Filomat 34:14 (2020), 4747–4756
https://doi.org/10.2298/FIL2014747P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# On the Tail Asymptotics of Supremum of Stationary $\chi$-Processes With Random Trend

**Goran Popivoda[a], Siniša Stamatović[a]**

[a]*University of Montenegro, Faculty of Science and Mathematics, Podgorica, Montenegro*

**Abstract.** Let $\chi_n(t)$, $t \geqslant 0$, be a chi-process with $n$ degrees of freedom. We derive the asymptotic exact result for

$$\mathbb{P}\left(\sup_{t\in[0,T]}(\chi_n(t)+\eta(t))>u\right),\ \text{as } u\to\infty,$$

where $\eta(t)$ is a certain random process independent of $\chi_n(t)$ and $T>0$ is a constant.

## 1. Introduction and main results

The tail asymptotic behaviour of the supremum of chi-processes (generated by stationary, non-stationary or self-similar Gaussian process) has been a subject of numerous papers: [1, 2, 5, 11, 12]. Recently, the papers [6, 9, 10] are dealing with the asymptotic behaviour of chi-processes with a trend. We will consider a chi-process with a random trend.

Let $\xi(t)$, $t\in[0,T]$ ($T>0$ is constant), be a centered stationary Gaussian process and let the covariance function $r(t)$ of process $\xi$ satisfies

$$r(t)=1-|t|^{\alpha}+o(|t|^{\alpha}),\ \text{as } t\to 0, \tag{1}$$

for some $\alpha\in(0,2]$, and

$$r(t)<1,\ \text{for all } t>0. \tag{2}$$

Let $\xi_i(t)$, $i=1,\ldots,n$, be independent copies of process $\xi$. The process

$$\chi_n(t):=\left(\xi_1^2(t)+\ldots+\xi_n^2(t)\right)^{\frac{1}{2}},\quad t\in[0,T],$$

is called a (stationary) *chi-process with n degrees of freedom*. Let $\eta(t)$ be another random process, independent of $\xi(t)$. The sum process $X(t):=\chi_n(t)+\eta(t)$ will be called a *chi-process with random trend*.

Let us first formulate the result of [6].

2010 *Mathematics Subject Classification*. Primary 60G15; Secondary 60G70
*Keywords*. chi-process with $n$ degrees of freedom, Gaussian process, random trend
Received: 11 February 2020; Revised: 30 April 2020; Accepted: 13 May 2020
Communicated by Marija Milošević
*Email addresses:* goranp@ucg.ac.me (Goran Popivoda), sstamatovic@ucg.ac.me (Siniša Stamatović)

**Theorem 1.1 (Theorem 2.3 of [6]).** *Suppose that the covariance function $r(t)$ of the centered stationary Gaussian process $\{\xi(t), t \geqslant 0\}$ satisfies assumptions (1) and (2). Assume further that $g(\cdot)$ be a non-negative bounded measurable function that attains its minimum $0$ over $[0,T]$ at the unique point $0$, and further there exist some positive constants $c, \beta$ such that*

$$g(t) = ct^{\beta}(1+o(1)), \quad t \to 0.$$

*Then*

$$\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) - g(t)) > u\right) = (1+o(1))\, M^{c}_{\alpha,\beta}\, u^{\left(\frac{2}{\alpha}-\frac{1}{\beta}\right)_+} \Upsilon_n(u), \quad u \to \infty.$$

*where,*

$$M^{c}_{\alpha,\beta} = \begin{cases} c^{-1/\beta}\Gamma(1/\beta+1)H_{\alpha}, & \text{if } \alpha < 2\beta, \\ P^{c}_{\alpha,\alpha/2}, & \text{if } \alpha = 2\beta, \\ 1, & \text{if } \alpha > 2\beta, \end{cases}$$

*and* $\Upsilon_n(x) := \dfrac{2^{(n-2)/2}}{\Gamma(n/2)}\, x^{n-2} \exp\left\{-\dfrac{x^2}{2}\right\}$.

Here, with $\Gamma(\cdot)$ we denoted the Gamma function, $H_{\alpha}$ denotes the *Pickands constant*

$$H_{\alpha} := \lim_{S\to\infty} \frac{1}{S}\mathbb{E}\left(\exp\left\{\max_{t\in[0,S]}\left(\sqrt{2}B_{\alpha/2}(t) - t^{\alpha}\right)\right\}\right) \in (0,\infty),$$

and $P^{c}_{\alpha,\alpha/2}$ is defined by

$$P^{c}_{\alpha,\alpha/2} := \lim_{S\to\infty} \mathbb{E}\left(\exp\left\{\max_{t\in[0,S]}\left(\sqrt{2}B_{\alpha/2}(t) - t^{\alpha} - c\cdot t^{\alpha/2}\right)\right\}\right) \in (0,\infty),$$

where $\{B_{\alpha/2}(t),\ t \in \mathbf{R}\}$ is a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0,1]$.

