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Lagrange Multiplier Characterizations of Constrained Best Approximation with Nonsmooth Nonconvex Constraints

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Abstract. In this paper, we consider the constraint set $K := \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ \forall \ j = 1, 2, \ldots, m\}$ of inequalities with nonsmooth nonconvex constraint functions $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(j = 1, 2, \cdots, m)$. We show that under Abadie's constraint qualification the "perturbation property" of the best approximation to any x in $\mathbb{R}^n$ from a convex set $\tilde{K} := C \cap K$ is characterized by the strong conical hull intersection property (strong CHIP) of C and K, where C is an arbitrary non-empty closed convex subset of $\mathbb{R}^n$. By using the idea of tangential subdifferential and a non-smooth version of Abadie's constraint qualification, we do this by first proving a dual cone characterization of the constraint set K. Moreover, we present sufficient conditions for which the strong CHIP property holds. In particular, when the set $\tilde{K}$ is closed and convex, we show that the Lagrange multiplier characterizations of constrained best approximation holds under a non-smooth version of Abadie's constraint qualification. The obtained results extend many corresponding results in the context of constrained best approximation. Several examples are provided to clarify the results.

1. Introduction

The problem of determining the best approximation to any $x \in \mathbb{R}^n$ from the set $\tilde{D} := C \cap D$, where C and D are closed convex subsets of $\mathbb{R}^n$, has been of substantial interest in constrained best approximation and interpolation [5]. A central question to this problem is whether the best approximation to x from $\tilde{D}$ can be characterized by the best approximation to a perturbation $x - l$ of x from the set C for some l in a certain cone in $\mathbb{R}^n$. From the point of view of applications, finding suitable conditions for this "perturbation property" is of great significance, as it is often easier to compute the best approximation from C than from $\tilde{D}$ (see [5]). The merit and motivation for such characterization (perturbation property) is inspired from [5, Chapter 10]. "Characterizing constrained interpolation from a convex set" is one of the applications of the "perturbation property" (for more details, see [5]).

For many years, a great deal of attention has been focusing on the case where the constraint set $K := \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ \forall \ j = 1, 2, \ldots, m\}$ is a closed convex set and has a convex representation in the sense that $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(j = 1, 2, \cdots, m)$ is a convex function [5–7, 9, 12, 13]. Various characterizations of the

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perturbation property have been given by using local constraint qualifications such as the strong conical hull intersection property (strong CHIP) of C and K at the best approximation [4, 7, 9, 12], where C is a non-empty closed convex subset of $\mathbb{R}^n$.

In this paper, we study the problem of whether the best approximation to any $x \in \mathbb{R}^n$ from the *closed convex* set $\tilde{K} := C \cap K$ can be characterized by the best approximation to a perturbation $x - x^*$ of x from a closed convex set $C \subseteq \mathbb{R}^n$ for some x^* in a certain cone in $\mathbb{R}^n$, where $K := \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ \forall \ j = 1, 2, \ldots, m\}$ with $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(j = 1, 2, \cdots, m)$ is a tangentially convex function at the best approximation. We show that the strong CHIP of C and K at the best approximation continues to completely characterize the perturbation property of the best approximation from the closed convex set $\tilde{K} := C \cap K$ under Abadie's constraint qualification.

Indeed, by using the idea of tangential subdifferential and a non-smooth version of Abadie's constraint qualification (which is the weakest qualification among the other well known constraint qualifications), we prove this by first establishing a dual cone characterization of the constraint set K. In the special case when $\tilde{K}$ is a closed convex set, we show that the Lagrange multipliers characterizations of constrained best approximation holds under a non-smooth version of Abadie's constraint qualification. Our results recapture the corresponding known results of [4, 6, 7, 9–14, 16]. Several illustrative examples are presented to clarify our results.

The paper has the following structure. In Section 2, we provide the basic results on tangentially convex functions and a non-smooth version of the constraint qualifications. Dual cone characterizations of the constraint set K and sufficient conditions for which the strong CHIP holds are presented in Section 3. In Section 4, we first show that the strong CHIP completely characterizes the perturbation property of the best approximation. Finally, we show that under a non-smooth version of Abadie's constraint qualification the Lagrange multipliers characterizations of constrained best approximation holds. Also, several examples are presented to illustrate our results.

2. Preliminaries

We start this section by fixing notations and preliminaries that will be used later. Recall [3] that for a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, the directional derivative of f at a point $\bar{x} \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ is defined by

$$f'(\bar{x}, v) := \lim_{\alpha \longrightarrow 0^+} \frac{f(\bar{x} + \alpha v) - f(\bar{x})}{\alpha}, \tag{1}$$

if the limit exists. Recall [15] that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called tangentially convex at a point $\bar{x} \in \mathbb{R}^n$, if $f'(\bar{x}, \cdot)$ is a real valued convex function.

It should be noted that if the function f is tangentially convex at a point $\bar{x} \in \mathbb{R}^n$, then, since $f'(\bar{x}, \cdot)$ is a positively homogeneous function, we conclude that $f'(\bar{x}, \cdot)$ is a sublinear function on $\mathbb{R}^n$.

The tangential subdifferential of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at a point $\bar{x} \in \mathbb{R}^n$ is defined by

$$\partial_T f(\bar{x}) := \{x^* \in \mathbb{R}^n : \langle x^*, v \rangle \leq f'(\bar{x}, v), \ \forall \ v \in \mathbb{R}^n\}. \tag{2}$$

If f is tangentially convex at $\bar{x}$, then, $\partial_T f(\bar{x}) \neq \emptyset$, and moreover, $f'(\bar{x}, \cdot)$ is the support functional of $\partial_T f(\bar{x})$, i.e., for each $v \in \mathbb{R}^n$, we have

$$f'(\bar{x}, v) = \max_{x^* \in \partial_T f(\bar{x})} \langle x^*, v \rangle. \tag{3}$$

It should be noted that if f is a convex function, then, $\partial_T f(x) = \partial f(x)$ for each $x \in \mathbb{R}^n$, where $\partial f(x)$ is the classical convex subdifferential of f at x.

Remark 2.1. *Note that if the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is tangentially convex at a point $\bar{x} \in \mathbb{R}^n$, then, $f'(\bar{x}, \cdot)$ is a real valued convex function on $\mathbb{R}^n$, and hence, $f'(\bar{x}, \cdot)$ is a continuous function on $\mathbb{R}^n$.*

Now, let $K \subseteq \mathbb{R}^n$ be defined by

$$K := \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ \forall \ j = 1, 2, \ldots, m\}, \tag{4}$$

where $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(j = 1, 2, \cdots, m)$ is a tangentially convex function at a given point $\bar{x} \in K$. Let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$, and let $S := \mathbb{R}^m_+$. Note that K is not necessarily a closed or a convex set. Let

$$\tilde{K} := C \cap K, \tag{5}$$

and

$$I := \{1, 2, \cdots, m\}. \tag{6}$$

For a point $\bar{x} \in K$, we define

$$I(\bar{x}) := \{j \in I : g_j(\bar{x}) = 0\}. \tag{7}$$

For a set $W \subseteq \mathbb{R}^n$, let

$$W^\circ := \{\lambda \in \mathbb{R}^n \ : \ \langle \lambda, y \rangle \leq 0, \ \forall \ y \in W\}, \tag{8}$$

where we denote $\langle \cdot, \cdot \rangle$ for the inner product of $\mathbb{R}^n$. The normal cone to a convex set $H \subseteq \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined by

$$N_H(x) := \{u \in \mathbb{R}^n : \langle u, t - x \rangle \leq 0, \ \forall \ t \in H\}. \tag{9}$$

It is clear that

$$N_H(x) = (H - x)^\circ, \ \ (x \in \mathbb{R}^n).$$

Let U be a subset of $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. We recall [2, 3] that the contingent cone of U at x is defined by

$$\begin{aligned} & T_U(x) \\ & := \{x^* \in \mathbb{R}^n : \exists \, \alpha_k > 0, \exists \, x_k^* \in \mathbb{R}^n \ni \alpha_k \longrightarrow 0^+, x_k^* \longrightarrow x^*, x + \alpha_k x_k^* \in U, \forall k \geq 1\}. \end{aligned} \tag{10}$$

We now introduce a non-smooth version of the linearized tangential cone:

$$D(\bar{x}) := \{x^* \in \mathbb{R}^n : \langle x^*, \eta_j \rangle \leq 0, \ \forall \ \eta_j \in \partial_T g_j(\bar{x}), \ \forall \ j \in I(\bar{x})\}, \tag{11}$$

where $\bar{x} \in K$.

