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Closedness, Separation and Connectedness in Pseudo-Quasi-Semi Metric Spaces

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Abstract. In this paper, we give the characterization of closed and strongly closed subsets of an extended pseudo-quasi-semi metric space and show that they induce closure operator. Moreover, we characterize each of T_i, $i = 0, 1, 2$ and connected extended pseudo-quasi-semi metric spaces and investigate the relationship among them. Finally, we introduce the notion of irreducible objects in a topological category and examine the relationship among each of irreducible, $T_i, i = 1, 2$, and connected extended pseudo-quasi-semi metric spaces.

1. Introduction

The extended pseudo-quasi-semi metric spaces defined in 1988 by E. Lowen and R. Lowen [10] with their corresponding non-expansive mappings. They are the most general category of metric spaces which is cartesian closed and hereditary topological [10].

Baran, in [2, 3], introduced the notion of (strong) closedness in set-based topological categories and used these notions to generalize topological concepts such as separation properties and connectedness. There are various generalizations of the notion of connectedness in a topological category [4–6, 13, 14].

Note that if (X, d) is an extended pseudo-quasi-semi metric space, then d does not induce a topology on X since d does not fulfil the triangle inequality. It will be useful to give topological concepts in the category **pqsMet** of extended pseudo-quasi-semi metric spaces and non-expansive mappings.

In this paper, we give the characterization of closed and strongly closed subsets of an extended pseudo-quasi-semi metric space and show that they form closure operators of **pqsMet** in the sense of Dikranjan and Giuli [7]. We also show that if (X, d) is a pseudo-quasi-semi metric space (with image in $[0, \infty)$), then the only strongly closed subsets of X are X and $\emptyset$, and so the strong closure becomes the trivial closure [8]. Moreover, we characterize each of T_i, $i = 0, 1, 2$ extended pseudo-quasi-semi metric spaces and show that the subcategories $\mathbf{T_i}$**pqsMet** of T_i-extended pseudo-quasi-semi metric spaces, $i = 0, 1, 2$ are quotient-reflective in **pqsMet** as well as investigate the relationship among these subcategories. Furthermore, we characterize various connected extended pseudo-quasi-semi metric spaces and investigate the relationship among these various forms. Finally, we introduce the notion of irreducible objects in a topological category and investigate the relationship among each of irreducible, $T_i, i = 1, 2$, and connected extended pseudo-quasi-semi metric spaces.

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2. Preliminaries

Recall, in [10], that an extended pseudo-quasi-semi metric space is a pair (X, d), where X is a set and $d : X \times X \to [0, \infty]$ is a function fulfills $d(x, x) = 0$ for all $x \in X$.

A mapping $f : (X, d) \to (Y, e)$ between extended pseudo-quasi-semi metric spaces is said to be a non-expansive if it fulfills the property $e(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.

Let **pqsMet** be the category of extended pseudo-quasi-semi metric spaces and non-expansive mappings and **Set** be the category of sets and functions. Note that **pqsMet** is cartesian closed and hereditary [10].

Proposition 2.1. *(1) Let I be an index set, $(X_i, d_i), i \in I\}$ be a class of extended pseudo-quasi-semi metric spaces, X be a nonempty set, and $\{f_i : X \to X_i, i \in I\}$ be a source in the category* **Set**. *A source $\{f_i : (X, d) \to (X_i, d_i), i \in I\}$ in* **pqsMet** *is an initial lift if and only if for all $x, y \in X, d(x, y) = \sup_{i \in I}(d_i(f_i(x), f_i(y)))$ [10, 12].*

(2) Let $(X_i, d_i), i \in I\}$ be a class of extended pseudo-quasi-semi metric spaces and X be a nonempty set. A sink $\{f_i : (X_i, d_i) \to (X, d), i \in I\}$ is final in **pqsMet** *if and only if for all $x, y \in X$,*
$d(x, y) = \inf\{(d_i(x_i, y_i))$: there exist $x_i, y_i \in X_i$ such that $f_i(x_i) = x$ and $f_i(y_i) = y, i \in I$ \} [10, 12].

(3) The discrete extended pseudo-quasi-semi metric structure d on X is given by

$$d_{dis}(a, b) = \begin{cases} 0 & \text{if } a = b \\ \infty & \text{if } a \neq b \end{cases}$$

for all $a, b \in X$.

3. Closed and Strongly Closed Subsets of an Extended Pseudo-Quasi-Semi Metric Spaces

Let B be a set and $p \in B$. The infinite wedge product $\bigvee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times ...$ be the countable cartesian product of B. Define $A_p^\infty : \bigvee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, ..., x, p, p, ...)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, ..., x, p, p, ...)$ and $\triangledown_p^\infty : \bigvee_p^\infty B \to B$ by $\triangledown_p^\infty(x_i) = x$ for all i [2, 3].

The skewed p-axis map $S_p : B \bigvee_p B \to B^2$ is given by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, ∇_p: $B \bigvee_p B \to B$ is given by $\nabla_p(x_i) = x$ for $i = 1, 2$ [2].

Let $U : \mathcal{E} \to$ **Set** be topological functor [1, 14] and X be an object in $\mathcal{E}$ with $U(X) = B$. Let M be a nonempty subset of B. We denote by X/M the final lift of the epi U-sink $q : U(X) = B \to B/M = (B\backslash M) \cup \{*\}$, where q is the epi map that is the identity on $B\backslash M$ and identifying M with a point * [2, 3].

Let p be a point in B.

Definition 3.1. ([2, 3]) (1) X is T_1 at p iff the initial lift of the U-source $\{S_p : B \bigvee_p B \to U(X^2) = B^2$ and $\nabla_p : B \bigvee_p B \to UD(B) = B\}$ is discrete, where D is the discrete functor which is a left adjoint to U.

(2) $\{p\}$ is closed iff the initial lift of the U-source $\{A_p^\infty : \bigvee_p^\infty B \to B^\infty = U(X^\infty)$ and $\triangledown_p^\infty : \bigvee_p^\infty B \to UD(B) = B\}$ is discrete.