If there exist some positive constants $c, \beta$ such that

$$g(t) = g(t_0) + c|t-t_0|^{\beta}(1+o(1)), \quad t \to t_0,$$

where $t_0 = argmin_{t\in[0,T]} g(t) \in (0,T)$ is unique, then in the previous asymptotic relation $u$ will be replaced by $u + g(t_0)$, $\Gamma(\cdot)$ will be replaced by $2\Gamma(\cdot)$ and $P^{c}_{\alpha,\alpha/2}$ will be replaced by

$$\tilde{P}^{c}_{\alpha,\alpha/2} := \lim_{S\to\infty} \mathbb{E}\left(\exp\left\{\max_{t\in[-S,S]}\left(\sqrt{2}B_{\alpha/2}(t) - |t|^{\alpha} - c\cdot |t|^{\alpha/2}\right)\right\}\right).$$

Our main results are the next two theorems.

Firstly, let us consider

$$\eta(t) := \lambda - \zeta t^{\beta},$$

where $\lambda$ and $\zeta$ are random variables independent of $\xi(\cdot)$, $\zeta > 0$ almost surely, and $\beta > 0$ is a constant. With the notation $\sigma(G) := \sup\{x : \mathbb{P}(G \leqslant x) < 1\}$ for any real valued random variable $G$, we further assume that $\sigma(\lambda)$, $\sigma(\zeta)$ are finite.

**Theorem 1.2.** *Let $\xi(t)$ and $\eta(t)$, $t \in [0,T]$, be above introduced random processes and let the tail $\bar{F}_{\lambda}(x) = 1 - F_{\lambda}(x)$ satisfy*

$$\bar{F}_{\lambda}(\sigma - 1/u) = u^{-\tau}\mathcal{L}(u)$$

*for some $\tau > 0$ and $\mathcal{L}$ is a slowly varying function.*

*Suppose that the functions* $m_1(x) := \mathbb{E}\left(\zeta^{-\frac{1}{\beta}} \,|\, \lambda = x\right)$ *and* $m_2(x) := \mathbb{E}\left(P^{\zeta}_{\alpha,\alpha/2} \,|\, \lambda = x\right)$ *exist and are continuous at* $x = \sigma(\lambda)$.

*Then*

$$\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u\right) = (1 + o(1))\, W_{\alpha,\beta}\, \Gamma(\tau + 1)\, u^{-\tau+\left(\frac{2}{\alpha}-\frac{1}{\beta}\right)_+} \mathcal{L}(u)\, \Upsilon_n(u - \sigma(\lambda)),$$

*as* $u \to \infty$, *where*

$$W_{\alpha,\beta} = \begin{cases} m_1(\sigma(\lambda))\, \Gamma\left(\frac{1}{\beta} + 1\right) H_\alpha, & \text{if } \alpha < 2\beta, \\ m_2(\sigma(\lambda)), & \text{if } \alpha = 2\beta, \\ 1, & \text{if } \alpha > 2\beta, \end{cases}$$

*and* $(x)_+ = \max\{0, x\}$.

**Example.** There are numerous examples of Gaussian processes which satisfy the assumptions of Theorem 1.2. We will give one simple example.

Let $\xi(t)$, $t \in [0, T]$ be the Ornstein-Uhlenbeck process with a covariance function $r(t) = e^{-|t|}$, and $\eta(t) = \lambda - \zeta t$, where $\lambda$ is uniformly distributed on $(0, 1)$, and $\zeta \,|\, \lambda = x$ is uniformly distributed on $\left(\frac{x}{2}, x\right)$. Then,

$$\alpha = 1 < 2\beta = 2, \;\; H_1 = 1$$

$$\sigma(\lambda) = 1, \;\; \bar{F}_\lambda\left(1 - \frac{1}{u}\right) = u^{-1},$$

and

$$m_1(1) := \mathbb{E}\left(\zeta^{-1} \,|\, \lambda = 1\right) = 2\ln(2).$$

It follows

$$\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u\right) = (1 + o(1))\, 2\ln(2)\, \Upsilon_n(u - 1), \;\; u \to \infty.$$

□

Now, let us consider a smooth process $\eta(t)$ which satisfies the next four conditions.

***η*1.** $0 < \sigma := \sigma(\eta(t)) < \infty$.

***η*2.** For some $\varepsilon, \delta > 0$ there exists $\eta''(t)$ for all $t$ with $(t, \eta(t)) \in K(\delta, \varepsilon) := [-\delta, T + \delta] \times [\sigma - \varepsilon, \sigma]$, and that

$$\sup_{(t,\eta(t))\in K(\delta,\varepsilon)} |\eta''(t)| \leqslant c,$$

for some constant $c$. Moreover, assume that for all $t$ with $(t, \eta(t)) \in K(\delta, \varepsilon)$ $\eta''(t)$ is equicontinuous in the following sense

$$\omega(h) := \sup_{(t,\eta(t))\in K(\delta,\varepsilon)} \; \sup_{s\in[0,h]:(t+s,\eta(t+s))\in K(\delta,\varepsilon)} \sigma(|\eta''(t + s) - \eta''(t)|) \to 0, \text{ as } h \to 0.$$

***η*3.** For some $\varepsilon, \delta > 0$ the vector $\mathbf{X}_t = (\eta(t), \eta'(t), \eta''(t))$ has a density $f_{\mathbf{X}_t}(x, y, z)$, $x \in [\sigma - \varepsilon, \sigma]$, which is bounded for any $t \in [-\delta, T + \delta]$.