Note that the non-smooth linearized tangential cone $D(\bar{x})$ reduces to its counterpart in the case of differentiability [2, 3]. Moreover, $D(\bar{x})$ is a convex cone.

We now present the definition of the near convexity which has been given in [11]. Let V be a non-empty subset of $\mathbb{R}^n$ and $x \in V$.

Definition 2.1. (**Nearly Convex at** $x \in V$). *The set V is nearly convex at the point $x \in V$ if for each $y \in V$ there exists a sequence $\{t_k\}_{k \geq 1}$ of positive real numbers with $t_k \longrightarrow 0^+$ such that $x + t_k(y - x) \in V$ for all sufficiently large $k \in \mathbb{N}$.*
The set V is called nearly convex whenever it is nearly convex at each of its points. It is easy to check that if V is convex, then it is nearly convex at each $x \in V$. As shown in [8], the near convexity may hold at a point for a non-convex set (for more details and illustrative examples related to the near convexity, see [8, 11]).

Lemma 2.1. *Let K be closed, given by* (4), *and let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$. Let $\tilde{K} := C \cap K$, and $\bar{x} \in \tilde{K}$. Assume that K is nearly convex at the point $\bar{x}$. Then, $T_{\tilde{K}}(\bar{x}) \subseteq D(\bar{x})$, where $T_{\tilde{K}}(\bar{x})$ and $D(\bar{x})$ defined by* (10) *and* (11), *respectively.*

Proof: Let $x^* \in T_{\tilde{K}}(\bar{x})$ be arbitrary. Then there exist sequences $\{\alpha_k\}_{k\geq 1} \subset \mathbb{R}_{++}$ and $\{x_k^*\}_{k\geq 1} \subset \mathbb{R}^n$ such that $\alpha_k \longrightarrow 0^+$, $x_k^* \longrightarrow x^*$ and $\bar{x} + \alpha_k x_k^* \in \tilde{K}$ for all $k \geq 1$. Since, by the hypothesis, K is nearly convex at the point $\bar{x}$ and $\bar{x} + \alpha_k x_k^* \in K$ for all $k \geq 1$, it follows from Definition 2.1 that, for each $k \geq 1$, there exists a sequence $\{\beta_{k,p}\}_{p\geq 1} \subset \mathbb{R}_{++}$ with $\beta_{k,p} \longrightarrow 0^+$ (as $p \longrightarrow +\infty$) such that $\bar{x} + \beta_{k,p}(\bar{x} + \alpha_k x_k^* - \bar{x}) \in K$ for all sufficiently large $p \in \mathbb{N}$. This implies that

$$g_j(\bar{x} + \beta_{k,p}\alpha_k x_k^*) \leq 0, \text{ for all sufficiently large } p \in \mathbb{N}, \ \forall\, k \geq 1, \ \forall\, j \in I. \tag{12}$$

Since g_j ($j \in I$) is tangentially convex at $\bar{x}$, it follows, by the definition, that $g'_j(\bar{x}, \cdot)$ is a real valued positively homogeneous and convex function on $\mathbb{R}^n$. Therefore, for each $j \in I(\bar{x})$, in view of (12) we have

$$g'_j(\bar{x}, \alpha_k x_k^*) = \lim_{p\longrightarrow+\infty} \frac{g_j(\bar{x} + \beta_{k,p}\alpha_k x_k^*) - g_j(\bar{x})}{\beta_{k,p}} = \lim_{p\longrightarrow+\infty} \frac{g_j(\bar{x} + \beta_{k,p}\alpha_k x_k^*)}{\beta_{k,p}} \leq 0, \ \forall\, k \geq 1,$$

and hence,

$$g'_j(\bar{x}, x_k^*) \leq 0, \ \forall\, k \geq 1. \tag{13}$$

Since $x_k^* \longrightarrow x^*$ and $g'_j(\bar{x}, \cdot)$ is continuous on $\mathbb{R}^n$ (see Remark 2.1), we conclude from (13) that

$$g'_j(\bar{x}, x^*) \leq 0, \ \forall\, j \in I(\bar{x}).$$

This together with (3) implies that

$$\langle x^*, \eta_j \rangle \leq 0, \ \forall\, \eta_j \in \partial_T g_j(\bar{x}), \ \forall\, j \in I(\bar{x}),$$

and so, $x^* \in D(\bar{x})$, which completes the proof. ■

Now, let us define the non-smooth versions of Robinson's constraint qualification and Abadie's constraint qualification.

Definition 2.2. (**Non-smooth Version of Robinson's Constraint Qualification** (*NRCQ*)). *Let $K = \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ \forall\, j = 1, 2, \cdots, m\}$ be as in (4), and let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$. Let $\tilde{K} := C \cap K$, and $\bar{x} \in \tilde{K}$. We say that non-smooth Robinson's constraint qualification holds at $\bar{x}$ if there exists $0 \neq \nu \in \mathbb{R}^n$ such that for each $j \in I(\bar{x})$ and each $\eta_j \in \partial_T g_j(\bar{x})$, one has $\langle \eta_j, \nu \rangle < 0$, where $\partial_T g_j(\bar{x})$ is the tangential subdifferential of g_j at $\bar{x}$.*

Definition 2.3. (**Non-smooth Version of Abadie's Constraint Qualification** (*NACQ*)). *Let $K = \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ \forall\, j = 1, 2, \cdots, m\}$ be as in (4), and let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$. Let $\tilde{K} := C \cap K$, and $\bar{x} \in \tilde{K}$. We say that non-smooth Abadie's constraint qualification holds at $\bar{x}$ if $D(\bar{x}) \subseteq T_{\tilde{K}}(\bar{x})$.*

Obviously, the above definitions of non-smooth version of constraint qualifications reduce to their counterparts in the case of differentiability [1, 3].

Clearly, in view of Lemma 2.1, Definition 2.2 and Definition 2.3, the following implication holds.

$$(NRCQ) \Longrightarrow (NACQ). \tag{14}$$

The following example shows that non-smooth Abadie's constraint qualification is weaker than non-smooth Robinson's Constraint Qualification.

