$ is a base of (X, \mathfrak{C}) .

Proof. Necessity. Since $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -embedding, we can obtain that f is injective, L -CP and L -almost CC. For each $C \in \mathfrak{C}$, then there exists $D \in \mathfrak{D}$ such that $C = f_L^{-1}(D)$. Note that $\mathfrak{B}_{\mathfrak{D}}$ is a base of (Y, \mathfrak{D}) , so we know that there is a directed family $\{B_i\}_{i \in I} \subseteq \mathfrak{B}_{\mathfrak{D}}$ such that $D = \bigvee_{i \in I} B_i$. It implies that

$$C = f_L^{-1}(\bigvee_{i \in I} B_i) = \bigvee_{i \in I} f_L^{-1}(B_i).$$

One can readily verify that $\{f_L^{-1}(B_i)\}_{i \in I}$ is a directed subset of $f_L^{-1}(\mathfrak{B}_{\mathfrak{D}})$. Hence $f_L^{-1}(\mathfrak{B}_{\mathfrak{D}})$ is a base of (X, \mathfrak{C}) .

Sufficiency. We need to show that $f|^{f^{-1}(X)} : (X, \mathfrak{C}) \rightarrow (f^{-1}(X), \mathfrak{D}|_{f^{-1}(X)})$ is an L -isomorphism. That $f|^{f^{-1}(X)}$ is bijective is obvious. By Lemma 4.11, we can get that $\mathfrak{B}_{\mathfrak{D}}|_{f^{-1}(X)}$ is a base of $(f^{-1}(X), \mathfrak{D}|_{f^{-1}(X)})$. For each $B \in \mathfrak{B}_{\mathfrak{D}}|_{f^{-1}(X)}$, there exists $C \in \mathfrak{B}_{\mathfrak{D}}$ such that $B = C|_{f^{-1}(X)}$. Since

$$\begin{aligned} (f|^{f^{-1}(X)})_L^{-1}(C|_{f^{-1}(X)})(x) &= (C|_{f^{-1}(X)})(f|^{f^{-1}(X)}(x)) \\ &= (C|_{f^{-1}(X)})(f(x)) \\ &= C(f(x)) \\ &= f_L^{-1}(C)(x) \end{aligned}$$

for all $x \in X$, we obtain $(f|^{f^{-1}(X)})_L^{-1}(B) = f_L^{-1}(C)$. Note that $f_L^{-1}(\mathfrak{B}_{\mathfrak{D}})$ is a base of (X, \mathfrak{C}) , so we have

$$(f|^{f^{-1}(X)})_L^{-1}(B) = f_L^{-1}(C) \in f_L^{-1}(\mathfrak{B}_{\mathfrak{D}}) \subseteq \mathfrak{C}.$$

Hence $(f|^{f^{-1}(X)})_L^{-1}(B) \in \mathfrak{C}$. Therefore, $f|^{f^{-1}(X)}$ is L -CP.

Now we show that $f|^{f^{-1}(X)}$ is L -CC. For each $C \in f_L^{-1}(\mathfrak{B}_{\mathfrak{D}})$, there exists $D \in \mathfrak{B}_{\mathfrak{D}} \subseteq \mathfrak{D}$ such that $C = f_L^{-1}(D)$. Similar to Theorem 4.10' proof, we get that $(f|^{f^{-1}(X)})_L^{-1}(f_L^{-1}(D)) = D|_{f^{-1}(X)}$. Hence $(f|^{f^{-1}(X)})_L^{-1}(C) \in \mathfrak{D}|_{f^{-1}(X)}$. This means that $f|^{f^{-1}(X)}$ is L -CC. Therefore, f is L -embedding. \square

Theorem 4.13. Let (X, \mathfrak{C}) and (Y, \mathfrak{D}) be two L -convex spaces, $\mathfrak{C}_{\mathfrak{D}}$ a subbase of (Y, \mathfrak{D}) . Then the mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is L -embedding iff it is injective and $f_L^{-1}(\mathfrak{C}_{\mathfrak{D}})$ is a subbase of (X, \mathfrak{C}) .