***η*4.** For some $\varepsilon, \delta, \kappa > 0$ almost surely $\eta''(t) \leqslant -\kappa$ for any $(t, x) \in K(\delta, \varepsilon)$ such that $\eta'(t) = 0$ and $\eta''(t) < 0$. Moreover, assume that the function

$$m(t, x) := \int_{-c}^{-\kappa} |z|^{1/2} f_{\eta'(t),\eta''(t) \,|\, \eta(t)=x}(0, z)dz$$

is continuous in $x = \sigma$ uniformly on $t$, with $\int_0^T m(t, \sigma)dt > 0$.

**Theorem 1.3.** *Let $\xi(t)$, $t \in [0,T]$, $T > 0$, be a stationary Gaussian process with the expectation of zero and with a covariance function $r(t)$ that satisfies (1) and (2) and let $\eta(t)$ be a process being independent of the process $\xi(t)$ that satisfies conditions $\boldsymbol{\eta 1} - \boldsymbol{\eta 4}$.*

*Let for any fixed $t \in [0,T]$ the tail $\bar{F}_{\eta(t)}(x) = 1 - F_{\eta(t)}(x)$ of the distribution function of the random variable $\eta(t)$ is regularly varying at $\sigma$, i.e., $\bar{F}_{\eta(t)}(\sigma - 1/u) = u^{-\tau}\mathcal{L}_t(u)$ for some $\tau > 0$ and $\mathcal{L}_t$ is a slowly varying function.*

*Then*

$$\mathbb{P}\left(\sup_{t\in[0,T]}(\chi_n(t) + \eta(t)) > u\right) = (1 + o(1))\sqrt{\pi}\,\Gamma(\tau+1)\,H_\alpha\,u^{\frac{2}{\alpha}-\frac{1}{2}-\tau}\,\Upsilon_n(u-\sigma)\int_0^T \mathcal{L}_t(u)\,m(t,\sigma)dt,$$

*as $u \to \infty$.*

## 2. Proofs

### 2.1. *Main lemma*

In the proofs of Theorem 1.2 and Theorem 1.3 we will use the next lemma.

**Lemma 2.1.** *Let $X$ be a positive random variable with the distribution function $F$ which has an upper endpoint $\sigma < \infty$. Suppose that tail $\bar{F}(x) = 1 - F(x)$ satisfy $\bar{F}(\sigma - 1/u) = u^{-\tau}\mathcal{L}(u)$ for some positive $\tau$ and a slowly varying function $\mathcal{L}$. Let $h$ be a non-negative measurable function such that $\mathbb{E}(h(X)) < \infty$ and suppose that $h$ is continuous and strictly positive at $\sigma$. Then, for any $s \in [0,\sigma)$ we have*

$$\int_s^\sigma h(t)\Upsilon_n(u-t)\,dF(t) \sim \Gamma(\tau+1)\,h(\sigma)\,\mathcal{L}(u)\,u^{-\tau}\,\Upsilon_n(u-\sigma), \quad u \to \infty.$$

*Proof.*
The following asymptotic relation is proved in [16] (Lemma 1)

$$\int_s^\sigma h(t)\Psi(u-t)\,dF(t) \sim \Gamma(\tau+1)\,h(\sigma)\,\mathcal{L}(u)\,u^{-\tau}\,\Psi(u-\sigma), \quad u \to \infty,$$

where $\Psi(u) := \dfrac{1}{\sqrt{2\pi}u}\exp\{-u^2/2\}$ and in the proof we are using the asymptotic result of Theorem 3.1 of [7]. By using the equality

$$\Upsilon_n(x) = \frac{2^{(n-2)/2}\sqrt{2\pi}}{\Gamma(n/2)}\,x^{n-1}\,\Psi(x),$$

and the first part it follows

$$\begin{aligned}\int_s^\sigma h(t)\Upsilon_n(u-t)\,dF(t) &= \frac{2^{(n-2)/2}\sqrt{2\pi}}{\Gamma(n/2)}\int_s^\sigma h(t)\,(u-t)^{n-1}\,\Psi(u-t)\,dF(t)\\ &\sim \frac{2^{(n-2)/2}\sqrt{2\pi}}{\Gamma(n/2)}\,\Gamma(\tau+1)\,h(\sigma)\,(u-\sigma)^{n-1}\,u^{-\tau}\,\mathcal{L}(u)\,\Psi(u-\sigma)\\ &\sim \Gamma(\tau+1)\,h(\sigma)\,u^{-\tau}\,\mathcal{L}(u)\,\Upsilon_n(u-\sigma), \quad u \to \infty.\end{aligned}$$

