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Solvability of the System of Implicit Generalized Order Complementarity Problems

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Abstract. In this paper, we introduce the notion of exceptional family for the system of implicit generalized order complementarity problems in vector lattice. We present some alternative existence results of the solutions for the system of implicit generalized order complementarity problems via topological degree aspects. The new developments in this paper generalize and improve some known results in the literature.

1. Introduction

The problems of optimization, game theory, mechanics, engineering, etc., can be transformed and designed in the form of complementarity problems (CPs) [5–8], among which, order complementarity problems (OCPs) are the particular class of complementarity problems. Borwein and Dempster [3], and Isac and Kostreva [9] studied the linear type OCP and the generalized order complementarity problems (GOCPs), respectively. The reader can find various results related to OCP in [2, 8, 10]. In [11], Isac et al. introduced and studied the existence consequences of multi-valued generalized order complementarity problems (MGOCPs). Later, Huang and Fang [1], and Fang et al. [12] considered the system of MGOCPs. Applications of order complementarity problems and numerical results are discussed in [5, 13–16].

The CPs and variational inequality problems (VIPs) had been considered extensively via the approach of exceptional family of elements (EFE) [17–22, 26–30]. Smith [4] first introduced the concept of EFE, which is an efficient method for the solvability of CP and VIP. As a result of which, several kinds of development of EFE were studied for various types of CPs and VIPs in [17–22, 26–30]. The nonexistence of EFE shows the solvability of the CP and VIP. In 2010, Németh [23] gave an advanced circulation of EFE, i.e., the ordered exceptional family of elements (OEFE) to OCP, and proved an alternative existence theorem for OCP. Huang and Ma [31] also discussed the existence of solutions of OCP by using the concept of topological degree and proved that Theorem 8 of [31] is weaker than Theorem 3.1 [23].

Recently, Zhao et al. [24] considered the OEFE for the system of generalized order complementarity problems (SGOCPs), which are the generalizations of the general order complementarity problems [16, 23]. In [23] and [24], it has been shown that the nonexistence of OEFE implies the solution to OCP and SGOCP,

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respectively. Therefore, it is a highly challenging task to obtain the new significant conditions in the sense of the existence of solutions to CP and VIP.

Inspired by the above research, we propose the concept of OEFE for the system of implicit generalized order complementarity problems (SIGOCPs). We show that nonexistence of OEFE suffices the existence of solutions to the SIGOCP. Moreover, we also present two remarkable sufficient conditions for generating solutions to the SIGOCPs.

The following is a description of the subsequent sections. In section 2, the preliminaries and some fundamental concepts are discussed. Section 3 presents the concept of the OEFE for SIGOCP with existence results and new significant settings for the nonexistence of OEFE. Finally, section 4 concludes the findings.

2. Preliminaries

Let $(X, \|.\|)$ be a Banach space with $C \subset X$ be a closed, pointed convex cone. We say a relation $\leq$ on X is an order relation if $\leq$ is reflexive, antisymmetric as well as transitive. The cone $C \subset X$ can induce a natural relation $\leq$ on X, i.e, $y \leq z$ iff $z - y \in C$. So, the cone $C = \{y \in X : y \geq 0\}$, and the triplet $(X, \|.\|, C)$ is called an ordered Banach space (OBS). The relation is induced by a cone $C \subset X$ on X iff it is invariant under translation (i.e, if $a \leq b$, then $a + d \leq b + d \ \forall \ d \in X$), scaling (i.e, if $a \leq b$, then $\alpha a \leq \alpha b \ \forall \ \alpha > 0$) and continuity (i.e, if any two convergent sequence $\{a_n\}$ and $\{b_n\}$ with limit point a and b, respectively, managing $\{a_n\} \leq \{b_n\} \ \forall \ n \in \mathbb{N}$, then $a \leq b$).

Definition 2.1. *We say an OBS $(X, \|.\|, C)$ is a vector lattice if for every $z, x \in X$, the $z \wedge x = inf\,\{z, x\}$ (equivalently $z \vee x = sup\,\{z, x\}$) exists through the order induced by C. We denote it as $(X, \|.\|, C_{\leq})$.*

Unless otherwise specified, we regard $(X, \|.\|, C_{\leq})$ as the vector lattices with latticial cone $C \subset X$. We denote $z^+ = 0 \vee z = -0 \wedge (-z)$ and $\wedge_{i=1}^{m}\{z_r^i\} = z_r^1 \wedge z_r^2, ..., \wedge z_r^m$. The following are some useful properties of the notion z^+, $\vee$ and $\wedge$, which are used throughout this paper.