Proof. Necessity. It is obvious that the mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is injective. Since $\mathfrak{C}_{\mathfrak{D}}$ is a subbase of (Y, \mathfrak{D}) , $\mathfrak{B}_{\mathfrak{C}_{\mathfrak{D}}}$ defined by $\mathfrak{B}_{\mathfrak{C}_{\mathfrak{D}}} = \{\bigwedge_{i \in I} B_i : \{B_i\}_{i \in I} \subseteq \mathfrak{C}_{\mathfrak{D}}\}$ is a base of (Y, \mathfrak{D}) . It follows that

$$\begin{aligned} f_L^{-1}(\mathfrak{B}_{\mathfrak{C}_{\mathfrak{D}}}) &= \{f_L^{-1}(\bigwedge_{i \in I} B_i) : \{B_i\}_{i \in I} \subseteq \mathfrak{C}_{\mathfrak{D}}\} \\ &= \{\bigwedge_{i \in I} f_L^{-1}(B_i) : \{B_i\}_{i \in I} \subseteq \mathfrak{C}_{\mathfrak{D}}\} \\ &= \{\bigwedge_{i \in I} C_i : \{C_i\}_{i \in I} \subseteq f_L^{-1}(\mathfrak{C}_{\mathfrak{D}})\}. \end{aligned}$$

Note that $\mathfrak{B}_{\mathfrak{C}_{\mathfrak{D}}}$ is a base of (Y, \mathfrak{D}) and f is L -embedding, by Theorem 4.12, we know that $f_L^{-1}(\mathfrak{B}_{\mathfrak{C}_{\mathfrak{D}}})$ is a base of (X, \mathfrak{C}) . It implies that $f_L^{-1}(\mathfrak{C}_{\mathfrak{D}})$ is a subbase of (X, \mathfrak{C}) .

Sufficiency. Since $f_L^{-1}(\mathfrak{C}_{\mathfrak{D}})$ is a subbase of (X, \mathfrak{C}) , $\mathfrak{B}_{f_L^{-1}(\mathfrak{C}_{\mathfrak{D}})}$ defined by $\mathfrak{B}_{f_L^{-1}(\mathfrak{C}_{\mathfrak{D}})} = \{\bigwedge_{j \in J} D_j : \{D_j\}_{j \in J} \subseteq f_L^{-1}(\mathfrak{C}_{\mathfrak{D}})\}$ is a base of (X, \mathfrak{C}) . Note that $\mathfrak{B}_{\mathfrak{C}_{\mathfrak{D}}}$ is a base of (Y, \mathfrak{D}) and $\mathfrak{B}_{f_L^{-1}(\mathfrak{C}_{\mathfrak{D}})} = f_L^{-1}(\mathfrak{B}_{\mathfrak{C}_{\mathfrak{D}}})$, by Theorem 4.12, we know that f is L -embedding. \square

As a straightforward result of Theorems 4.10, 4.12 and 4.13, we can easily get the following proposition.

Proposition 4.14. For a mapping $f : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ between two L -convex spaces, the following statements are equivalent:

- (1) f is L -embedding.
- (2) f is injective, L -CP and L -almost CC.
- (3) f is injective, $\mathfrak{B}_{\mathfrak{D}}$ is a base of (Y, \mathfrak{D}) and $f_L^{-1}(\mathfrak{B}_{\mathfrak{D}})$ is a base of (X, \mathfrak{C}) .
- (4) f is injective, $\mathfrak{S}_{\mathfrak{D}}$ is a subbase of (Y, \mathfrak{D}) and $f_L^{-1}(\mathfrak{S}_{\mathfrak{D}})$ is a subbase of (X, \mathfrak{C}) .

Proposition 4.15. Let (X, \mathfrak{C}) and (Y, \mathfrak{D}) be two L -convex spaces. We have the following statements:

- (1) If $f : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ is an L -isomorphism and $A \in 2^X$, then $f|A : (A, \mathfrak{C}|_A) \longrightarrow (f^{-1}(A), \mathfrak{D}|_{f^{-1}(A)})$ is an L -isomorphism.
- (2) If (X, \mathfrak{C}) can be embedded in (Y, \mathfrak{D}) , then any subspace of (X, \mathfrak{C}) can also be embedded in (Y, \mathfrak{D}) .

Proof. (1) Since $f : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ is an L -isomorphism, we obtain that f is bijective, L -CP and L -CC.