□

*2.2. Proof of Theorem 1.2*

By using the total probability rule we have

$$
\begin{aligned}
&\mathbb{P}\left(\max_{t\in[0,T]}\left(\chi_n(t)+\lambda-\zeta\cdot t^{\beta}\right)>u\right)\\
&=\mathbb{E}\left(\mathbb{P}\left(\max_{t\in[0,T]}\left(\chi_n(t)-\zeta\cdot t^{\beta}\right)>u-\lambda\,\middle|\,\lambda,\zeta\right)\right)\\
&=\int_{-\infty}^{\sigma(\lambda)}\int_{0}^{\sigma(\zeta)}\mathbb{P}\left(\max_{t\in[0,T]}\left(\chi_n(t)-\zeta\cdot t^{\beta}\right)>u-\lambda\,\middle|\,\lambda=b,\zeta=a\right)f_{\lambda,\zeta}(b,a)\,db\,da\\
&=\int_{-\infty}^{\sigma(\lambda)-\varepsilon}\int_{0}^{\sigma(\zeta)}\mathbb{P}\left(\max_{t\in[0,T]}\left(\chi_n(t)-at^{\beta}\right)>u-b\right)f_{\lambda}(b)\cdot f_{\zeta|\lambda=b}(a)\,db\,da+\\
&\quad+\int_{\sigma(\lambda)-\varepsilon}^{\sigma(\lambda)}\int_{0}^{\sigma(\zeta)}\mathbb{P}\left(\max_{t\in[0,T]}\left(\chi_n(t)-at^{\beta}\right)>u-b\right)f_{\lambda}(b)\cdot f_{\zeta|\lambda=b}(a)\,db\,da,
\end{aligned}
$$

for some small $\varepsilon>0$. Here, $f_{\lambda,\zeta}(b,a)$ denotes density function of random vector $(\lambda,\zeta)$, $f_{\zeta|\lambda=b}(a)$ is a density function of $\zeta\,|\,\lambda=b$, and $f_{\lambda}(b)$ is a density function of random variable $\lambda$.

The first integral in the previous equality we can estimate in the following way:

$$
\begin{aligned}
0&\leqslant\int_{-\infty}^{\sigma(\lambda)-\varepsilon}\int_{0}^{\sigma(\zeta)}\mathbb{P}\left(\max_{t\in[0,T]}\left(\chi_n(t)-at^{\beta}\right)>u-b\right)f_{\lambda}(b)\cdot f_{\zeta|\lambda=b}(a)\,db\,da\leqslant\\
&\leqslant\int_{-\infty}^{\sigma(\lambda)-\varepsilon}\int_{0}^{\sigma(\zeta)}\mathbb{P}\left(\max_{t\in[0,T]}\chi_n(t)>u-(\sigma-\varepsilon)\right)f_{\lambda}(b)\cdot f_{\zeta|\lambda=b}(a)\,db\,da\\
&=O\left(u^{\frac{2}{\alpha}}\Upsilon_n(u-\sigma(\lambda)+\varepsilon)\right),
\end{aligned}
\tag{3}
$$

where the last equality follows by Corollary 7.3 in [13].

By applying left inequality (3) and Theorem 2.3 of [6] we obtain

$$
\begin{aligned}
&\mathbb{P}\left(\max_{t\in[0,T]}\left(\chi_n(t)-\zeta\cdot t^{\beta}\right)>u-\lambda\right)\geqslant\\
&\geqslant(1-\gamma(u))\times\begin{cases}
\Gamma\left(\frac{1}{\beta}+1\right)H_{\alpha}\,u^{\frac{2}{\alpha}-\frac{1}{\beta}}\displaystyle\int_{\sigma(\lambda)-\varepsilon}^{\sigma(\lambda)}\int_{0}^{\sigma(\zeta)}a^{-\frac{1}{\beta}}\cdot\Upsilon_n(u-b)\cdot f_{\lambda}(b)\cdot f_{\zeta|\lambda=b}(a)\,db\,da, & \text{if } \alpha<2\beta,\\
\displaystyle\int_{\sigma(\lambda)-\varepsilon}^{\sigma(\lambda)}\int_{0}^{\sigma(\zeta)}P^{a}_{\alpha,\alpha/2}\cdot\Upsilon_n(u-b)\cdot f_{\lambda}(b)\cdot f_{\zeta|\lambda=b}(a)\,db\,da, & \text{if } \alpha=2\beta,\\
\displaystyle\int_{\sigma(\lambda)-\varepsilon}^{\sigma(\lambda)}\int_{0}^{\sigma(\zeta)}\Upsilon_n(u-b)\cdot f_{\lambda}(b)\cdot f_{\zeta|\lambda=b}(a)\,db\,da, & \text{if } \alpha>2\beta,
\end{cases}\\
&=(1-\gamma(u))\times\begin{cases}
\Gamma\left(\frac{1}{\beta}+1\right)H_{\alpha}\,u^{\frac{2}{\alpha}-\frac{1}{\beta}}\displaystyle\int_{\sigma(\lambda)-\varepsilon}^{\sigma(\lambda)}m_1(b)\cdot\Upsilon_n(u-b)f_{\lambda}(b)\,db, & \text{if } \alpha<2\beta,\\
\displaystyle\int_{\sigma(\lambda)-\varepsilon}^{\sigma(\lambda)}m_2(b)\cdot\Upsilon_n(u-b)f_{\lambda}(b)\,db, & \text{if } \alpha=2\beta,\\
\displaystyle\int_{\sigma(\lambda)-\varepsilon}^{\sigma(\lambda)}\Upsilon_n(u-b)f_{\lambda}(b)\,db, & \text{if } \alpha>2\beta,
\end{cases}\\
&\geqslant(1-\gamma(u)-\nu(u))\,W_{\alpha,\beta}\,\Gamma(\tau+1)\,u^{-\tau+\left(\frac{2}{\alpha}-\frac{1}{\beta}\right)_{+}}\mathcal{L}(u)\,\Upsilon_n(u-\sigma(\lambda)),\quad u\to\infty.
\end{aligned}
$$

where the last inequality follows by using Theorem 2.1. Here, $\gamma(u),\nu(u)\to 0$, as $u\to\infty$.