Example 2.1. *Let $g_1, g_2, g_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by*

$$\begin{aligned} g_1(x_1, x_2) &:= |x_2| - x_1, \\ g_2(x_1, x_2) &:= 1 - x_1^2 - (x_2 - 1)^2, \\ g_3(x_1, x_2) &:= 1 - x_1^2 - (x_2 + 1)^2, \end{aligned}$$

for all $(x_1, x_2) \in \mathbb{R}^2$. Then, we have

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq x_1\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq -x_2 \leq x_1\}.$$

Let

$$C := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\},$$

and $\tilde{K} := C \cap K = K$. Let $\bar{x} := (0, 0) \in \tilde{K}$. Clearly, g_1, g_2, g_3 are tangentially convex at $\bar{x}$, and $g_1(\bar{x}) = g_2(\bar{x}) = g_3(\bar{x}) = 0$. Moreover, it is easy to check that

$$\begin{aligned} g_1'(\bar{x}, (t_1, t_2)) &= |t_2| - t_1, \\ g_2'(\bar{x}, (t_1, t_2)) &= 2t_2, \\ g_3'(\bar{x}, (t_1, t_2)) &= -2t_2, \end{aligned}$$

for all $(t_1, t_2) \in \mathbb{R}^2$. This together with (2) *implies that*

$$\begin{aligned} \partial_T g_1(\bar{x}) &= \mathrm{co}\{(-1, -1), (-1, 1)\}, \\ \partial_T g_2(\bar{x}) &= \{(0, 2)\}, \\ \partial_T g_3(\bar{x}) &= \{(0, -2)\}. \end{aligned}$$

It is clear that non-smooth Robinson's constraint qualification does not hold at $\bar{x}$. But, we have

$$T_{\tilde{K}}(\bar{x}) = D(\bar{x}) = \{(t_1, 0) \in \mathbb{R}^2 : t_1 \geq 0\},$$

and hence, non-smooth Abadie's constraint qualification holds at $\bar{x}$.

Remark 2.2. *It should be noted that in [16] the constraint functions are continuously Fréchet differentiable, while in this paper the constraint functions are only tangentially convex at the point of best approximation. Moreover, in view of* (14) *and Example 2.1,* (*NACQ*) *is weaker than* (*NRCQ*). *So, we obtain our results under non-smooth Abadie's constraint qualification, which extend the results in [16] and the corresponding results of [4, 6, 7, 9, 10, 12–14].*

For a non-empty subset W of $\mathbb{R}^n$ and an arbitrary point $x \in \mathbb{R}^n$, we define

$$d(x, W) := \inf_{w \in W} \|x - w\|.$$

We say that a point $x_0 \in W$ is a best approximation (a projection) of $x \in \mathbb{R}^n$ if $\|x - x_0\| = d(x, W)$ [18]. The set of all best approximations (projections) of x in W denoted by $P_W(x)$ and is given by:

$$P_W(x) := \{w \in W : \|x - w\| = d(x, W)\}.$$

The following characterization of best approximation in $\mathbb{R}^n$ is well known [5].

Lemma 2.2. *Let D be a non-empty closed convex subset of $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $x_0 \in D$. Then, $x_0 = P_D(x)$ if and only if $x - x_0 \in (D - x_0)^\circ$.*

In the following, we give the notion of strong CHIP. The definition of strong CHIP was first introduced in [7] (see also, [4, 5]).

Definition 2.4. (**Strong CHIP**). *Let $C_1, C_2, \dots, C_m$ be non-empty closed convex sets in $\mathbb{R}^n$, and let $x \in \bigcap_{j=1}^m C_j$. Then, the collection $\{C_1, C_2, \dots, C_m\}$ is said to have the strong CHIP (canonical hull intersection property) at x if*

$$\Big(\bigcap_{j=1}^m C_j - x\Big)^\circ = \sum_{j=1}^m \Big(C_j - x\Big)^\circ.$$

The collection $\{C_1, C_2, \dots, C_m\}$ is said to have the strong CHIP if it has the strong CHIP at each $x \in \bigcap_{j=1}^m C_j$.

We recall [2, 3] the following well known result from non-smooth analysis.

Theorem 2.1. *Let $C \subset \mathbb{R}^n$ be a non-empty convex set, and let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function such that $C \cap \operatorname{dom}(f) \neq \emptyset$. Assume that $\bar{x} \in C$ and f is continuous at $\bar{x}$. Then, $\bar{x}$ is a global minimizer of the function f over C if and only if*

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}),$$

where domain of the function f, $\operatorname{dom}(f)$, is defined by

$$\operatorname{dom}(f) := \{x \in \mathbb{R}^n : f(x) < +\infty\}.$$

3. Dual Cone Characterizations of the Constraint Set K

In this section, we give dual cone characterizations of the constraint set K at a point $x \in K$, where K is given by (4). Also, we present sufficient conditions for which the strong CHIP holds.

For each $x \in K$, put

$$M(x) := \bigcup_{\lambda \in S} \Big\{ \sum_{j=1}^m \lambda_j \partial_T g_j(x) : \lambda_j g_j(x) = 0,\ j = 1, 2, \cdots, m \Big\}, \tag{15}$$

where $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_m) \in S$ and $S := \mathbb{R}^m_+$.

Remark 3.1. *Throughout the paper, we assume that the constraint functions g_j, $j = 1, 2, \dots, m$, are tangentially convex at a given point $\bar{x} \in \tilde{K} := C \cap K$ and $M(\bar{x})$ is closed.*

We now give a dual cone characterization of the nearly convex constraint set K, which has a crucial role for characterizing best approximations by the set $\tilde{K} := C \cap K$.

Theorem 3.1. *Let K be closed, given by* (4), *and let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$. Let $\tilde{K} := C \cap K$, $\bar{x} \in \tilde{K}$ and $M(\bar{x})$ be as in* (15). *Assume that K is nearly convex at the point $\bar{x}$. If non-smooth Abadie's constraint qualification holds at $\bar{x}$, then, $M(\bar{x}) = (K - \bar{x})^\circ = (\tilde{K} - \bar{x})^\circ$.*

Proof: It is easy to see that $(K - \bar{x})^\circ \subseteq (\tilde{K} - \bar{x})^\circ$. Now, let $u \in (\tilde{K} - \bar{x})^\circ$ be arbitrary. Then, $\langle u, y - \bar{x} \rangle \leq 0$ for all $y \in \tilde{K}$, and so,

$$\langle -u, y - \bar{x} \rangle \geq 0,\ \forall\ y \in \tilde{K}. \tag{16}$$

Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by

$$h(y) := \langle -u, y \rangle,\ \forall\ y \in \mathbb{R}^n. \tag{17}$$

It is clear that h is a continuous convex function on $\mathbb{R}^n$. Now, we show that $h(y) \geq 0$ for all $y \in T_{\tilde{K}}(\bar{x})$. To this end, let $y \in T_{\tilde{K}}(\bar{x})$ be arbitrary. Then by (10) there exist $\{t_m\}_{m \geq 1} \subset \mathbb{R}_{++}$ and $\{y_m\}_{m \geq 1} \subset \mathbb{R}^n$ such that $t_m \longrightarrow 0^+$,