(3) $M \subset X$ is strongly closed iff X/M is T_1 at * or $M = \emptyset$.

(4) $M \subset X$ is closed iff $\{*\}$, the image of M, is closed in X/M or $M = \emptyset$.

(5) If $B = M = \emptyset$, then we define M to be both closed and strongly closed.

In **Top**, the category of topological spaces, the notion of closedness coincides with the usual closedness [2, 3] and M is strongly closed iff $M$is closed and for each $x \notin M$ there exists a neighbourhood of M missing x. If a topological space is T_1, then the notions of closedness and strong closedness coincide [2, 3].

Theorem 3.2. *Let (X, d) be an extended pseudo-quasi-semi metric space and $p \in X$. $\{p\}$ is closed in X if and only if for all $x \in X$ with $x \neq p$, $d(x, p) = \infty$ or $d(p, x) = \infty$.*

Proof. Suppose $\{p\}$ is closed and for $x \in X$ with $x \neq p$. We show that $d(x,p) = \infty$ or $d(p,x) = \infty$. Note that

$$d(\pi_1 A_p^\infty(x,p,p,...), \pi_1 A_p^\infty(p,x,p,p,...)) = d(x,p)$$

$$d(\pi_2 A_p^\infty(x,p,p,...), \pi_1 A_p^\infty(p,x,p,p,...)) = d(p,x)$$

$$d(\pi_i A_p^\infty(x,p,p,...), \pi_i A_p^\infty(p,x,p,p,...)) = d(p,p) = 0$$

for all $i \geq 3$, where $\pi_i : X^\infty \to X$ are the projection maps, for all $i \in I$, and $d_{dis}(\triangledown_p^\infty(x,p,p,...), \triangledown_p^\infty(p,x,p,p,...)) = d_{dis}(x,x) = 0$. Since $\{p\}$ is closed, by Proposition 2.1 and Definition 3.1, we have $\infty = \sup\{d_{dis}(\triangledown_p^\infty(x,p,p,...), \triangledown_p^\infty(p,x,p,p,...)), d(\pi_i A_p^\infty(x,p,p,...), \pi_i A_p^\infty(p,x,p,p,...)), i \in I\} = \sup\{d(x,p), d(p,x)\}$ and consequently, $d(x,p) = \infty$ or $d(p,x) = \infty$.

Conversely, suppose that for $x \in X$ with $x \neq p$, $d(x,p) = \infty$ or $d(p,x) = \infty$. We show that $\{p\}$ is closed.

Let $\bar{d}$ be the extended pseudo-quasi-semi metric structure on $\bigvee_p^\infty X$ induced by $A_p^\infty : \bigvee_p^\infty X \to (X^\infty, d^\infty)$ and $\triangledown_p^\infty : \bigvee_p^\infty X \to (X, d_{dis})$, where d^∞ and d_{dis} are the product extended pseudo-quasi-semi metric structure on X^∞ and the discrete extended pseudo-quasi-semi metric structure on X, respectively.

Let u and v be any points in $\bigvee_p^\infty X$. If $u = v$, then $\bar{d}(u,v) = 0$.

Suppose $\triangledown_p^\infty(u) = x = \triangledown_p^\infty(v)$ for some $x \in X$. If $x = p$, then $u = p_k = p_n = v$ for all $k, n \in I$, and consequently, $\bar{d}(u,v) = 0$.

If $x \neq p$, then it follows easily that $u = x_k$ and $v = x_n$ for some k and n. Note that

$$d(\pi_i A_p^\infty(u), \pi_i A_p^\infty(v)) = \begin{cases} d(x,p) & \text{if } i = k \\ d(p,x) & \text{if } i = n \\ d(p,p) = 0 & \text{if } i \notin \{k,n\} \end{cases}$$

and $d_{dis}(\triangledown_p^\infty(u), \triangledown_p^\infty(v)) = d(x,x) = 0$. Since $x \neq p$ and $d(x,p) = \infty$ or $d(p,x) = \infty$, by Definition 3.1, $\bar{d}(u,v) = \sup\{d_{dis}(\triangledown_p^\infty(u), \triangledown_p^\infty(v)), d(\pi_i A_p^\infty(u), \pi_i A_p^\infty(v)), i \in I\} = \sup\{d(x,p), d(p,x)\} = \infty$.

Suppose that $\triangledown_p^\infty(u) \neq \triangledown_p^\infty(v)$, then, by Proposition 2.1(3), $d_{dis}(\triangledown_p^\infty(u), \triangledown_p^\infty(v)) = \infty$, and consequently, by Proposition 2.1(1), $\bar{d}(u,v) = \sup\{d_{dis}(\triangledown_p^\infty(u), \triangledown_p^\infty(v)), d(\pi_i A_p^\infty(u), \pi_i A_p^\infty(v)), i \in I\} = \sup\{\infty, d(\pi_i A_p^\infty(u), \pi_i A_p^\infty(v)), i \in I\} = \infty$. Hence, by Definition 3.1, $\{p\}$ is closed. $\square$

Theorem 3.3. *Let (X,d) be an extended pseudo-quasi-semi metric space and $p \in X$. (X,d) is T_1 at $p \in X$ if and only if for all $x \in X$ with $x \neq p$, $d(x,p) = \infty = d(p,x)$.*

Proof. It is proved in [9]. $\square$

Theorem 3.4. *Let (X,d) be an extended pseudo-quasi-semi metric space and $\emptyset \neq M \subset X$.*
(1) M is strongly closed if and only if $d(M,x) = \infty = d(x,M)$ for all $x \in X$ with $x \notin M$.
(2) M is closed if and only if $d(M,x) = \infty$ or $d(x,M) = \infty$ for all $x \in X$ with $x \notin M$.