Obviously, the mapping $f|A : (A, \mathfrak{C}|_A) \longrightarrow (f^{-1}(A), \mathfrak{D}|_{f^{-1}(A)})$ is bijective for all $A \in 2^X$. By Proposition 3.1(5), we get that $f|A$ is L -CP. We need to show that $f|A$ is L -CC. For each $B \in \mathfrak{C}|_A$, there exists $C \in \mathfrak{C}$ such that $B = C|_A$. Note that f is L -CC, so we have $f_L^{-1}(C) \in \mathfrak{D}$. Since

$$\begin{aligned} (f|A)_L^{-1}(C|_A)(y) &= \bigvee \{(C|_A)(x) : x \in A, (f|A)(x) = y\} \\ &= \bigvee \{C(x) : x \in A, f(x) = y\} \\ &= \bigvee \{C(x) : x \in X, f(x) = y\} \quad (f \text{ is bijective}) \\ &= f_L^{-1}(C)(y) \\ &= (f_L^{-1}(C)|_{f^{-1}(A)})(y) \end{aligned}$$

for all $y \in f^{-1}(A)$, we have $(f|A)_L^{-1}(C|_A) = f_L^{-1}(C)|_{f^{-1}(A)}$. It means $(f|A)_L^{-1}(B) \in \mathfrak{D}|_{f^{-1}(A)}$. This implies that $f|A$ is L -CC. Hence $f|A$ is an L -isomorphism.

(2) Let $(A, \mathfrak{C}|_A)$ be a subspace of (X, \mathfrak{C}) . Since (X, \mathfrak{C}) can be embedded in (Y, \mathfrak{D}) , we know that $f|f^{-1}(X) : (X, \mathfrak{C}) \longrightarrow (f^{-1}(X), \mathfrak{D}|_{f^{-1}(X)})$ is an L -isomorphism. By (1), we obtain that

$$f|A : (A, \mathfrak{C}|_A) \longrightarrow (f^{-1}(A), (\mathfrak{D}|_{f^{-1}(X)})|_{f^{-1}(A)})$$

is an L -isomorphism. Note that $f^{-1}(A) \subseteq f^{-1}(X)$, so we have $(\mathfrak{D}|_{f^{-1}(X)})|_{f^{-1}(A)} = \mathfrak{D}|_{f^{-1}(A)}$. Therefore, $(A, \mathfrak{C}|_A)$ can be embedded in (Y, \mathfrak{D}) . \square

5. L -quotient mappings

In this section, we present the definition of L -quotient mappings from the point of view of L -convex sets, which is a special case of Definition 4.3 in [20]. Further, we discuss some properties of L -quotient mappings and L -quotient spaces.

Definition 5.1. A mapping $f : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ between two L -convex spaces is called an L -quotient mapping provided that f is surjective and for any $B \in L^Y$, B is an L -convex set in (Y, \mathfrak{D}) iff $f_L^{-1}(B)$ is an L -convex set in (X, \mathfrak{C}) .

Remark 5.2. From Definition 5.1, we can easily obtain the following statements:

- (1) The composition of two L -quotient mappings is also an L -quotient mapping.
- (2) According to Theorem 4.8, we know that an L -isomorphism is an L -quotient mapping and an injective L -quotient mapping is an L -isomorphism.

The following proposition show the elementary properties of L -quotient mappings.

Proposition 5.3. *Let $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ be an L-CP mapping between two L-convex space. We have the following statements:*

- (1) *If there is an L-CP mapping $h : (Y, \mathfrak{D}) \rightarrow (X, \mathfrak{C})$ such that $f \circ h = id_Y$, then f is an L-quotient mapping.*
- (2) *If f is surjective and L-CC, then f is an L-quotient mapping.*
- (3) *If f is surjective and almost L-CC, then f is an L-quotient mapping.*
- (4) *If $g : (Y, \mathfrak{D}) \rightarrow (Z, \mathfrak{G})$ is an L-CP mapping and $g \circ f : (X, \mathfrak{C}) \rightarrow (Z, \mathfrak{G})$ is an L-quotient mapping, then g is an L-quotient mapping.*
- (5) *If there exists $\emptyset \neq A \in 2^X$ such that $f|_A : (A, \mathfrak{C}|_A) \rightarrow (Y, \mathfrak{D})$ is an L-quotient mapping, then f is an L-quotient mapping.*

Proof. (1) Suppose $B \in L^Y$. Let $B \in \mathfrak{D}$, obviously, $f_L^{\leftarrow}(B) \in \mathfrak{C}$. Conversely, let $f_L^{\leftarrow}(B) \in \mathfrak{C}$. Note that $f \circ h = id_Y$, so we can obtain that f is surjective. By $h_L^{\leftarrow}(f_L^{\leftarrow}(B)) = (f \circ h)_L^{\leftarrow}(B) = (id_Y)_L^{\leftarrow}(B) = B$ and $h : (Y, \mathfrak{D}) \rightarrow (X, \mathfrak{C})$ is L-CP, we have $B \in \mathfrak{D}$. Hence f is an L-quotient mapping.