$y_m \longrightarrow y$ and $\bar{x} + t_m y_m \in \tilde{K}$ for all $m \geq 1$. Thus, in view of (16), we conclude that $\langle -u, y_m \rangle \geq 0$ for all $m \geq 1$. This together with the fact that $y_m \longrightarrow y$ implies that $\langle -u, y \rangle \geq 0$, and so,

$$h(y) \geq 0, \ \forall \ y \in T_{\tilde{K}}(\bar{x}). \tag{18}$$

Consider the following optimization problem:

$$\min h(y) \text{ subject to } y \in T_{\tilde{K}}(\bar{x}). \tag{19}$$

It follows from (18) that $y = 0 \in T_{\tilde{K}}(\bar{x})$ is a global minimizer of the problem (19) over $T_{\tilde{K}}(\bar{x})$. On the other hand, since K is nearly convex at the point $\bar{x}$, in view of Lemma 2.1 and the validity of non-smooth Abadie's constraint qualification at $\bar{x}$, we have, $D(\bar{x}) = T_{\tilde{K}}(\bar{x})$. Therefore, the problem (19) can be represented as the following convex optimization problem:

$$\min h(y) \text{ subject to } y \in D(\bar{x}). \tag{20}$$

Note that $D(\bar{x})$ is a closed convex subset of $\mathbb{R}^n$, and $y = 0 \in D(\bar{x})$ is a global minimizer of the problem (20) over $D(\bar{x})$. In view of (3) and (11), the problem (20) can be represented as the following convex optimization problem:

$$\min h(y) \text{ subject to } y \in \mathbb{R}^n, \text{ and } g'_j(\bar{x}, y) \leq 0, \ \forall \ j \in I(\bar{x}). \tag{21}$$

Note that $y = 0$ (because $g'_j(\bar{x}, 0) = 0, \ j \in I(\bar{x})$) is a global minimizer of the problem (21). Let

$$C_j := \{y \in \mathbb{R}^n : g'_j(\bar{x}, y) \leq 0\}, \ (j \in I(\bar{x})),$$

and let $H := \cap_{j \in I(\bar{x})} C_j$. Since $g'_j(\bar{x}, \cdot)$ is convex on $\mathbb{R}^n$ $(j \in I)$, it is easy to see that C_j is convex for each $j \in I(\bar{x})$, and hence, H is a convex set. Since, by (20), $y = 0 \in H$ (note that $g'_j(\bar{x}, 0) = 0 \leq 0, \ j \in I(\bar{x})$) is a global minimizer of the problem (21) over H, it follows from Theorem 2.1 that

$$0 \in \partial h(0) + N_H(0). \tag{22}$$

This together with [3, Section 3.3, p. 56] implies that

$$0 \in \partial h(0) + \sum_{j \in I(\bar{x})} N_{C_j}(0). \tag{23}$$

Let

$$M_j(\bar{x}) := \{\lambda_j \eta_j : \lambda_j \geq 0, \ \eta_j \in \partial g'_j(\bar{x}, \cdot)(0)\}, \ (j \in I(\bar{x})). \tag{24}$$

It is easy to check that $M_j(\bar{x})$ is a closed convex cone in $\mathbb{R}^n$ for each $j \in I(\bar{x})$. Now, we claim that

$$N_{C_j}(0) \subseteq M_j(\bar{x}), \ (j \in I(\bar{x})). \tag{25}$$

Assume if possible that there exists $x^* \in N_{C_j}(0)$ such that $x^* \notin M_j(\bar{x})$. Since $M_j(\bar{x})$ is a closed convex cone, by using the separation theorem there exists $0 \neq v \in \mathbb{R}^n$ such that

$$\langle v, u_j \rangle \leq 0 < \langle v, x^* \rangle, \ \forall \ u_j \in M_j(\bar{x}), \ (j \in I(\bar{x})). \tag{26}$$

For simplicity, put $h_j(\cdot) := g'_j(\bar{x}, \cdot)$ $(j \in I(\bar{x}))$. Since g_j is tangentially convex at $\bar{x}$, it follows that, for each $j \in I(\bar{x})$, h_j is a real valued positively homogeneous and convex function, and so,

$$h'_j(0, v) = \max_{\eta_j \in \partial h_j(0)} \langle \eta_j, v \rangle, \ (j \in I(\bar{x})).$$

This together with (26) implies that $h_j'(0,v) \leq 0$ ($j \in I(\bar{x})$). On the other hand, since h_j ($j \in I(\bar{x})$) is positively homogeneous, we conclude that $h_j'(0,v) = g_j'(\bar{x},v)$ ($j \in I(\bar{x})$). So, $g_j'(\bar{x},v) \leq 0$ ($j \in I(\bar{x})$), and hence, $v \in C_j$ ($j \in I(\bar{x})$). But, we have $x^* \in N_{C_j}(0)$ ($j \in I(\bar{x})$). Therefore, $\langle v, x^* \rangle \leq 0$, which contradicts (26). Then, the inclusion (25) holds. This together with (23) implies that

$$0 \in \partial h(0) + \sum_{j \in I(\bar{x})} M_j(\bar{x}).$$

So, for each $j \in I(\bar{x})$, there exists $\lambda_j \geq 0$ such that

$$0 \in \partial h(0) + \sum_{j \in I(\bar{x})} \lambda_j \partial g_j'(\bar{x},\cdot)(0). \tag{27}$$

It is not difficult to see that $\partial g_j'(\bar{x},\cdot)(0) = \partial_T g_j(\bar{x})$ ($j \in I(\bar{x})$). Thus, it follows from (27) that

$$0 \in \partial h(0) + \sum_{j \in I(\bar{x})} \lambda_j \partial_T g_j(\bar{x}). \tag{28}$$

But, in view of (17), we have $\partial h(0) = \{-u\}$. Now, for each $j \notin I(\bar{x})$, put $\lambda_j = 0$. Therefore, we obtain from (28) that

$$u \in \sum_{j \in I(\bar{x})} \lambda_j \partial_T g_j(\bar{x}) = \sum_{j=1}^{m} \lambda_j \partial_T g_j(\bar{x}), \text{ and } \lambda_j g_j(\bar{x}) = 0, \ j = 1,2,\cdots,m,$$

and so, by (15), $u \in M(\bar{x})$. Hence, $(\tilde{K} - \bar{x})^\circ \subseteq M(\bar{x})$.

Now, we show that $M(\bar{x}) \subseteq (K - \bar{x})^\circ$. To this end, let $u \in M(\bar{x})$ be arbitrary. Then, in view of (15), there exists $(\lambda_1, \lambda_2, \cdots, \lambda_m) \in S$ with $\lambda_j g_j(\bar{x}) = 0$ ($j = 1,2,\cdots,m$) such that

$$u \in \sum_{j=1}^{m} \lambda_j \partial_T g_j(\bar{x}).$$

This implies that, for each $j = 1,2,\cdots,m$, there exists $\eta_j \in \partial_T g_j(\bar{x})$ such that

$$u = \sum_{j=1}^{m} \lambda_j \eta_j. \tag{29}$$

Now, let $y \in K$ be arbitrary. Since $\bar{x} \in K$ and K is nearly convex at $\bar{x}$, it follows from Definition 2.1 that there exists a sequence $\{\alpha_k\}_{k\geq 1} \subset \mathbb{R}_{++}$ with $\alpha_k \longrightarrow 0^+$ such that $\bar{x} + \alpha_k(y - \bar{x}) \in K$ for all sufficiently large $k \in \mathbb{N}$. So, by (4),

$$g_j(\bar{x} + \alpha_k(y - \bar{x})) \leq 0, \text{ for all sufficiently large } k \in \mathbb{N} \text{ and all } j = 1,2,\cdots,m. \tag{30}$$