Proof. (1) Suppose M is strongly closed and $x \in X$ with $x \notin M$. Since $q(x) = x \neq * = q(M)$ and M is strongly closed, by Definition 3.1, $(X/M, d_1)$ is T_1 at $*$, where d_1 is the quotient structure on X/M induced from the epi map $q : X \to X/M$. By Theorem 3.3, $d_1(x,*) = \infty = d_1(*,x)$. By Proposition 2.1(2) $d_1(*,x) = \inf\{d(y,x) :$ there exists $y \in X$ such that $q(y) = * \} = d(M,x)$ and $d_1(x,*) = \inf\{d(x,y) :$ there exists $y \in X$ such that $q(y) = * \} = d(x,M)$. Consequently, $d(M,x) = \infty = d(x,M)$ for all $x \in X$ with $x \notin M$.

Conversely, suppose that $d(M,x) = \infty = d(x,M)$ for all $x \in X$ with $x \notin M$. We show that $(X/M, d_1)$ is T_1 at $*$. Let $a \in X/M$ with $a \neq *$ and d_1 be the quotient structure on X/M. Note that, by Proposition 2.1(2) and assumption, $d_1(a,*) = \inf\{d(a,y) :$ there exists $y \in X$ such that $q(y) = * \} = d(a,M) = \infty$ and $d_1(*,a) = \inf\{d(y,a)$: there exists $y \in X$ such that $q(y) = * \} = d(M,a) = \infty$. Hence, $d_1(*,a) = \infty = d_1(a,*)$ and by Theorem 3.3, $(X/M, d_1)$ is T_1 at $*$, and by Definition 3.1, M is strongly closed.

The proof of (2) is similar to the proof of (1). $\square$

Theorem 3.5. *Let $(X,d), (Y,e)$ be extended pseudo-quasi-semi metric spaces and $f : (X,d) \to (Y,e)$ be a non expansive mapping.*
(1) If $D \subset Y$ is (strongly) closed, then $f^{-1}(D)$ is (strongly) closed subset of X.
(2) If $N \subset X$ is (strongly) closed and $M \subset N$ is (strongly) closed, then $M \subset X$ is (strongly) closed.

Proof. (1) Suppose $D \subset Y$ is closed. If $f^{-1}(D) = \emptyset$, then by Definition 3.1, $f^{-1}(D)$ is closed. Suppose $f^{-1}(D) \neq \emptyset$ and $x \in X$ with $x \notin f^{-1}(D)$. Note that $f(x) \in Y$, $f(x) \notin D$ and $f(f^{-1}(D)) \subset D$ implies $e(f(x), D) \leq e(f(x), f(f^{-1}(D)))$ and $e(D, f(x)) \leq e(f(f^{-1}(D)), f(x))$. Since $D \subset Y$ is closed, by Theorem 3.4, $e(D, f(x)) = \infty$ or $e(f(x), D) = \infty$ and consequently, $e(f(x), f(f^{-1}(D))) = \infty$ or $e(f(f^{-1}(D)), f(x)) = \infty$. Since f is a non expansive mapping, it follows that $d(x, f^{-1}(D)) = \infty$ or $d(f^{-1}(D), x) = \infty$ and by Theorem 3.4, $f^{-1}(D)$ is a closed subset of X. The proof for strong closedness is similar.

(2) Suppose $N \subset X$ and $M \subset N$ are strongly closed. Let d_N be the initial extended pseudo-quasi-semi metric structure on N induced by the inclusion map $i : N \to (X, d)$ and d_M be the initial extended pseudo-quasi-semi metric structure on M induced by the inclusion map $i : M \to (N, d_N)$. Let $x \in X$, $x \notin M$ and $x \notin N$. By Proposition 2.1(1), $d_M(x, M) = d_N(x, M) = d(x, M)$ and $d_M(M, x) = d_N(M, x) = d(M, x)$ and by Theorem 3.4, $d(M, x) = \infty = d(x, M)$ since $N \subset X$ is strongly closed.

Suppose $x \in N$. Since $x \notin M$ and $M \subset N$ is strongly closed by Theorem 3.4, $d_N(x, M) = \infty = d_N(M, x)$ and by Proposition 2.1(1), $d(x, M) = \infty = d(M, x)$. Hence, by Theorem 3.4, $M \subset X$ is strongly closed. The proof for closedness is similar. □

Example 3.6. (1) Let (X, d) be an extended pseudo-quasi-semi metric space and $M \subset X$. If $M \subset X$ is strongly closed, then by Theorem 3.4, M is closed but the reverse implication is not true. For example, let $X = \{x, y, z\}$ and define a map $d : X^2 \to [0, \infty]$ as follows: $d(x, x) = d(y, y) = d(z, z) = 0$, $d(x, y) = d(x, z) = d(z, y) = \infty$, $d(y, x) = 1$, $d(y, z) = d(z, x) = 0$. Note that (X, d) is an extended pseudo-quasi-semi metric space and by Theorems 3.2 and 3.4, $\{x\}$ is closed but it is not strongly closed.

(2) Let $X = [0, 10]$, $A = \{(x, y) \in X^2 : x > y\} \setminus \{(10, 0)\}$ and define a map $d : X^2 \to [0, \infty]$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \text{ or } x = 10, y = 0 \\ \infty & \text{if } x < y \\ 1 & \text{if } (x,y) \in A \end{cases}$$

for all $x, y \in X$. Note that (X, d) is an extended pseudo-quasi-semi metric space. By Theorem 3.4, one can easily show that a subset $M \subset X$ is closed if and only if $M = \emptyset$, $\{x\}$, $[x, y]$, $[x, y)$, $(x, y]$, (x, y) for $0 \leq x < y \leq 10$ and the only strongly closed subsets of X are $\emptyset$ and X.

Theorem 3.7. *Let (X, d) be an extended pseudo-quasi-semi metric space.*

(1) If $M_i \subset X, i \in I$ is (strongly) closed for all $i \in I$, then $\bigcap_{i\in I} M_i$ is (strongly) closed.