(2) It is straightforward.

(3) For any $B \in L^Y$, we only to show that $f_L^{\leftarrow}(B) \in \mathfrak{C}$ implies $B \in \mathfrak{D}$. Since f is almost L-CC, we can obtain that there exists $D \in \mathfrak{D}$ such that $f_L^{\leftarrow}(B) = f_L^{\leftarrow}(D)$. Note that f is surjective, so we have

$$B = f_L^{\rightarrow}(f_L^{\leftarrow}(B)) = f_L^{\rightarrow}(f_L^{\leftarrow}(D)) = D.$$

It implies that $B \in \mathfrak{D}$. Hence f is an L-quotient mapping.

(4) Clearly, g is surjective. For any $C \in L^Z$, if the inverse image $g_L^{\leftarrow}(C) \in \mathfrak{D}$, then we obtain

$$f_L^{\leftarrow}(g_L^{\leftarrow}(C)) = (g \circ f)_L^{\leftarrow}(C) \in \mathfrak{C}.$$

Note that $g \circ f$ is an L-quotient mapping, so we have $C \in \mathfrak{G}$. Hence g is an L-quotient mapping.

(5) Note that $i|_A : (A, \mathfrak{C}|_A) \rightarrow (X, \mathfrak{C})$ is an L-CP mapping and $f|_A = f \circ (i|_A)$ is L-quotient mapping, by (4) we know that f is an L-quotient mapping. \square

In [20], the authors provided a method to generate a new convex space based on a convex space and a surjection in (L, M) -fuzzy setting. As its special case, when $M = \{\perp, \top\}$, we have the following proposition.

Proposition 5.4. *Let $f : X \rightarrow Y$ be a surjective mapping and \mathfrak{C} be an L-convex structure on X . We define $\mathfrak{D} = \{B \in L^Y : f_L^{\leftarrow}(B) \in \mathfrak{C}\}$. Then (Y, \mathfrak{D}) is an L-convex space.*

As we all know, if R is an equivalence relation over X , then the mapping $q : X \rightarrow X/R$ is surjective. For an L-convex structure \mathfrak{C} on X , we obtain that $\mathfrak{C}_R = \{B \in L^{X/R} : q_L^{\leftarrow}(B) \in \mathfrak{C}\}$ is an L-convex structure on X/R . One can readily verify that \mathfrak{C}_R is the finest L-convex structure on X/R such that $q : X \rightarrow X/R$ is L-CP. The L-convex structure \mathfrak{C}_R is called the L-quotient convex structure, the set X/R equipped with \mathfrak{C}_R is called the L-quotient space of (X, \mathfrak{C}) with respect to R , and $q : (X, \mathfrak{C}) \rightarrow (X/R, \mathfrak{C}_R)$ is called the natural L-quotient mapping.

There is a close relationship between L-quotient mappings and L-quotient spaces. If $(X/R, \mathfrak{C}_R)$ is the L-quotient space of (X, \mathfrak{C}) , then $q : (X, \mathfrak{C}) \rightarrow (X/R, \mathfrak{C}_R)$ is an L-quotient mapping. Conversely, we have the following proposition.

Proposition 5.5. *A mapping $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ between two L-convex spaces is an L-quotient mapping iff there is an equivalence relation R on X such that $g : (X/R, \mathfrak{C}_R) \rightarrow (Y, \mathfrak{D})$ is an L-isomorphism, where*

$$R = \{(x, y) \in X \times X : f(x) = f(y)\}$$

and $g : X/R \rightarrow Y$ is defined by $g([x]_R) = f(x)$ for all $x, y \in X$.

Proof. Necessity. It is easy to verify that g is bijective. Since f is an L -quotient mapping, we obtain that f is L -CP. Note that $f = g \circ q$ and q is natural L -quotient mapping, so we have $f_L^{\leftarrow}(B) = (g \circ q)_L^{\leftarrow}(B) = q_L^{\leftarrow}(g_L^{\leftarrow}(B)) \in \mathfrak{C}$ for all $B \in \mathfrak{D}$. Hence $g_L^{\leftarrow}(B) \in \mathfrak{C}_R$. This means that g is an L -CP mapping.