Similarly, by using Theorem 2.3 of [6], the right inequality in (3) and Theorem 2.1 we have

$$\mathbb{P}\left(\max_{t\in[0,T]}\left(\chi_n(t)-\zeta\cdot t^{\beta}\right)>u-\lambda\right)$$
$$\leqslant O\left(u^{\frac{2}{\alpha}}\Upsilon_n(u-\sigma(\lambda)+\varepsilon)\right)$$
$$+(1+\gamma(u)+\nu(u))W_{\alpha,\beta}\,\Gamma(\tau+1)\,u^{-\tau+\left(\frac{2}{\alpha}-\frac{1}{\beta}\right)_+}\mathcal{L}(u)\,\Upsilon_n(u-\sigma(\lambda)),\quad u\to\infty.$$

The assertion of theorem follows. □

### 2.3. *Proof of Theorem 1.3.*

`Upper bound.`

Following the idea of the proof of Theorem 2 from [8] which was used also in papers [14, 15], we will consider the points $t$ of local maxima of $\eta$ such that $\eta(t)\geqslant\sigma-\varepsilon(u)$ where $0<\varepsilon(u)<\varepsilon/2$ and $\varepsilon(u)\to 0$ as $u\to\infty$.

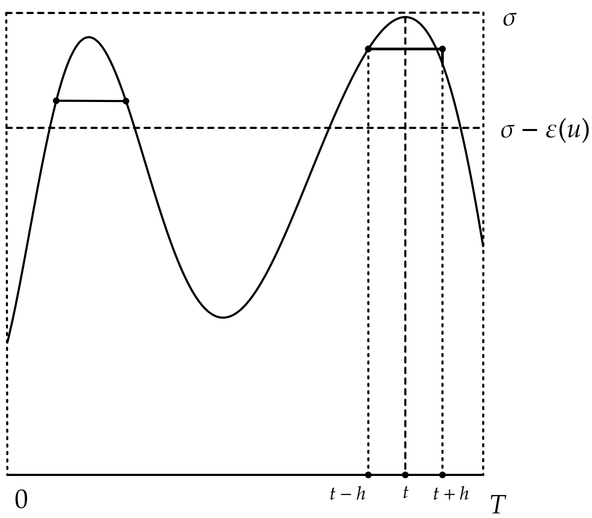


Figure 1. The illustration of the trajectory of process $\eta$.

Every two points of local maxima in $K(\delta,\varepsilon(u))$ are separated by at least $2h$, for some small $h>0$. For such $h$ and for $s$ be a point such that $|s-t|<h$ one can obtain

$$\eta(t)+\frac{(s-t)^2}{2}(\eta''(t)-\omega(h^*))\leqslant\eta(s)\leqslant\eta(t)+\frac{(s-t)^2}{2}(\eta''(t)+\omega(h^*)).\tag{4}$$

Let $s_1$ be the first local maximum of $\eta$ in $[0,T]$ with $\eta(s_1)\geqslant\sigma-\varepsilon(u)$ and $s_M$ the last one. We introduce the random set

$$L_+:=\left([0,T]\cap\bigcup_{s\in\mathcal{M}(\varepsilon(u))}[s-\delta(u),s+\delta(u)]\right)\cup[0,s_1\mathbf{1}_{A_1}]\cup[s_M\mathbf{1}_{A_M},T\mathbf{1}_{A_M}],$$

where $\mathcal{M}(\varepsilon(u))$ is a set of local maximum points of the process $\eta(t)$ which are above $\sigma-\varepsilon(u)$ and $\delta(u):=2\sqrt{\frac{\varepsilon(u)}{\kappa}}$, $A_1:=\{\eta(0)\geqslant\sigma-\varepsilon(u),\eta'(0)<0\}$ and $A_M:=\{\eta(T)\geqslant\sigma-\varepsilon(u),\eta'(T)>0\}$.

If $t\in[0,T]\backslash L_+$, then $\eta(t)<\sigma-\varepsilon(u)$, so we have

$$\mathbb{P}\left(\max_{t\in[0,T]\backslash L_+}(\chi_n(t)+\eta(t))>u\,|\,\eta\right)\leqslant\mathbb{P}\left(\max_{t\in[0,T]\backslash L_+}\chi_n(t)>u-(\sigma-\varepsilon(u))\right)$$

$$\leqslant \mathbb{P}\left(\max_{t\in[0,T]} \chi_n(t) > u - (\sigma - \varepsilon(u))\right)$$
$$= O\left(u^{\frac{2}{\alpha}} \Upsilon_n(u - (\sigma - \varepsilon(u)))\right), \text{ as } u \to \infty.$$

where the last equality follows from Corollary 7.3 in [13] (or Proposition 2.1 of [6]).