(i) $(z + x) \wedge (y + x) = x + z \wedge y$,

(ii) $(z + x) \vee (y + x) = x + z \vee y$,

(iii) $z \wedge x = x - (x - z)^+$.

Definition 2.2. *[23] A continuous mapping $f : S \subset X \to X$ is completely continuous, if for every bounded set $D \subset S$, the image set $f(D)$ is relatively compact.*

Remark 2.3. From Definition 2.2, it is easy to prove that addition of two completely continuous mappings is completely continuous and the composition of a continuous function with completely continuous mapping is also completely continuous.

Definition 2.4. *[24] Let E be a nonempty subset of the real Banach space X and $G : E \subset X \to X$. A point $z^* \subset E$ is called a fixed point of G iff $G(z^*) = z^*$.*

We briefly introduce some topological degree concepts for the implementation of our main result. Let V be an open bounded set of X. Denote $\overline{V}$ and ∂V as the closure and boundary of V. Denote $I : X \to X$ as the identity mapping. For any arbitrary $p \in V$ and a completely continuous mapping $f : \overline{V} \to X$ with $p \notin (I - f)(\partial V)$, we write $\deg(A, V, p)$ as the topological degree associated to $A = I - f$, V and p. Let us recall the Poincaré-Bohl and the Kronecker theorems (see, [30]).

Theorem 2.5. *(Poincaré-Bohl theorem [30]) Let $V \subset X$ be an open bounded set. Assume $H : \overline{V} \times [0, 1] \to X$ be a completely continuous mapping with $p \notin h(z, t) = z - H(z, t) \ \forall \ (z, t) \in \partial V \times [0, 1]$. Then, $deg(h(z, t), V, p)$ is constant for all $t \in [0, 1]$.*

Theorem 2.6. *(Kronecker theorem, [30]) Let $V \subset X$ be an open bounded set and $f : \overline{V} \to X$ be a completely continuous mapping. If $deg(I - f, V, p) \neq 0$, then $z - f(z) = p$ has at least a solution in V.*

Theorem 2.7. *([25]) Let $V \subset X$ be an open bounded set. Then, $deg(I, V, p) = 1 \;\forall\; p \in V$.*

Let $S \subset X$ be a nonempty closed and convex set. Suppose $f_1, f_2, ..., f_m : X \times X \to X$ are m ($m \in \mathbb{N}$) mappings of the form $f_i(y,z) = y - S_i(y,z)$, where S_i is a mapping from $X \times X$ to X for each i. We introduce the system of implicit generalized order complementarity problem, which is to find $(y_*, z_*) \in S \times S$ such that

$$\text{SIGOCP}(\{f_i\}_{i=1}^{n}, S) \begin{cases} \inf\{f_1(y_*, z_*), f_2(y_*, z_*), ..., f_m(y_*, z_*)\}=0, \\ \inf\{f_1(z_*, y_*), f_2(z_*, y_*), ..., f_m(z_*, y_*)\}=0. \end{cases} \tag{1}$$

The SIGOCP $(\{f_i\}_{i=1}^{n}, S)$ contains a large class of problems as special cases. So, it is very interesting to study the existence consequence of problem 1.

If $S_1(y,z) = 0$ for all $(y,z) \in X \times X$ then $f_1(y,z) = y$ and $f_1(z,y) = z$. So, the problem 1 reduces to system of generalized order complementarity problem, which is to find $(y_*, z_*) \in S \times S$ such that

$$\text{SGOCP}(\{f_i\}_{i=1}^{n}, S) \quad \begin{cases} \inf\{y_*, f_2(y_*, z_*), ..., f_m(y_*, z_*)\}=0, \\ \inf\{z_*, f_2(z_*, y_*), ..., f_m(z_*, y_*)\}=0. \end{cases} \tag{2}$$

The problem SGOCP was first studied by Zhao et al. in [24] via the notion of exceptional family by applying Leray-Schauder alternative theorem [16, 24]. The existence of the set valued version of problem 2 was investigated by Huang and Fang [1].

If $S_1(y,z) = 0$ for all $(y,z) \in X \times X$, and for $i \neq 1, S_i(y,z) = S_i(z)$ for all $(y,z) \in X \times X$, then problem 1 becomes the the following order complementarity problem, which is to find $(y_*, z_*) \in S \times S$ such that

$$\begin{cases} \inf\{y_*, y_* - S_2(z_*), ..., y_* - S_m(z_*)\}=0, \\ \inf\{z_*, z_* - S_2(y_*), ..., z_* - S_m(y_*)\}=0. \end{cases} \tag{3}$$

Also, some existence results related to the set-valued version of problem 3 has been discussed in [12].