Next, we show that g is L -CC. Let $C \in \mathfrak{C}_R$. Since $q : (X, \mathfrak{C}) \rightarrow (X/R, \mathfrak{C}_R)$ is an L -quotient mapping, we obtain that

$$\begin{aligned} f_L^{\leftarrow}(g_L^{\rightarrow}(C)) &= (g \circ q)_L^{\leftarrow}(g_L^{\rightarrow}(C)) \\ &= q_L^{\leftarrow}(g_L^{\leftarrow}(g_L^{\rightarrow}(C))) \\ &= q_L^{\leftarrow}(C) \end{aligned}$$

and $f_L^{\leftarrow}(g_L^{\rightarrow}(C)) \in \mathfrak{C}$. Note that f is an L -quotient mapping, so we have $g_L^{\rightarrow}(C) \in \mathfrak{D}$. It implies that g is an L -CC mapping. Hence $g : (X/R, \mathfrak{C}_R) \rightarrow (Y, \mathfrak{D})$ is an L -isomorphism.

Sufficiency. Let $g : (X/R, \mathfrak{C}_R) \rightarrow (Y, \mathfrak{D})$ is an L -isomorphism. Then g is an L -quotient mapping. Since $q : (X, \mathfrak{C}) \rightarrow (X/R, \mathfrak{C}_R)$ is an L -quotient mapping and $f = g \circ q$, we know that $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ is an L -quotient mapping. \square

Proposition 5.6. *Let R be an equivalent relation on X and S an equivalent relations on Y . Let $f : (X, \mathfrak{C}) \rightarrow (Y, \mathfrak{D})$ be an L -CP mapping between two L -convex spaces such that $(x, y) \in R$ implies $(f(x), f(y)) \in S$ for all $x, y \in X$. Define $f^* : (X/R, \mathfrak{C}_R) \rightarrow (Y/S, \mathfrak{D}_S)$ by $f^*([x]_R) = [f(x)]_S$. We have the following statements:*

- (1) f^* is an L -CP mapping and $f^* \circ q_X = q_Y \circ f$, where $q_X : (X, \mathfrak{C}) \rightarrow (X/R, \mathfrak{C}_R)$ and $q_Y : (Y, \mathfrak{D}) \rightarrow (Y/S, \mathfrak{D}_S)$ are natural L -quotient mappings.
- (2) If both f and q_Y are L -CC mappings, then f^* is also an L -CC mapping.
- (3) If f is an L -quotient mapping, then f^* is also an L -quotient mapping.

Proof. (1) For any $[x]_R = [y]_R$, it implies $(x, y) \in R$. Thus, we have $(f(x), f(y)) \in S$ which implies $[f(x)]_S = [f(y)]_S$. This means that f^* is well defined.

It is easy to verify that $f^* \circ q_X = q_Y \circ f$. Let $D \in \mathfrak{D}_S$. Then $(q_Y)_L^{\leftarrow}(D) \in \mathfrak{D}$. Since f is an L -CP mapping, we obtain that $f_L^{\leftarrow}((q_Y)_L^{\leftarrow}(D)) \in \mathfrak{C}$. Note that

$$f_L^{\leftarrow}((q_Y)_L^{\leftarrow}(D)) = (q_Y \circ f)_L^{\leftarrow}(D) = (f^* \circ q_X)_L^{\leftarrow}(D) = (q_X)_L^{\leftarrow}((f^*)_L^{\leftarrow}(D)),$$

so we get that $(q_X)_L^{\leftarrow}((f^*)_L^{\leftarrow}(D)) \in \mathfrak{C}$. It follows that $(f^*)_L^{\leftarrow}(D) \in \mathfrak{C}_R$. This implies that f^* is an L -CP mapping.

(2) For any $C \in \mathfrak{C}_R$, $(q_X)_L^{\leftarrow}(C) \in \mathfrak{C}$. Since f and q_Y are L -CC mappings, it follows that $f_L^{\rightarrow}((q_X)_L^{\leftarrow}(C)) \in \mathfrak{D}$ and $(q_Y)_L^{\rightarrow}(f_L^{\rightarrow}((q_X)_L^{\leftarrow}(C))) \in \mathfrak{D}_S$.