By using the total probability rule and the previous inequality it follows

$$\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u\right) = \mathbb{E}\left(\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u \,|\, \eta\right)\right)$$
$$= \mathbb{E}\left(\mathbb{P}\left(\max_{t\in L_+} (\chi_n(t) + \eta(t)) > u \,|\, \eta\right)\right) + O\left(u^{\frac{2}{\alpha}} \Upsilon_n(u - (\sigma - \varepsilon(u)))\right),$$

so we obtain the bound

$$\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u\right) \leqslant \mathbb{E}\left(\sum_{t\in\mathcal{M}(\varepsilon(u))\cap[0,T]} \mathbb{P}\left(\max_{s\in[t-h,t+h]} (\chi_n(s) + \eta(s)) > u \,|\, \eta\right)\right)$$
$$+ \mathbb{E}\left(\mathbb{P}\left(\left(\max_{s\in[0,s_1-h]} (\chi_n(s) + \eta(s)) > u\right) \cap A_1 \,|\, \eta\right)\right)$$
$$+ \mathbb{E}\left(\mathbb{P}\left(\left(\max_{s\in[s_M+h,T]} (\chi_n(s) + \eta(s)) > u\right) \cap A_M \,|\, \eta\right)\right)$$
$$+ O\left(u^{\frac{2}{\alpha}} \Upsilon_n(u - (\sigma - \varepsilon(u)))\right).$$

Now, by setting

$$\mathcal{M} = \mathcal{M}(\varepsilon(u)) \cap [-\delta(u), T + \delta(u)],$$

and choosing $\varepsilon(u) = \dfrac{\ell + \frac{2}{\alpha}}{u - \sigma} \cdot \ln u$, with a large positive $\ell$ ($> \frac{1}{2} + \tau - \frac{2}{\alpha}$), such that

$$u^{\frac{2}{\alpha}} \Upsilon_n(u - (\sigma - \varepsilon(u)) \sim u^{-\ell} \Upsilon_n(u - \sigma), \text{ as } u \to \infty,$$

we get

$$\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u\right)$$
$$\leqslant \mathbb{E}\left(\sum_{t\in\mathcal{M}} \mathbb{P}\left(\max_{s\in[t-h,t+h]} (\chi_n(s) + \eta(s)) > u \,|\, \eta\right)\right) + O\left(u^{-\ell} \Upsilon_n(u - \sigma)\right).$$

Using Theorem 2.3 in [6] we obtain

$$\mathbb{P}\left(\max_{s\in[t-h,t+h]} (\chi_n(s) + \eta(s)) > u \,\middle|\, \eta\right)$$
$$\leqslant \mathbb{P}\left(\max_{s\in[t-h,t+h]} \left(\chi_n(s) + \eta(t) + \frac{(s-t)^2}{2}(\eta''(t) + \omega(h))\right) > u \,\middle|\, \eta\right)$$
$$\leqslant \mathbb{P}\left(\max_{s\in[t-h,t+h]} \left(\chi_n(s) - \frac{(s-t)^2}{2}(-\eta''(t))\left(1 - \frac{\omega(h)}{\kappa}\right)\right) > u - \eta(t) \,\middle|\, \eta\right)$$
$$\leqslant \sqrt{\pi} H_\alpha \left(-\eta''(t)\left(1 - \frac{\omega(h)}{\kappa}\right)\right)^{-\frac{1}{2}} u^{\frac{2}{\alpha} - \frac{1}{2}} \Upsilon_n(u - \eta(t)) (1 + \gamma(u)),$$

where $\gamma(u)$ ($\downarrow 0$ as $u \to \infty$) can be chosen to be deterministic (see [8, 14]).

Let us consider the point process of local maxima $\{(t, \eta(t), \eta''(t)),\ t \in \mathcal{M}(\varepsilon(u))\}$ as a point process in $[-\delta(u), T+\delta(u)] \times [\sigma - \varepsilon(u), \sigma] \times [-c, -\kappa]$. Its intensity is

$$\nu(t,x,z) = |z| \mathbf{1}_{\{z<0\}} f_{\mathbf{X}_t}(x, 0, z)$$

(see Chapter 3 in [3] for more details) and for any bounded function $F(t, x, z)$ we have (Campbell's Formula, see for instance Theorem 2.2 in [4])

$$\mathbb{E}\left(\sum_{\mathcal{M}(\varepsilon(u))\cap[0,T]} F(t, \eta(t), \eta''(t))\right) = \int_{-\delta(u)}^{T+\delta(u)} \int_{\sigma-\varepsilon(u)}^{\sigma} \int_{-c}^{-\kappa} F(t,x,z)\, \nu(t,x,z) dt\, dx\, dz.$$