Since g_j ($j = 1,2,\cdots,m$) is tangentially convex at $\bar{x}$, it follows from (3), (29), (30) and the fact that $\lambda_j = 0$ for each $j \notin I(\bar{x})$ (because $\lambda_j g_j(\bar{x}) = 0, \ j = 1,2,\ldots,m$) that

$$\begin{aligned}
\langle u, y - \bar{x} \rangle \quad &= \langle \sum_{j=1}^{m} \lambda_j \eta_j, y - \bar{x} \rangle = \sum_{j=1}^{m} \lambda_j \langle \eta_j, y - \bar{x} \rangle \leq \sum_{j=1}^{m} \lambda_j g_j'(\bar{x}, y - \bar{x}) \\
&= \sum_{j \in I(\bar{x})} \lambda_j g_j'(\bar{x}, y - \bar{x}) = \sum_{j \in I(\bar{x})} \lambda_j \Big\{ \lim_{k \longrightarrow +\infty} \frac{g_j(\bar{x} + \alpha_k(y - \bar{x})) - g_j(\bar{x})}{\alpha_k} \Big\} \\
&= \sum_{j \in I(\bar{x})} \lambda_j \Big\{ \lim_{k \longrightarrow +\infty} \frac{g_j(\bar{x} + \alpha_k(y - \bar{x}))}{\alpha_k} \Big\} \leq 0, \ \forall \, y \in K.
\end{aligned}$$

Hence, $u \in (K - \bar{x})^\circ$, which completes the proof. ■

Remark 3.2. *It should be noted that in Theorem 3.1, for proving the inclusion $M(\bar{x}) \subseteq (K-\bar{x})^\circ$, we only require that the set K is nearly convex at the point $\bar{x}$ without the validity of non-smooth Abadie's constraint qualification at $\bar{x}$.*

Remark 3.3. *We now give some conditions on the constraint functions g_j that guarantee the set K, given by (4), is nearly convex at a given point $\bar{x} \in K$. For example, we give the following two cases.*
(i): If each function g_j $(j = 1, 2, \dots, m)$ is quasi-convex, then the set K is convex and so it is nearly convex at each point $\bar{x} \in K$.
(ii): Fix $\bar{x} \in K$. If for each $y \in K \setminus \{\bar{x}\}$ and each $j \in I(\bar{x})$, $g'_j(\bar{x}, y - \bar{x}) < 0$, and g_j is continuous at $\bar{x}$ for each $j \notin I(\bar{x})$, then the set K is nearly convex at $\bar{x}$. To this end, let $y \in K \setminus \{\bar{x}\}$ be arbitrary. Since, for each $j \in I(\bar{x})$, we have $g'_j(\bar{x}, y - \bar{x}) < 0$, it follows from the definition of the directional derivative of g_j that there exists $t_j > 0$ such that $g_j(\bar{x} + t(y - \bar{x})) < 0$ for all $t \in (0, t_j)$ with $j \in I(\bar{x})$. On the other hand, one has $g_j(\bar{x}) < 0$ for all $j \notin I(\bar{x})$. Therefore, since g_j is continuous at $\bar{x}$ for each $j \notin I(\bar{x})$, there exists $s_j > 0$ such that $g_j(\bar{x} + t(y - \bar{x})) < 0$ for all $t \in (0, s_j)$. Put $\bar{t} := \min\{\min\{t_j : j \in I(\bar{x})\}, \min\{s_j : j \notin I(\bar{x})\}\}$. Thus, we conclude that $g_j(\bar{x} + t(y - \bar{x})) < 0$ for all $t \in (0, \bar{t})$ and all $j = 1, 2, \dots, m$. This together with (4) implies that there exists a sequence $\{t_k\}_{k \geq 1}$ of positive real numbers with $t_k \longrightarrow 0^+$ such that $\bar{x} + t_k(y - \bar{x}) \in K$ for all sufficiently large $k \in \mathbb{N}$. Hence, in view of Definition 2.1, K is nearly convex at $\bar{x}$.

Corollary 3.1. *Let K be closed, given by (4), and let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$. Let $\tilde{K} := C \cap K$, and $\bar{x} \in \tilde{K}$. Assume that K is nearly convex at the point $\bar{x}$ and non-smooth Abadie's constraint qualification holds at $\bar{x}$. Then, $\{C, K\}$ has the strong CHIP at $\bar{x}$.*

Proof: We first note that one always has

$$(C-\bar{x})^\circ + (K-\bar{x})^\circ \subseteq (\tilde{K}-\bar{x})^\circ.$$

For the converse inclusion, in view of Theorem 3.1, we conclude that $(\tilde{K}-\bar{x})^\circ = (K-\bar{x})^\circ$. Therefore, since $0 \in (C-\bar{x})^\circ$, we have

$$(\tilde{K}-\bar{x})^\circ \subseteq (C-\bar{x})^\circ + (K-\bar{x})^\circ,$$

which completes the proof. ■

The following example shows that the converse statement to Corollary 3.1 is not valid.

Example 3.1. *Let $g(x) := |x| - x$ for all $x \in \mathbb{R}$. Thus, $K = \{x \in \mathbb{R} : g(x) \leq 0\} = [0, +\infty)$, which is closed. Let $C := (-\infty, 0]$, and $\bar{x} = 0$. It is clear that g is tangentially convex at $\bar{x}$, $g'(\bar{x}, v) = |v| - v$ for all $v \in \mathbb{R}$, and $C \cap K = \{0\}$. Also, we have $g(\bar{x}) = 0$ and $\partial_T g(\bar{x}) = [-2, 0]$. Note that K is nearly convex at the point $\bar{x}$. It is easy to check that $(K-\bar{x})^\circ = (-\infty, 0]$ and $(C-\bar{x})^\circ = [0, +\infty)$. Let $\tilde{K} := C \cap K = \{0\}$. Then,*

$$(\tilde{K}-\bar{x})^\circ = \{0\}^\circ = \mathbb{R} = (-\infty, 0] + [0, +\infty) = (K-\bar{x})^\circ + (C-\bar{x})^\circ.$$

Thus, $\{C, K\}$ has the strong CHIP at $\bar{x}$. But, on the other hand, we have

$$D(\bar{x}) = [0, +\infty) \text{ and } T_{\tilde{K}}(\bar{x}) = \{0\}.$$

This implies that

$$D(\bar{x}) \not\subseteq T_{\tilde{K}}(\bar{x}),$$

and hence, non-smooth Abadie's constraint qualification does not hold at $\bar{x}$.

By the following example we show that in Theorem 3.1 the validity of the near convexity of K at the point $\bar{x} \in K$ cannot be omitted.

Example 3.2. *Let* $g_1, g_2 : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by*

$$g_1(x) := 8 - x^3, \text{ and } g_2(x) := -x^2 + 6x - 8, \ \forall\, x \in \mathbb{R}.$$

Then, we have

$$K := \{x \in \mathbb{R} : g_j(x) \le 0, \ j = 1, 2\} = \{2\} \cup [4, +\infty),$$

which is closed. Let $C := [2, 3]$, $\tilde{K} := C \cap K = \{2\}$ *and* $\bar{x} := 2$. *It is easy to see that* g_1 *and* g_2 *are tangentially convex at* $\bar{x}$, $g_1(\bar{x}) = g_2(\bar{x}) = 0$, $\partial_T g_1(\bar{x}) = \{-12\}$ *and* $\partial_T g_2(\bar{x}) = \{2\}$. *Clearly,*

$$D(\bar{x}) = \{0\}, \text{ and } T_{\tilde{K}}(\bar{x}) = \{0\}.$$

Thus, $D(\bar{x}) = T_{\tilde{K}}(\bar{x})$, *and so, non-smooth Abadie's constraint qualification holds at* $\bar{x}$, *while it is clear that* K *is not nearly convex at the point* $\bar{x}$.