(2) If $M_i \subset X, i \in I$ is strongly closed for all $i \in I$, then $\bigcup_{i\in I} M_i$ is strongly closed.

(3) If M_1 and M_2 are closed, then $M_1 \cup M_2$ may not be closed.

(4) If $M_i \subset (X_i, d_i) i \in I$ is (strongly) closed for all $i \in I$, then $\prod_{i\in I} M_i$ is (strongly) closed in $\prod_{i\in I} X_i$.

Proof. (1) Suppose $M_i \subset X, i \in I$ is (strongly) closed for all $i \in I$ and $x \in X$ with $x \notin M = \bigcap_{i\in I} M_i$. It follows that there exists $k \in I$ such that $x \notin M_k$. Since M_k is (strongly) closed, by Theorem 3.4, $d(M_k, x) = \infty$ or $d(x, M_k) = \infty$ (resp. $d(M_k, x) = \infty = d(x, M_k)$) and $M = \bigcap_{i\in I} M_i \subset M_k$ implies $d(x, M) = \infty$ or $d(M, x) = \infty$ (resp. $d(x, M) = \infty = d(M, x)$) . Hence, by Theorem 3.4, M is (strongly) closed.

The proof for (2) can be done similarly.

(3) Take (X, d) in Example 3.6(1), $M_1 = \{x\}$, and $M_2 = \{y\}$. By Theorem 3.4, M_1 and M_2 are closed but $M_1 \cup M_2 = \{x, y\}$ is not closed since $d(z, M_1 \cup M_2) = \inf\{d(z, x), d(z, y)\} = \inf\{0, \infty\} = 0$ and $d(M_1 \cup M_2, z) = \inf\{d(x, z), d(y, z)\} = \inf\{\infty, 0\} = 0$.

(4) Suppose $M_i \subset (X_i, d_i) i \in I$ is (strongly) closed for all $i \in I$ and $x \in X = \prod_{i\in I} X_i$ with $x \notin M = \prod_{i\in I} M_i$. It follows that there exists $k \in I$ such that $x_k \notin M_k$. Since M_k is (strongly) closed, by Theorem 3.4, $d(M_k, x_k) = \infty = d(x_k, M_k)$. Since the projection map π_k is non-expansive, we have $d_k(x_k, M_k) \leq d^*(x, M)$ and $d_k(M_k, x_k) \leq d^*(M, x)$ which imply $d^*(x, M) = \infty$ or $d^*(M, x) = \infty$ (resp. $d^*(x, M) = \infty = d^*(M, x)$), where d^* is the product structure on X. Hence, by Theorem 3.4, M is (strongly) closed. □

Let $\mathcal{E}$ be a topological category and let C be a closure operator of $\mathcal{E}$ in the sense of [7].

Set $\mathcal{E}_{0C} = \{ X \in \mathcal{E} : x \in C(\{y\})$ and $y \in C(\{x\})$ implies $x = y\}$.

$\mathcal{E}_{1C} = \{ X \in \mathcal{E} : C(\{x\}) = \{x\}$, for each $x \in X\}$.

$\mathcal{E}_{2C} = \{X \in \mathcal{E} : C(\Delta) = \Delta$, the diagonal$\}$.

Let $\mathcal{E}$ = **Top** and C be the ordinary closure. Then we obtain the class of T_0-spaces, T_1-spaces and T_2-spaces, respectively.

Definition 3.8. Let (X, d) be an extended pseudo-quasi-semi metric space and $M \subset X$. The (strong) closure of M is the intersection of all (strongly) closed subsets of X containing M and it is denoted by $cl(M)$ (resp. $scl(M)$).

Theorem 3.9. *Both scl and cl are idempotent, weakly hereditary, productive, and hereditary closure operators of* **pqsMet**.

Proof. It follows from Theorem 3.5 and Exercise 2.D, Theorems 2.3 and 2.4, Propositions 2.5 and 3.6 of [8]. □

Let $\mathcal{E}$ = **pqsMet** and $\mathbf{pqsMet}_{iC}$, $i = 0, 1, 2$ be the full subcategory of **pqsMet** consisting of all T_i, $i = 0, 1, 2$ extended pseudo-quasi-semi metric spaces, where $C = cl$ or scl.

Theorem 3.10.

Let (X, d) be an extended pseudo-quasi-semi metric space.

(1) $(X, d) \in \mathbf{pqsMet}_{0cl}$ if and only if for every distinct pair x and y in X, there exists a closed subset M of X such that $y \in M$ and $x \notin M$ or there exists a closed subset N of X such that $x \in N$ and $y \notin N$.

(2) $(X, d) \in \mathbf{pqsMet}_{0scl}$ if and only if for every distinct pair x and y in X, there exists a strongly closed subset M of X such that $y \in M$ and $x \notin M$ or there exists a strongly closed subset N of X such that $x \in N$ and $y \notin N$.

(3) $(X, d) \in \mathbf{pqsMet}_{icl}$, $i = 1, 2$ if and only if for every distinct pair x and y in X, $d(x, y) = \infty$ or $d(y, x) = \infty$.

(4) $(X, d) \in \mathbf{pqsMet}_{iscl}$, $i = 1, 2$ if and only if for every distinct pair x and y in X, $d(x, y) = \infty$ and $d(y, x) = \infty$.

Proof. (1) Suppose $(X, d) \in \mathbf{pqsMet}_{0cl}$ and $x, y \in X$ with $x \neq y$. It follows that $x \notin cl(\{y\})$ or $y \notin cl(\{x\})$. If $x \notin cl(\{y\})$, then by Definition 3.8, there exists a closed subset M of X such that $y \in M$ and $x \notin M$. If $y \notin cl(\{x\})$, then there exists a closed subset N of X such that $x \in N$ and $y \notin N$.
Suppose the condition holds and $x, y \in X$ with $x \neq y$. If the first part of assumption holds, then $x \notin cl(\{y\})$ and if the second part of assumption holds, then $y \notin cl(\{x\})$. Hence, $(X, d) \in \mathbf{pqsMet}_{0cl}$.