It follows that

$$\begin{aligned}
&\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u\right) \\
&\leqslant (1+\gamma(u)) \sqrt{\pi} H_\alpha\, u^{\frac{2}{\alpha}-\frac{1}{2}} \left(1 - \frac{\omega(h)}{\kappa}\right)^{-\frac{1}{2}} \int_{-\delta(u)}^{T+\delta(u)} \int_{\sigma-\varepsilon(u)}^{\sigma} \int_{-c}^{-\kappa} |z|^{\frac{1}{2}} \Upsilon_n(u-x)\, f_{\mathbf{X}_t}(x,0,z) dt dx dz \\
&\quad + O\left(u^{-\ell} \Upsilon_n(u-\sigma)\right) \\
&\leqslant (1+\gamma(u)) \sqrt{\pi} H_\alpha\, u^{\frac{2}{\alpha}-\frac{1}{2}} \left(1 - \frac{\omega(h)}{\kappa}\right)^{-\frac{1}{2}} \int_{-\delta(u)}^{T+\delta(u)} \int_{\sigma-\varepsilon}^{\sigma} \int_{-c}^{-\kappa} |z|^{\frac{1}{2}} \Upsilon_n(u-x)\, f_{\mathbf{X}_t}(x,0,z) dt dx dz \\
&\quad + O\left(u^{-\ell} \Upsilon_n(u-\sigma)\right).
\end{aligned}$$

By the equality

$$f_{\mathbf{X}_t}(x,0,z) = f_{\eta(t)}(x) f_{\eta'(t), \eta''(t) \mid \eta(t)=x}(0,z)$$

and Lemma 2.1 we derive the bound

$$\begin{aligned}
&\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u\right) \\
&\leqslant (1+\gamma(u)+\gamma_1(u)) \sqrt{\pi}\, \Gamma(\tau+1) H_\alpha\, u^{\frac{2}{\alpha}-\frac{1}{2}-\tau} \left(1 - \frac{\omega(h)}{\kappa}\right)^{-\frac{1}{2}} \Upsilon_n(u-\sigma) \int_{-\delta(u)}^{T+\delta(u)} \mathcal{L}_t(u) \cdot m(t,\sigma) dt \\
&\quad + O\left(u^{-\ell} \Upsilon_n(u-\sigma)\right),
\end{aligned}$$

where $\gamma_1(u) \to 0$ as $u \to \infty$.

Finally, we have

$$\limsup_{u\to\infty} \frac{\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u\right)}{u^{\frac{2}{\alpha}-\frac{1}{2}-\tau} \Upsilon_n(u-\sigma) \int_0^T \mathcal{L}_t(u) \cdot m(t,\sigma) dt} \to \sqrt{\pi}\, \Gamma(\tau+1) H_\alpha.$$

as $h \to 0$.

`Lower Bound.`

If $(s, \eta(s)), (t, \eta(t)) \in K(\delta, \varepsilon(u))$ and $t$ and $s$ are points of local maximum of $\eta$, then $|t-s| \geqslant 2h$. It implies that there are at most $\lfloor \frac{T}{2h} \rfloor$ points of such local maximum in the $[0,T]$. By setting $\mathcal{M}_1 := \mathcal{M}(\varepsilon(u)) \cap [\delta(u), T-\delta(u)]$ we have

$$\mathbb{E}\left(\mathbb{P}\left(\max_{t\in[0,T]} (\chi_n(t) + \eta(t)) > u \,|\, \eta\right)\right) \geqslant \mathbb{E}\left(\mathbb{P}\left(\bigcup_{t\in\mathcal{M}_1} \left\{\max_{s\in[t-h/2, t+h/2]} (\chi_n(s) + \eta(s)) > u\right\} \,\middle|\, \eta\right)\right)$$

$$\geqslant \mathbb{E}\left(\sum_{t\in\mathcal{M}_1} \mathbb{P}\left(\max_{s\in[t-h/2,t+h/2]}(\chi_n(s)+\eta(s)) > u \mid \eta\right)\right)$$
$$- \mathbb{E}\left(\sum_{\substack{s,t\in\mathcal{M}_1\\ s\neq t}} \mathbb{P}\left(\max_{v\in[t-h/2,t+h/2]}(\chi_n(v)+\eta(v)) > u, \max_{v\in[s-h/2,s+h/2]}(\chi_n(v)+\eta(v)) > u \mid \eta\right)\right). \tag{5}$$

Using the left inequality (4), and Theorem 2.3 in [6], we get

$$\begin{aligned}
&\mathbb{P}\left(\max_{s\in[t-h/2,t+h/2]}(\chi_n(s)+\eta(s)) > u \mid \eta\right)\\
&\geqslant \mathbb{P}\left(\max_{s\in[t-h/2,t+h/2]}\left(\chi_n(s)+\eta(t)+\frac{(s-t)^2}{2}(\eta''(t)-\omega(h))\right) > u \mid \eta\right)\\
&\geqslant \mathbb{P}\left(\max_{s\in[t-h/2,t+h/2]}\left(\chi_n(s)-\frac{(s-t)^2}{2}(-\eta''(t))\left(1+\frac{\omega(h)}{\kappa}\right)\right) > u-\eta(t) \mid \eta\right)\\
&\geqslant \sqrt{\pi}\, H_\alpha \left(-\eta''(t)\left(1+\frac{\omega(h)}{\kappa}\right)\right)^{-\frac{1}{2}} u^{\frac{2}{\alpha}-\frac{1}{2}}\, \Psi_n(u-\eta(t))\,(1-\nu(u)),
\end{aligned}$$