On the other hand, it is not difficult to check that $M(\bar{x}) = \mathbb{R}$, $(\tilde{K} - \bar{x})^\circ = \mathbb{R}$ *and* $(K - \bar{x})^\circ = (-\infty, 0]$. *Hence, Theorem 3.1 does not hold.*

The following example shows that non-smooth Abadie's constraint qualification in Theorem 3.1 cannot be omitted.

Example 3.3. *Let* $g_1, g_2 : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by*

$$g_1(x) := \begin{cases} x^{\frac{3}{2}}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

and

$$g_2(x) := \begin{cases} -x^{\frac{3}{2}}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Thus, we have

$$K := \{x \in \mathbb{R} : g_j(x) \le 0, \ j = 1, 2\} = (-\infty, 0],$$

which is closed. Let $C := [0, 1]$, $\tilde{K} := C \cap K = \{0\}$ *and* $\bar{x} := 0$. *It is easy to check that* g_1 *and* g_2 *are tangentially convex at* $\bar{x}$, $g_1(\bar{x}) = g_2(\bar{x}) = 0$, $\partial_T g_1(\bar{x}) = \{0\}$ *and* $\partial_T g_2(\bar{x}) = \{0\}$. *Also, one can see that*

$$D(\bar{x}) = \mathbb{R}, \text{ and } T_{\tilde{K}}(\bar{x}) = \{0\}.$$

Therefore, $D(\bar{x}) \nsubseteq T_{\tilde{K}}(\bar{x})$, *and hence, non-smooth Abadie's constraint qualification does not hold at* $\bar{x}$, *while* K *is nearly convex at the point* $\bar{x}$.

Furthermore, it is easy to see that $M(\bar{x}) = \{0\}$, $(\tilde{K} - \bar{x})^\circ = \mathbb{R}$ *and* $(K - \bar{x})^\circ = [0, +\infty)$, *and so, Theorem 3.1 does not hold.*

4. Characterizations of Constrained Best Approximation

In this section, we give characterizations of constrained best approximations under non-smooth Abadie's constraint qualification. Let K be as in (4), given by,

$$K := \{x \in \mathbb{R}^n : g_j(x) \le 0, \ j = 1, 2, \cdots, m\},$$

where $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ ($j = 1, 2, \cdots, m$) is a tangentially convex function at a given point $\bar{x} \in K$. Let $S := \mathbb{R}^m_+$, and let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$. Note that K is not necessarily a closed or a convex set.

The following theorem shows that under non-smooth Abadie's constraint qualification the "perturbation property" (for details, see [5]) is characterized by the strong conical hull intersection property (Strong CHIP).

Theorem 4.1. *Let K be closed, given by* (4), *and let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$. Let $\bar{x} \in \tilde{K} := C \cap K$. Assume that $\tilde{K}$ is closed and convex. If K is nearly convex at the point $\bar{x}$ and non-smooth Abadie's constraint qualification holds at $\bar{x}$, then the following assertions are equivalent.*
(*i*) *$\{C, K\}$ has the strong CHIP at $\bar{x}$,*
(*ii*) *For any $x \in \mathbb{R}^n$, $\bar{x} = P_{\tilde{K}}(x)$ if and only if there exist $(\lambda_1, \lambda_2, \cdots, \lambda_m) \in S$ with $\lambda_j g_j(\bar{x}) = 0$ and $\eta_j \in \partial_T g_j(\bar{x})$ such that*

$$\bar{x} = P_C(x - \sum_{j=1}^{m} \lambda_j \eta_j), \ j = 1, 2, \cdots, m. \tag{31}$$

Proof: [(*i*) $\Longrightarrow$ (*ii*)]. Suppose that (*i*) holds. Then, by Definition 2.4,

$$(\tilde{K} - \bar{x})^\circ = (C - \bar{x})^\circ + (K - \bar{x})^\circ. \tag{32}$$

Also, in view of the hypotheses and Theorem 3.1, we have $M(\bar{x}) = (K - \bar{x})^\circ$. So, it follows from (32) that

$$(\tilde{K} - \bar{x})^\circ = (C - \bar{x})^\circ + M(\bar{x}). \tag{33}$$

Now, for any $x \in \mathbb{R}^n$, assume that $\bar{x} = P_{\tilde{K}}(x)$. Thus, by Lemma 2.2, one has $x - \bar{x} \in (\tilde{K} - \bar{x})^\circ$. Therefore, in view of (15) and (33), there exist $\ell \in (C - \bar{x})^\circ$ and $(\lambda_1, \lambda_2, \cdots, \lambda_m) \in S$ with $\lambda_j g_j(\bar{x}) = 0$ such that

$$x - \bar{x} - \ell \in \sum_{j=1}^{m} \lambda_j \partial_T g_j(\bar{x}), \ j = 1, 2, \cdots, m.$$

So, for each $j = 1, 2, \cdots, m$, there exists $\eta_j \in \partial_T g_j(\bar{x})$ such that

$$x - \bar{x} - \ell = \sum_{j=1}^{m} \lambda_j \eta_j.$$

Then, we conclude that

$$[x - \sum_{j=1}^{m} \lambda_j \eta_j] - \bar{x} = \ell \in (C - \bar{x})^\circ \text{ with } \lambda_j g_j(\bar{x}) = 0 \text{ and } \eta_j \in \partial_T g_j(\bar{x}).$$

So, by using Lemma 2.2, it follows that $\bar{x} = P_C(x - \sum_{j=1}^{m} \lambda_j \eta_j)$. Hence, the following implication holds.

$$\bar{x} = P_{\tilde{K}}(x) \Longrightarrow \bar{x} = P_C(x - \sum_{j=1}^{m} \lambda_j \eta_j) \text{ with } \lambda_j g_j(\bar{x}) = 0 \text{ and } \eta_j \in \partial_T g_j(\bar{x}).$$

Conversely, assume that there exist $(\lambda_1, \lambda_2, \cdots, \lambda_m) \in S$ with $\lambda_j g_j(\bar{x}) = 0$ and $\eta_j \in \partial_T g_j(\bar{x})$ such that

$$\bar{x} = P_C(x - \sum_{j=1}^{m} \lambda_j \eta_j), \ j = 1, 2, \cdots, m.$$

By using Lemma 2.2,

$$[x - \sum_{j=1}^{m} \lambda_j \eta_j] - \bar{x} \in (C - \bar{x})^\circ.$$

Then,

$$x - \bar{x} \in (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \eta_j \text{ with } \lambda_j g_j(\bar{x}) = 0 \text{ and } \eta_j \in \partial_T g_j(\bar{x}). \tag{34}$$

Therefore, it follows from (15), (33) and (34) that

$$\begin{aligned} x - \bar{x} &\in (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \eta_j \subseteq (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \partial_T g_j(\bar{x}) \subseteq (C - \bar{x})^\circ + M(\bar{x}) \\ &= (\tilde{K} - \bar{x})^\circ. \end{aligned}$$

Again, by using Lemma 2.2, $\bar{x} = P_{\tilde{K}}(x)$.
[$(ii) \Longrightarrow (i)$]. Let $y \in (\tilde{K} - \bar{x})^\circ$ be arbitrary, and let $x := \bar{x} + y$. Thus, by Lemma 2.2, $\bar{x} = P_{\tilde{K}}(x)$. Then, in view of the hypothesis (ii), there exist $(\lambda_1, \lambda_2, \cdots, \lambda_m) \in S$ with $\lambda_j g_j(\bar{x}) = 0$ and $\eta_j \in \partial_T g_j(\bar{x})$ such that

$$\bar{x} = P_C(x - \sum_{j=1}^{m} \lambda_j \eta_j), \ j = 1, 2, \cdots, m.$$

Again, by Lemma 2.2,

$$y - \sum_{j=1}^{m} \lambda_j \eta_j \in (C - \bar{x})^\circ \text{ with } \lambda_j g_j(\bar{x}) = 0 \text{ and } \eta_j \in \partial_T g_j(\bar{x}). \tag{35}$$