The proof for (2) is similar.

(3) Suppose $(X, d) \in \mathbf{pqsMet}_{1cl}$ and $x, y \in X$ with $x \neq y$. It follows that $cl(\{x\}) = \{x\}$ for all $x \in X$, i.e., $\{x\}$ is closed and by Theorem 3.2, for all $y \in X$ with $x \neq y$, $d(x, y) = \infty$ or $d(y, x) = \infty$.

Suppose $d(x, y) = \infty$ or $d(y, x) = \infty$ for all $x, y \in X$ with $x \neq y$. By Theorem 3.2, $\{x\}$ is closed for all $x \in X$, i.e., $cl(\{x\}) = \{x\}$. Hence, $(X, d) \in \mathbf{pqsMet}_{1cl}$.

Suppose $(X, d) \in \mathbf{pqsMet}_{2cl}$ and $x, y \in X$ with $x \neq y$. Note that $(x, y) \notin \Delta$ and Δ is cl-closed, i.e., Δ is closed. By Theorem 3.4, $d^2((x, y), \Delta) = \infty$ or $d^2(\Delta, (x, y)) = \infty$, where d^2 is the product structure on X^2. If $d^2((x, y), \Delta) = \infty$, then it follows that $d^2((x, y), (y, y)) = \infty$ and by Proposition 2.1(1), $d(x, y) = \infty$. If $d^2(\Delta, (x, y)) = \infty$, then $d^2((y, y), (x, y)) = \infty$ and by Proposition 2.1(1), $d(y, x) = \infty$.

Suppose the condition holds and $(x, y) \in X^2$ with $(x, y) \notin \Delta$. It follows that $x \neq y$ and by assumption, $d(x, y) = \infty$ or $d(y, x) = \infty$. Note that, $sup\{d(a, x), d(a, y)\} = \infty$ or $sup\{d(x, a), d(y, a)\} = \infty$ and by Proposition 2.1(1), $d^2((a, a), (x, y)) = \infty$ or $d^2((x, y), (a, a)) = \infty$ for all $a \in X$. Hence, $d^2((x, y), \Delta) = \infty$ or $d^2(\Delta, (x, y)) = \infty$ and by Theorem 3.4, Δ is closed, i.e., $(X, d) \in \mathbf{pqsMet}_{2cl}$.

(4) Suppose $(X, d) \in \mathbf{pqsMet}_{1scl}$ and $x, y \in X$ with $x \neq y$. It follows that $scl(\{x\}) = \{x\}$ for all $x \in X$ and by Theorem 3.4, for all $y \in X$ with $x \neq y$, $d(x, y) = \infty$ and $d(y, x) = \infty$.

Suppose $d(x, y) = \infty$ and $d(y, x) = \infty$ for all $x, y \in X$ with $x \neq y$. By Theorem 3.4, $\{x\}$ is strongly closed, i.e., $scl(\{x\}) = \{x\}$. Hence, $(X, d) \in \mathbf{pqsMet}_{1scl}$.

Suppose $(X, d) \in \mathbf{pqsMet}_{2scl}$ and $x, y \in X$ with $x \neq y$. Note that $(x, y) \notin \Delta$ and Δ is scl-closed, by Theorem 3.4, $d^2((x, y), \Delta) = \infty$ and $d^2(\Delta, (x, y)) = \infty$. It follows that $d^2((x, y), (y, y)) = \infty = d^2((y, y), (x, y))$ and by Proposition 2.1(1), $d(x, y) = \infty = d(y, x)$.

Suppose the condition holds and $(x, y) \in X^2$ with $(x, y) \notin \Delta$. By assumption, $d(x, y) = \infty = d(y, x)$ since $x \neq y$. It follows that $sup\{d(a, x), d(a, y)\} = \infty = sup\{d(x, a), d(y, a)\}$ and by Proposition 2.1(1), $d^2((a, a), (x, y)) = \infty = d^2((x, y), (a, a))$ for all $a \in X$ with $x \neq y$. Hence, $d^2((x, y), \Delta) = \infty = d^2(\Delta, (x, y))$ and by Theorem 3.4, Δ is strongly closed, i.e., $(X, d) \in \mathbf{pqsMet}_{2scl}$. □

Example 3.11. Let $X = \{x, y\}$ and define a map $d : X^2 \to [0, \infty]$ by $d(x, x) = d(y, y) = 0, d(x, y) = \infty, d(y, x) = 1$. By Theorem 3.10, $(X, d) \in \mathbf{pqsMet}_{icl}, i = 0, 1, 2$ but $(X, d) \notin \mathbf{pqsMet}_{iscl}, i = 0, 1, 2$ and by Theorems 3.2 and 3.4, both $\{x\}$ and $\{y\}$ are closed but they are not strongly closed.

Remark 3.12. (1) It follows easily from Theorem 3.10 that each of the subcategories $\mathbf{pqsMet}_{icl}, i = 0, 1, 2$ and $\mathbf{pqsMet}_{iscl}, i = 0, 1, 2$ are quotient-reflective [14] in **pqsMet**, i.e., they are full, isomorphism-closed, closed under formation of subspaces, products, and finer structures (i.e., if $(X, d) \in \mathcal{S}$, $(X, e) \in$ **pqsMet** and the map $f : (X, e) \to (X, d)$ is a non-expansive map morphism, where $f : X \to X$ is the identity map, then $(X, e) \in \mathcal{S}$, where $\mathcal{S}$=$\mathbf{pqsMet}_{icl}, i = 0, 1, 2$ or $\mathbf{pqsMet}_{iscl}, i = 0, 1, 2$).