where $\nu(u)$ ($\to 0$ as $u\to\infty$) can be chosen non-randomly. Now, by the arguments for the upper bound we get

$$\liminf_{u\to\infty} \frac{\mathbb{E}\left(\sum_{t\in\mathcal{M}_1} \mathbb{P}\left(\max_{s\in[t-h/2,t+h/2]}(\chi_n(s)+\eta(s)) > u \mid \eta\right)\right)}{u^{\frac{2}{\alpha}-\frac{1}{2}-\tau}\Upsilon_n(u-\sigma)\int_0^T \mathcal{L}_t(u)\cdot m(t,\sigma)dt} \to \sqrt{\pi}\,\Gamma(\tau+1)\,H_\alpha,$$

as $h\to 0$.

Using the "Appendix" of paper [6] we obtain upper bound for the double sum, i.e.

$$\begin{aligned}
&\mathbb{P}\left(\max_{v\in[t-h/2,t+h/2]}(\chi_n(v)+\eta(v)) > u, \max_{v\in[s-h/2,s+h/2]}(\chi_n(v)+\eta(v)) > u \mid \eta\right)\\
&\leqslant o\left(u^{\frac{2}{\alpha}-\frac{1}{2}-\tau}\Upsilon_n(u-\sigma)\right) \text{ as } u\to\infty.
\end{aligned}$$

Thus, we get

$$\liminf_{u\to\infty} \frac{\mathbb{P}\left(\max_{t\in[0,T]}(\chi_n(t)+\eta(t)) > u\right)}{u^{\frac{2}{\alpha}-\frac{1}{2}-\tau}\Upsilon_n(u-\sigma)\int_0^T \mathcal{L}_t(u)\cdot m(t,\sigma)dt} \to \sqrt{\pi}\,\Gamma(\tau+1)\,H_\alpha.$$

as $h\to 0$.

□

**Acknowledgment.** The authors express their profound gratitude to Professors Vladimir I. Piterbarg and Enkelejd Hashorva for their constant encouragement and support. We are also very grateful to the reviewers of the first version of this paper for their valuable comments, which substantially helped to improve the text.

## References

[1] J. M. P. Albin. On extremal theory for stationary processes. *Ann. Probab.*, 18:92–128, 1990.
[2] J. M. P. Albin. On extremal theory for self-similar processes. *Ann. Probab.*, 26:743–793, 1998.
[3] J.-M. Azais and M. Wschebor. *Level Sets and Extrema of Random Processes and Fields*. Wiley, New York, 1 edition, 2009.

[4] A. Baddeley, I. Barany, R. Schneider, and W. Weil. *Stochastic Geometry: Lectures given at the C.I.M.E. Summer School held in Martina Franca, Italy 2004*. Springer-Verlag, Berlin, Heidelberg, 2007.
[5] S. M. Berman. Sojourns and extremes of stationary processes. *Ann. Probab.*, 10:1–46, 1982.
[6] E. Hashorva and L. Ji. Piterbarg theorems for chi-processes with trend. *Extremes*, 18(1):37–64, 2015.
[7] E. Hashorva, A. G. Pakes, and Q. Tang. Asymptotics of random contractions. *Insurance Math. Econom.*, 47(3):405–414, 2010.
[8] J. Hüsler, V. I. Piterbarg, and E. Rumyantseva. Extremes of Gaussian processes with a smooth random variance. *Stoch. Proc. Appl.*, 121(11):2592–2605, 2011.
[9] P. Liu and L. Ji. Extremes of chi-square processes with trend. *Probability and Mathematical Statistics*, 36(1):1–20, 2016.
[10] P. Liu and L. Ji. Extremes of locally stationary chi-square processes with trend. *Stochastic Processes and their Applications*, 127(2): 497–525, 2017.
[11] V. I. Piterbarg. High deviations for multidimensional stationary Gaussian processes with independent components. In *Stability Problems for Stochastic Models*, pages 197–230, 1994.
[12] V. I. Piterbarg. High excursions for nonstationary generalized chi-square processes. *Stochastic Processes and their Applications*, 53: 307–337, 1994.
[13] V. I. Piterbarg. *Asymptotic Methods in the Theory of Gaussian Processes and Fields*. AMS Translations of Mathematical Monographs 148, Providence, Rhode Island, 1 edition, 1996.
[14] V. I. Piterbarg, G. Popivoda, and S. Stamatović. Extremes of Gausssian processes with a smooth random trend. *Filomat*, 31(8): 2267–2279, 2017.
[15] V. I. Piterbarg, G. Popivoda, and S. Stamatović. Extremes of Gausssian processes with a smooth random expectation and a smooth random variance. *Lithuanian Mathematical Journal*, 57(1):128–141, 2017.
[16] G. Popivoda and S. Stamatović. On probability of high extremes of Gaussian fields with a smooth random trend. *Statistics & Probability Letters*, 147:29–35, 2019.