Therefore, it follows from (15), (35) and Theorem 3.1 that

$$\begin{aligned} y &\in (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \eta_j \subseteq (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \partial_T g_j(\bar{x}) \subseteq (C - \bar{x})^\circ + M(\bar{x}) \\ &= (C - \bar{x})^\circ + (K - \bar{x})^\circ. \end{aligned}$$

Hence,

$$(\tilde{K} - \bar{x})^\circ \subseteq (C - \bar{x})^\circ + (K - \bar{x})^\circ.$$

On the other hand, one always has,

$$(C - \bar{x})^\circ + (K - \bar{x})^\circ \subseteq (\tilde{K} - \bar{x})^\circ,$$

which implies that (i) holds. ■

The following examples illustrate Theorem 4.1. Moreover, these examples justify how one can use best approximations to check the strong CHIP without explicitly proving the strong CHIP.

Example 4.1. *Let $g_1, g_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by*

$$g_1(x_1, x_2) := |x_2| - x_1 - x_1^2 - x_1^3,$$
$$g_2(x_1, x_2) := |x_1 - x_2| - x_1 - x_1 x_2 - x_2^3,$$

for all $(x_1, x_2) \in \mathbb{R}^2$. Let $S := \mathbb{R}^2_+$ and $C := \mathbb{R}^2$. It is easy to see that

$$K := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq x_2 \geq 0\},$$

which is closed and convex. Let $\tilde{K} := C \cap K = K$ and $\bar{x} := (0, 0) \in \tilde{K}$. Note that $\tilde{K}$ is closed and convex. It is clear that g_1 and g_2 are tangentially convex at $\bar{x}$, but not convex. Moreover, $g_1(\bar{x}) = g_2(\bar{x}) = (0, 0)$, and

$$g_1'(\bar{x}, (t_1, t_2)) = |t_2| - t_1, \ g_2'(\bar{x}, (t_1, t_2)) = |t_1 - t_2| - t_1,$$

for all $(t_1, t_2) \in \mathbb{R}^2$. Therefore, we have

$$\partial_T g_1(\bar{x}) = \text{co}\{(-1, 1), (-1, -1)\}, \text{ and } \partial_T g_2(\bar{x}) = \text{co}\{(-2, 1), (0, -1)\}.$$

So, it is easy to check that

$$D(\bar{x}) = \{(t_1, t_2) \in \mathbb{R}^2 : t_1 \geq t_2 \geq 0\} = T_{\tilde{K}}(\bar{x}),$$

and hence, Abadie's constraint qualification holds at $\bar{x}$. Also, it is clear that K is nearly convex at the point $\bar{x}$.

Now, for any $x := (0, x_2) \in \mathbb{R}^2$ with $x_2 \leq 0$, it is easy to see that

$$P_{\tilde{K}}(x) = \bar{x} = (0,0) = P_C((0,0)) = P_C(x - (\lambda_1\eta_1 + \lambda_2\eta_2)),$$

where $(\lambda_1 := 0, \lambda_2 := -x_2) \in S$, $\lambda_j g_j(\bar{x}) = 0$ $(j = 1,2)$, and $\eta_1 := (-1,1) \in \partial_T g_1(\bar{x})$, $\eta_2 := (0,-1) \in \partial_T g_2(\bar{x})$. Then, in view of Theorem 4.1 (the implication $[(ii) \Longrightarrow (i)]$), we conclude that $\{C, K\}$ has the strong CHIP at $\bar{x}$. Indeed, one can see that

$$\begin{aligned}(\tilde{K} - \bar{x})^\circ &= \{(t_1, t_2) \in \mathbb{R}^2 : t_1 \leq -t_2,\ t_1 \leq 0\} \cup (\mathbb{R}_- \times \mathbb{R}_-)\\ &= \{(0,0)\} + \{(t_1, t_2) \in \mathbb{R}^2 : t_1 \leq -t_2,\ t_1 \leq 0\} \cup (\mathbb{R}_- \times \mathbb{R}_-)\\ &= (C - \bar{x})^\circ + (K - \bar{x})^\circ.\end{aligned}$$

Example 4.2. *Let $g_1, g_2 : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by*

$$g_1(x) := 1 - x^3,\ g_2(x) := x^3 - 3x^2 + x - 3,$$

for all $x \in \mathbb{R}$. Let $S := \mathbb{R}^2_+$ and $C := [1, +\infty)$. Clearly, we have

$$K = \{x \in \mathbb{R} : g_j(x) \leq 0,\ j = 1,2\} = [1,3],$$

which is closed and convex. Let $\tilde{K} := C \cap K = [1,3]$ and $\bar{x} := 1 \in \tilde{K}$. Thus, $\tilde{K}$ is closed and convex, g_1, g_2 are tangentially convex at $\bar{x}$ (but not convex), $g_1(\bar{x}) = 0$, $g_2(\bar{x}) = -4 \neq 0$, and

$$g_1'(\bar{x}, t) = -3t,\ g_2'(\bar{x}, t) = -2t,$$

for all $t \in \mathbb{R}$. This implies that

$$\partial_T g_1(\bar{x}) = \{-3\}, \text{ and } \partial_T g_2(\bar{x}) = \{-2\}.$$

Moreover, it is not difficult to show that

$$D(\bar{x}) = [0, +\infty) = T_{\tilde{K}}(\bar{x}),$$

and so, non-smooth Abadie's constraint qualification holds at $\bar{x}$. Note that K is nearly convex at the point $\bar{x}$. It is easy to see that, for any $x \in \mathbb{R}$ with $x \leq 1$, we have

$$P_{\tilde{K}}(x) = \bar{x} = 1 = P_C(1) = P_C(x - (\lambda_1\eta_1 + \lambda_2\eta_2)),$$

where $(\lambda_1 := \frac{1-x}{3}, \lambda_2 := 0) \in S$, $\lambda_j g_j(\bar{x}) = 0$ $(j = 1,2)$, and $\eta_1 := -3 \in \partial_T g_1(\bar{x})$, $\eta_2 := -2 \in \partial_T g_2(\bar{x})$. Hence, by using Theorem 4.1 (the implication $[(ii) \Longrightarrow (i)]$), we conclude that $\{C, K\}$ has the strong CHIP at $\bar{x}$. Indeed, one can see that

$$(\tilde{K} - \bar{x})^\circ = (-\infty, 0] = (-\infty, 0] + (-\infty, 0] = (C - \bar{x})^\circ + (K - \bar{x})^\circ.$$

Now, let $x \in \mathbb{R}^n$ be fixed, and define the function $h : \mathbb{R}^n \longrightarrow [0, +\infty)$ by

$$h(y) := \|y - x\|,\ \forall\ y \in \mathbb{R}^n.$$

For $\bar{x} \in \mathbb{R}^n$, we recall that $\partial h(\bar{x}) := \partial\|\cdot - x\|(\bar{x})$ is given by

$$\partial\|\cdot - x\|(\bar{x}) = \{x^* \in \mathbb{R}^n : \|x^*\| = 1,\ \langle x^*, \bar{x} - x\rangle = \|\bar{x} - x\|\}. \tag{36}$$

In the following, we give the Lagrange multipliers characterizations of constrained best approximation under non-smooth Abadie's constraint qualification.