(2) If $(X, d) \in \mathbf{pqsMet}_{iscl}, i = 0, 1, 2$, then, by Theorems 3.4 and 3.10, $(X, d) \in \mathbf{pqsMet}_{icl}, i = 0, 1, 2$. By Example 3.11, Theorems 3.4 and 3.10, $(X, d) \in \mathbf{pqsMet}_{icl}, i = 0, 1, 2$, but $(X, d) \notin \mathbf{pqsMet}_{iscl}, i = 0, 1, 2$ and by Example 3.6(2) and Theorem 3.10, $(X, d) \in \mathbf{pqsMet}_{0cl}$ but $(X, d) \notin \mathbf{pqsMet}_{0scl}$.

(3) By Theorem 3.10, the subcategories $\mathbf{pqsMet}_{1cl}$ and $\mathbf{pqsMet}_{2cl}$ (resp. $\mathbf{pqsMet}_{1scl}$ and $\mathbf{pqsMet}_{2scl}$) are isomorphic.

(4) By Theorems 3.2 and 3.10, $(X, d) \in \mathbf{pqsMet}_{1cl}$ if and only if for every $x \in X$, $\{x\}$ is closed.

(5) By Theorem 3.4 and Example 3.11, if $(X, d) \in \mathbf{pqsMet}_{1cl}$, then one-point sets in X may not be strongly closed.

(6) By Theorem 3.4 and Theorem 3.10, if $(X, d) \in \mathbf{pqsMet}_{1scl}$, then all subsets of X are closed. But by Example 3.11 all subsets of X are closed and by Theorem 3.10 $(X, d) \notin \mathbf{pqsMet}_{1scl}$.

(7) If (X, d) is a pseudo-quasi-semi metric space (with image in $[0, \infty)$), then by Theorem 3.4, it is clear the only strongly closed subsets of X are X and $\emptyset$, and the strong closure becomes the trivial closure [8].

4. Connected Extended Pseudo-Quasi-Semi Metric Spaces

There are various generalizations of the notion of connectedness in a topological category [4–6, 13, 14]. In this section, we characterize each of these various connected extended pseudo-quasi-semi metric spaces and investigate the relationship among them.

Definition 4.1. ([4]) Let $U : \mathcal{E} \to$ **Set** be a topological functor, X be an object in $\mathcal{E}$ and M be a nonempty subset of X.

(1) If M^C, the complement of M is strongly closed, then $M \subset X$ is said to be strongly open.

(2) If M^C, the complement of M is closed, then $M \subset X$ is said to be open.

(3) If the only subsets of X both strongly open and strongly closed are X and $\emptyset$, then X is said to be connected.

(4) If the only subsets of X both open and closed are X and $\emptyset$, then X is said to be strongly connected.

In **Top**, the notion of openness (resp. strong connectedness) coincides with the usual openness (resp. connectedness) [4] and if a topological space is T_1, then the notions of connectedness and strong connectedness coincide [4].

Theorem 4.2. *Let (X, d) be an extended pseudo-quasi-semi metric space and $\emptyset \neq M \subset X$.*

(1) M is strongly open if and only if $d(M^C, x) = \infty = d(x, M^C)$ for all $x \in X$ with $x \in M$.

(2) M is open if and only if $d(M^C, x) = \infty$ or $d(x, M^C) = \infty$ for all $x \in X$ with $x \in M$.

Proof. It follows from Theorem 3.4 and Definition 4.1. □

Theorem 4.3. *An extended pseudo-quasi-semi metric space (X, d) is (strongly) connected if and only if for any nonempty proper subset M of X, either the conditions (I) or (II) holds.*

(I) There exists $x \in X$ with $x \notin M$, $d(M, x) < \infty$ or (resp. and) $d(x, M) < \infty$.

(II) There exists $x \in X$ with $x \in M$, $d(M^C, x) < \infty$ or (resp. and) $d(x, M^C) < \infty$.

Proof. Combine Theorems 3.4, 4.2, and Definition 4.1. □

Definition 4.4. ([4, 5, 13, 14]) Let $U : \mathcal{E} \rightarrow$ **Set** be a topological functor and X be an object in $\mathcal{E}$. If any morphism from X to discrete object is constant, then X is said to be D-connected.

Theorem 4.5. *An extended pseudo-quasi-semi metric space (X, d) is D-connected if and only if for any nonempty proper subset M of X, $d(y, x) < \infty$ or $d(x, y) < \infty$ for some $x \in M$ and $y \in M^C$.*

Proof. Suppose (X, d) is D-connected and there is a nonempty proper subset M of X, $d(y, x) = \infty$ and $d(x, y) = \infty$ for all $x \in M$ and $y \in M^C$. Let (Y, e) be a discrete extended pseudo-quasi-semi metric space with card $Y > 1$. Define $f : (X, d) \rightarrow (Y, e)$ by

$$f(x) = \begin{cases} a & \text{if } x \in M \\ b & \text{if } x \notin M \end{cases}$$

for $x \in X$. Let $x, y \in X$. If $x, y \in M$ or $x, y \in M^C$, then $e(f(x), f(y)) = 0 = e(f(y), f(x)) \leq d(x, y) \wedge d(y, x)$. If $x \in M$ and $y \in M^C$ (resp. $y \in M$ and $x \in M^C$), then by Proposition 2.1(3) , $e(f(x), f(y)) = \infty = e(f(y), f(x)) \leq d(x, y) \wedge d(y, x) = \infty$. Hence, f is a non-expansive mapping but it is not constant, a contradiction.