Note that $f^* \circ q_X = q_Y \circ f$ and

$$\begin{aligned} (q_Y)_L^{\rightarrow}(f_L^{\rightarrow}((q_X)_L^{\leftarrow}(C))) &= (q_Y \circ f)_L^{\rightarrow}((q_X)_L^{\leftarrow}(C)) \\ &= (f^* \circ q_X)_L^{\rightarrow}((q_X)_L^{\leftarrow}(C)) \\ &= ((f^*)_L^{\rightarrow} \circ (q_X)_L^{\leftarrow})((q_X)_L^{\leftarrow}(C)) \\ &= (f^*)_L^{\rightarrow}((q_X)_L^{\leftarrow}((q_X)_L^{\leftarrow}(C))) \\ &= (f^*)_L^{\rightarrow}(C), \end{aligned}$$

so we have $(f^*)_L^{\rightarrow}(C) \in \mathfrak{D}_S$. This means that f^* is an L -CC mapping.

(3) Suppose $D \in \mathfrak{D}_S$. Let $D \in \mathfrak{D}_S$. Then we have $(q_Y)_L^{\leftarrow}(D) \in \mathfrak{D}$. Note that f is L -CP and $f^* \circ q_X = q_Y \circ f$, so we have

$$(q_X)_L^{\leftarrow}((f^*)_L^{\leftarrow}(D)) = f_L^{\leftarrow}((q_Y)_L^{\leftarrow}(D)) \in \mathfrak{C}.$$

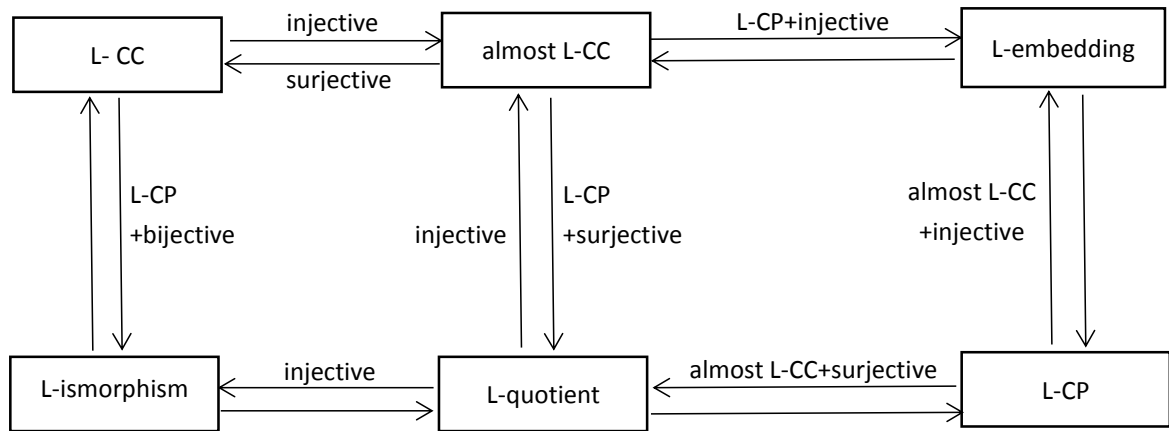
It implies that $(f^*)_L^{\leftarrow}(D) \in \mathfrak{C}_R$.

Conversely, if $(f^*)_L^{\leftarrow}(D) \in \mathfrak{C}_R$, then we obtain that

$$f_L^{\leftarrow}((q_Y)_L^{\leftarrow}(D)) = (q_X)_L^{\leftarrow}((f^*)_L^{\leftarrow}(D)) \in \mathfrak{C}.$$

Since f is an L -quotient mapping, we obtain that $(q_Y)_L^{\leftarrow}(D) \in \mathfrak{D}$, implying that $D \in \mathfrak{D}_S$. Hence f^* is an L -quotient mapping. \square

The relationships among the above-mentioned mappings can be described in the following figure:



6. Conclusions

It is well known that CP and CC mappings are two basic mappings between convex spaces. They play key roles in the theory of convex spaces. In this paper, we mainly studied some new properties of six kinds of special mappings in the L -fuzzy case, including L -CP mappings, L -CC mappings, almost L -CC mapping, L -isomorphic mappings, L -embedding mappings and L -quotient mappings. In particular, we introduced the definition of almost L -CC mapping and presented some characterizations of L -embedding mappings. Finally, we gave a diagram to show the relationships among the above-mentioned mappings. These related results are novel, and they are helpful for us to further perfect L -convex space theory.

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