Theorem 4.2. *Let K be closed, given by* (4), *and let C be a non-empty closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$. Let $\bar{x} \in \tilde{K} := C \cap K$ and $x \in \mathbb{R}^n$. Assume that $\tilde{K}$ is closed and convex. If K is nearly convex at the point $\bar{x}$ and non-smooth Abadie's constraint qualification holds at $\bar{x}$, then the following assertions are equivalent.*
(i) $\bar{x} = P_{\tilde{K}}(x)$.
(ii) There exist $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S$ *with* $\lambda_j g_j(\bar{x}) = 0$ *and* $\eta_j \in \partial_T g_j(\bar{x})$ *such that*

$$\bar{x} = P_C(x - \sum_{j=1}^{m} \lambda_j \eta_j), \; j = 1, 2, \dots, m.$$

(iii) There exist $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S$ *with* $\lambda_j g_j(\bar{x}) = 0$ *and* $\eta_j \in \partial_T g_j(\bar{x})$ *such that*

$$0 \in \partial \| \cdot - x \| (\bar{x}) + (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \eta_j, \; j = 1, 2, \dots, m,$$

where we denote $\partial f(x_0)$ for the convex subdifferential of a convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at the point $x_0 \in \mathbb{R}^n$.

Proof: [*(i)* $\Longleftrightarrow$ *(ii)*]. Since, by the hypothesis, non-smooth Abadie's constraint qualification holds at $\bar{x}$ and K is nearly convex at the point $\bar{x}$, it follows from Corollary 3.1 that $\{C, K\}$ has the strong CHIP at $\bar{x}$. Therefore, the implication [*(i)* $\Longleftrightarrow$ *(ii)*] follows from Theorem 4.1.
[*(i)* $\Longrightarrow$ *(iii)*]. We may assume without loss of generality that $x \neq \bar{x}$. Suppose that *(i)* holds. Then, we have $\bar{x} = P_{\tilde{K}}(x)$. This together with Lemma 2.2 implies that $x - \bar{x} \in (\tilde{K} - \bar{x})^\circ$. But, in view of Theorem 3.1, one has

$$(\tilde{K} - \bar{x})^\circ = M(\bar{x}).$$

Hence, $x - \bar{x} \in M(\bar{x})$. Since $M(\bar{x})$ is a cone, we conclude that

$$\frac{x - \bar{x}}{\|\bar{x} - x\|} \in M(\bar{x}).$$

Therefore, it follows from (15) that there exist $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S$ with $\lambda_j g_j(\bar{x}) = 0$ and $\eta_j \in \partial_T g_j(\bar{x})$ such that

$$-u := \frac{x - \bar{x}}{\|\bar{x} - x\|} = \sum_{j=1}^{m} \lambda_j \eta_j, \; j = 1, 2, \dots, m, \tag{37}$$

where

$$u := \frac{\bar{x} - x}{\|\bar{x} - x\|}.$$

Then, $u \in \mathbb{R}^n$, $\|u\| = 1$ and

$$\langle u, \bar{x} - x \rangle = \|\bar{x} - x\|.$$

This together with (36) implies that

$$u \in \partial \| \cdot - x \| (\bar{x}). \tag{38}$$

Thus, it follows from (37) and (38) that

$$0 \in \partial \| \cdot - x \| (\bar{x}) + \sum_{j=1}^{m} \lambda_j \eta_j \text{ with } \lambda_j g_j(\bar{x}) = 0 \text{ and } \eta_j \in \partial_T g_j(\bar{x}). \tag{39}$$

On the other hand, since $0 \in (C - \bar{x})^\circ$, we have

$$\partial \| \cdot - x \| (\bar{x}) + \sum_{j=1}^{m} \lambda_j \eta_j \subseteq \partial \| \cdot - x \| (\bar{x}) + (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \eta_j.$$

Hence, in view of (39), one has

$$0 \in \partial \| \cdot - x \|(\bar{x}) + (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \eta_j \text{ with } \lambda_j g_j(\bar{x}) = 0 \text{ and } \eta_j \in \partial_T g_j(\bar{x}),$$

which implies that (*iii*) holds.
[(*iii*) $\Longrightarrow$ (*i*)]. Suppose that (*iii*) holds. Then there exist $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in S$ with $\lambda_j g_j(\bar{x}) = 0$ and $\eta_j \in \partial_T g_j(\bar{x})$ such that

$$0 \in \partial \| \cdot - x \|(\bar{x}) + (C - \bar{x})^\circ + \sum_{j=1}^{m} \lambda_j \eta_j, \ j = 1, 2, \ldots, m. \tag{40}$$

Now, let $y \in \tilde{K}$ be arbitrary. So, $y \in K$. Since K is nearly convex at the point $\bar{x}$, it follows from Definition 2.1 that there exists a sequence $\{\alpha_k\}_{k \geq 1} \subset \mathbb{R}_{++}$ with $\alpha_k \longrightarrow 0^+$ such that $\bar{x} + \alpha_k(y - \bar{x}) \in K$ for all sufficiently large $k \in \mathbb{N}$. So, by (4),

$$g_j(\bar{x} + \alpha_k(y - \bar{x})) \leq 0, \text{ for all sufficiently large } k \in \mathbb{N} \text{ and all } j = 1, 2, \cdots, m. \tag{41}$$

Since g_j $(j = 1, 2, \cdots, m)$ is tangentially convex at $\bar{x}$, it follows from (3), (40) and (41) with some $\nu \in (C - \bar{x})^\circ$ that

$$\begin{aligned}
\|\bar{x} - x\| - \|y - x\| &\leq \langle \sum_{j=1}^{m} \lambda_j \eta_j + \nu, y - \bar{x} \rangle = \langle \sum_{j=1}^{m} \lambda_j \eta_j, y - \bar{x} \rangle + \langle \nu, y - \bar{x} \rangle \\
&\leq \langle \sum_{j=1}^{m} \lambda_j \eta_j, y - \bar{x} \rangle \leq \sum_{j=1}^{m} \lambda_j g_j'(\bar{x}, y - \bar{x}) = \sum_{j \in I(\bar{x})} \lambda_j g_j'(\bar{x}, y - \bar{x}) \\
&= \sum_{j \in I(\bar{x})} \lambda_j \Big\{ \lim_{k \longrightarrow +\infty} \frac{g_j(\bar{x} + \alpha_k(y - \bar{x})) - g_j(\bar{x})}{\alpha_k} \Big\} \\
&= \sum_{j \in I(\bar{x})} \lambda_j \Big\{ \lim_{k \longrightarrow +\infty} \frac{g_j(\bar{x} + \alpha_k(y - \bar{x}))}{\alpha_k} \Big\} \leq 0, \ \forall \ y \in \tilde{K}.
\end{aligned} \tag{42}$$

Note that in the above we used the fact that $\lambda_j = 0$ for each $j \notin I(\bar{x})$, because $\lambda_j g_j(\bar{x}) = 0$ for all $j = 1, 2, \ldots, m$. Therefore, we conclude from (42) that $\|\bar{x} - x\| = \inf_{y \in \tilde{K}} \|y - x\| = d(x, \tilde{K})$, and so, $\bar{x} = P_{\tilde{K}}(x)$, i.e., (*i*) holds. ■

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