Suppose for any nonempty proper subset M of X, $d(y, x) < \infty$ or $d(x, y) < \infty$ for some $x \in M$ and $y \in M^C$. Let (Y, e) be a discrete extended pseudo-quasi-semi metric space and $f : (X, d) \rightarrow (Y, e)$ be a non-expansive mapping. If card Y = 1, then f is constant. Suppose that card $Y > 1$ and f is not constant. There exists $a, b \in X$ with $a \neq b$ such that $f(a) \neq f(b)$. Let $M = f^{-1}\{f(a)\}$. Note that M is a proper subset of X, $a \in M$ and $b \notin M$. By assumption, there exist $x \in M$ and $y \in M^C$ such that $d(y, x) < \infty$ or $d(x, y) < \infty$. By Proposition 2.1(3), $e(f(x), f(y)) = \infty = e(f(y), f(x))$ which implies f is not a non-expansive mapping, a contradiction. Hence, f must be constant and by Definition 4.4, (X, d) is D-connected. □

Definition 4.6. ([6]) Let $\mathcal{E}$ be a complete category [1, 14] and C be a closure operator of $\mathcal{E}$ in the sense of [7]. An object X of $\mathcal{E}$ is called C-connected if the diagonal morphism $\delta_X = \langle 1_X, 1_X \rangle : X \rightarrow X \times X$ is C-dense [6].

In **Top**, if $C = q$, the quasi-component closure operator, then a topological space X is q-connected if and only if X is connected. If $C = K$, the usual Kuratowski closure operator, then a topological space X is K-connected if and only if X is irreducible, i.e., if M, N are closed subsets of X and $X = M \cup N$, then $M = X$ or $N = X$ [6].

Theorem 4.7. *An extended pseudo-quasi-semi metric space (X, d) is scl-connected if and only if for any $x, y \in X$ with $x \neq y$, there exist $a, b \in X$ with $a \neq x$ and $b \neq y$ such that either $d(x, a) < \infty$ and $d(y, b) < \infty$ or $d(a, x) < \infty$ and $d(b, y) < \infty$ holds.*

Proof. Suppose (X, d) is *scl*-connected and there exist $x, y \in X$ with $x \neq y$, ($d(x, a) = \infty$ or $d(y, b) = \infty$) and ($d(a, x) = \infty$ or $d(b, y) = \infty$) for all $a, b \in X$ with $a \neq x$ or $b \neq y$. Let $M = \{(a, b) : a, b \in X, a \neq x$ or $b \neq y\}$. Note that $\Delta \subset M$, $(x, y) \notin M$ and by Proposition 2.1(1), $d^2((x, y), (a, b)) = sup\{d(x, a), d(y, b)\} = \infty$ and $d^2((a, b), (x, y)) = sup\{d(a, x), d(b, y)\} = \infty$ for all $a, b \in X$ with $a \neq x$ and $b \neq y$, where d^2 is the product structure on X^2.

Hence, $d^2(M, (x, y)) = \infty = d^2((x, y), M)$. By Theorem 3.4, M is strongly closed and by Definition 3.8, $(x, y) \notin scl(\Delta) = X^2$, a contradiction.

Suppose the condition holds and (X, d) is not *scl*-connected, then there exists $(x, y) \in X^2$ such that $(x, y) \notin scl(\Delta)$. By Definition 3.8, there is a strongly closed subset M of X such that $\Delta \subset M$ and $(x, y) \notin M$. By Theorem 3.4, $d^2((x, y), M) = \infty = d^2(M, (x, y))$ and by Proposition 2.1(2), ($d(x, a) = \infty$ or $d(y, b) = \infty$) and ($d(a, x) = \infty$ or $d(b, y) = \infty$) for all $a, b \in X$ with $a \neq x$ or $b \neq y$, a contradiction. Hence, (X, d) is *scl*-connected. □

Theorem 4.8. *(X, d) is cl-connected if and only if for any $x, y \in X$ with $x \neq y$, there exist $a, b \in X$ with $a \neq x$ and $b \neq y$ such that both ($d(x, a) < \infty$ and $d(y, b) < \infty$) and ($d(a, x) < \infty$ and $d(b, y) < \infty$) hold.*

Proof. The proof is similar to the proof of Theorem 4.7. □

Theorem 4.9. *(1) (X, d) is connected if and only if (X, d) is D-connected.*
(2) If (X, d) is strongly connected, then (X, d) is connected.
(3) If (X, d) is cl-connected, then (X, d) is scl-connected.

Proof. (1) Suppose (X, d) is connected, M is any non-empty proper subset of X and $x \in X$. If $x \notin M$, then by Theorem 4.3, $d(M, x) < \infty$ or $d(x, M) < \infty$ and consequently, there exist $y, z \in M$ such that $d(y, x) < \infty$ or $d(x, z) < \infty$.

If $x \in M$, then by Theorem 4.3, $d(M^C, x) < \infty$ or $d(x, M^C) < \infty$ and consequently, there exist $y, z \in M^C$ such that $d(y, x) < \infty$ or $d(x, z) < \infty$. Hence, by Theorem 4.5, (X, d) is D-connected.

Suppose (X, d) is D-connected and M is any non-empty proper subset of X. Then by Theorem 4.5, there exist $x \in M$ and $y \in M^C$ such that $d(y, x) < \infty$ or $d(x, y) < \infty$. It follows that $d(y, M) < d(y, x) < \infty$ or $d(x, M^C) < d(x, y) < \infty$ and by Theorem 4.3, (X, d) is connected.

(2) follows from Theorem 4.3. But the converse is not true. For example, let (X, d) be in Example 3.11 and take $M = \{x\}$ in Theorem 4.3, (X, d) is connected but it is not strongly connected.

(3) follows from Theorems 4.7 and 4.8. □

Theorem 4.10. *If $X \neq \emptyset$ and $(X, d) \in$ $\mathbf{pqsMet}_{iscl}$, $i = 1, 2$, then the following are equivalent:*
(1) (X, d) is connected.
(2) (X, d) is scl-connected.
(3) (X, d) is cl-connected.
(4) (X, d) is strongly connected.
(5) (X, d) is D-connected.
(6) (X, d) is a one-point space.

Proof. Combine Theorems 3.10, 4.3, 4.5, 4.7, and 4.8. □

Remark 4.11. (1) Let (X, d) be in Example 3.6(1) and by Theorems 4.3, 4.5, 4.7, and 4.8, (X, d) is neither strongly connected nor *cl*-connected but it is connected, *scl*-connected, and *D*-connected.

(2) Let (X, d) be an extended pseudo-quasi-semi metric space and $M \subset X$. By Theorems 3.4 and 4.2, M is strongly closed if and only if strongly open. This implies there is a partition of (X, d) consisting of strongly open (strongly closed) subsets. It follows that (X, d) is connected if and only if X and $\emptyset$ are the only strongly open (strongly closed) subsets.

5. Irreducible Extended Pseudo-Quasi-Semi Metric Spaces

In this section, we introduce the notion of irreducible objects in a topological category and investigate the relationship among each of irreducible, $T_i, i = 1, 2$, and connected extended pseudo-quasi-semi metric spaces.

Definition 5.1. Let $U : \mathcal{E} \to \mathbf{Set}$ be a topological functor and X be an object in $\mathcal{E}$.

(1) X is said to be strongly irreducible if M, N are strongly closed subobjects of X and $X = M \cup N$, then $M = X$ or $N = X$.

(2) X is said to be irreducible if M, N are closed subobjects of X and $X = M \cup N$, then $M = X$ or $N = X$.

In **Top**, the notion of irreducibility coincides with the usual irreducibility [6]. Irreducible spaces play an important role in model algebraic geometry. The Zariski topologies are irreducible.

We state some well known properties of irreducible topological spaces and examine the validity of them for extended pseudo-quasi-semi metric spaces.

Theorem 5.2. *Let (X, τ) be a topological space.*
(1) If (X, τ) is irreducible, then (X, τ) is connected.
(2) If (X, τ) is T_1, then the notions of irreducible spaces and strongly irreducible spaces coincide.
(3) If (X, τ) is nonempty irreducible and T_2, then (X, τ) must be a one-point space.

Proof. (1) follows from Definitions 4.1 and 5.1 and 3-1 of [11].

If a topological space is T_1, then the notions of closedness and strong closedness coincide [2, 3]. Hence, (2) follows from Definition 5.1.

The proof of (3) follows easily. □

Example 5.3. (1) Let (X, d) be in Example 3.11 and take $M = \{x\}$, $N = \{y\}$ which are closed but not strongly closed. Since $X = M \cup N$, by 5.1, (X, d) is strongly irreducible but it is not irreducible.

(2) Let (X, d) be in Example 3.6(2) and take $M = [0, 2]$, $N = (2, 10]$ which are closed. Since $X = M \cup N$, by 4.3 and 5.1, (X, d) is neither irreducible nor strongly connected but it is connected and strongly irreducible.

(3) If $(X, d) \in$ **pqsMet** and d is finite, i.e., $d(x, y) < \infty$ and $d(y, x) < \infty$ for all $x, y \in X$, then by Theorem 3.4 and Definition 5.1, (X, d) is irreducible and strongly irreducible.

Theorem 5.4. *Let (X, d) be an extended pseudo-quasi-semi metric space.*

(1) If (X, d) is irreducible, then (X, d) is strongly irreducible.

(2) If (X, d) is irreducible, then (X, d) is strongly connected.

(3) If (X, d) is strongly irreducible, then (X, d) is connected.

(4) If (X, d) is strongly irreducible, then (X, d) is D-connected.

Proof. (1) Suppose M, N are strongly closed subsets of X and $X = M \cup N$. Then by Theorem 3.4, M and N are closed and by 5.1, then $M = X$ or $N = X$ since (X, d) is irreducible.

(2) Suppose (X, d) is irreducible but it is not strongly connected. By Theorem 4.3, there is a nonempty proper subset M of X satisfying for each $x \in X$ if $x \notin M$, then $d(M, x) = \infty$ or $d(x, M) = \infty$ and if $x \in M$, then $d(M^C, x) = \infty$ or $d(x, M^C) = \infty$. By Theorem 3.4, M and M^C are closed and $X = M \cup M^C$, a contradiction.

The proof of (3) is similar.

(4) follows from Theorem 4.9(1) and Part (3). □

Theorem 5.5. *Let (X, d) be an extended pseudo-quasi-semi metric space.*

(1) If (X, d) is a nonempty (strongly) irreducible and $(X, d) \in$ $\mathbf{pqsMet}_{2scl}$, then (X, d) must be a one-point space.

(2) If (X, d) is strongly irreducible and $(X, d) \in$ $\mathbf{pqsMet}_{2cl}$, then (X, d) may not be a one-point space.

Proof. (1) Suppose that (X, d) is nonempty (strongly) irreducible, $(X, d) \in \mathbf{pqsMet}_{2scl}$ and X has at least two points, x and y. Let $M = \{x\}$. By Theorem 3.10 and Remark 3.12(6), M and M^C are proper (strongly) closed and $X = M \cup M^C$, a contradiction. Hence, (X, d) must be a one-point space.

(2) Let (X, d) be in Example 3.6(2). By Example 5.3(2), (X, d) is strongly irreducible and by Remark 3.12(2), $(X, d) \in \mathbf{pqsMet}_{2cl}$ but (X, d) is not a one-point space. □

Theorem 5.6. *(A) If $X \neq \emptyset$ and $(X, d) \in$ $\mathbf{pqsMet}_{1scl}$, then the following are equivalent:*

(1) (X, d) is connected.

(2) (X, d) is irreducible.

(3) (X, d) is strongly irreducible.

(4) (X, d) is strongly connected.

(5) (X, d) is D-connected.

(6) (X, d) is a one-point space.

(B) If $(X, d) \in$ $\mathbf{pqsMet}_{1cl}$ and (X, d) is strongly irreducible, then (X, d) may not be irreducible.

Proof. (A) Combine Theorems 4.10, 5.3, and 5.4.

(B) Let (X, d) be in Example 3.6(2). By Example 5.3(2), (X, d) is strongly irreducible and by Remark 3.12(2), $(X, d) \in \mathbf{pqsMet}_{1cl}$ but by Example 5.3(2), (X, d) is not irreducible. □

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