Setting $S_i(y,z) = S_i(y) \;\forall\; y,z \in X$ for each i in problem 1, then SIGOCP $(\{f_i\}_{i=1}^{n}, S)$ takes the form of OCP [23], which is to find $y_* \in S$ such that

$$\inf\{f_1(y_*), f_2(y_*), ..., f_m(y_*)\} = 0. \tag{4}$$

Németh had studied the existence results for the OCP via OEFE [23]. OCP was also called the implicit order complementarity problem in [16]. Setting $S_i(y,z) = S_i(y) \;\forall\; y,z \in X$ for each $i \neq 1$ in problem 2, then the SGOCP reduces to GOCP, which was considered by Isac and Kostreva [9]. The GOCP is to find $y_* \in S$ such that

$$\inf\{y_*, f_2(y_*), ..., f_m(y_*)\} = 0. \tag{5}$$

From the above, one can clearly observe that the SIGOCP $(\{f_i\}_{i=1}^{n}, S)$ can control all the above problems.

3. Order Exceptional family and Existence Results

This section introduces the notion of OEFE for SIGOCP $(\{f_i\}_{i=1}^{m}, S)$.

Definition 3.1. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Let $f_1, f_2, ..., f_m : X \times X \to X$ are m mappings with the form $f_i(y,z) = y - S_i(y,z)$, where $S_i : X \times X \to X$ for each i. A sequence of elements $(y_r, z_r) \in S \times S$ is said to be an OEFE for SIGOCP $(\{f_i\}_{i=1}^{m}, S)$, if for each $r > 0$, $\exists$ a real number $\mu_r > 0$, such that the following conditions are satisfied:*

(i) $\|(y_r, z_r)\| \to \infty$ $(r \to \infty)$;

(ii) $\wedge_{i=1}^{m}\{u_r^i\} = 0$ *and* $\wedge_{i=1}^{m}\{v_r^i\} = 0$, *where* $u_r^i = \mu_r y_r + f_i(y_r, z_r)$ *and* $v_r^i = \mu_r z_r + f_i(z_r, y_r)$ *for each* i.

If $S_i(y,z) = S_i(y)\ \forall\ y,z \in X$, then, Definition 3.1 is reduced to Definition 3.1 in [23]. Now, we present an important alternative existence results for SIGOCP $(\{f_i\}_{i=1}^m, S)$.

Theorem 3.2. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings with the form $f_i(y,z) = y - S_i(y,z)$, where $S_i : X \times X \to X$ is completely continuous for each i satisfying $S_m(S \times S) + C \subset S$. Then, either SIGOCP $(\{f_i\}_{i=1}^m, S)$ has a solution or an OEFE.*

Proof. Consider the mapping $\Psi : X \times X \to X$, which is defined as:

$$\Psi(y,z) = y - f_1(y,z) \wedge f_2(y,z) \wedge ... \wedge f_m(y,z)\ \forall\ \ y,z \in X.$$

Define $G : X \times X \to X \times X$ as:

$$\begin{aligned} G(y,z) &= (\Psi(y,z), \Psi(z,y)), \\ &= (y - f_1(y,z) \wedge f_2(y,z) \wedge ... \wedge f_m(y,z),\ z - f_1(z,y) \wedge f_2(z,y) \wedge ... \wedge f_m(z,y)). \end{aligned} \tag{6}$$

From (6), $(y_*, z_*) \in S \times S$ is a solution of SIGOCP $(\{f_i\}_{i=1}^m, S)$ iff (y_*, z_*) is a fixed point of G.

Denote $\Psi_i(y,z) = y - f_1(y,z) \wedge f_2(y,z) \wedge ... \wedge f_i(y,z)\ \ \forall\ \ y,z \in X$ and $i = 1,2,...,m$. We prove that $\Psi_i(y,z)$ is completely continuous for each i and $\Psi(y,z) \subset S$.

As, $S_i : X \times X \to X$ is completely continuous mapping for each i. So, for $i = 1$, we get

$$\Psi_1(y,z) = y - f_1(y,z) = S_1(y,z),$$

which is completely continuous. Suppose $\Psi_i(y,z)$ is completely continuous mapping for $i = 1,2,...,m-1$.

$$\begin{aligned} \Psi_{i+1}(y,z) &= y - f_1(y,z) \wedge f_2(y,z) \wedge ... \wedge f_m(y,z), \\ &= y - (y - \Psi_i(y,z)) \wedge f_{i+1}(y,z), \\ &= y - (y - \Psi_i(y,z)) \wedge (x - S_{i+1}(y,z)), \\ &= y - [(y - S_{i+1}(y,z)) - (x - S_{i+1}(y,z) - x + \Psi_i(y,z))_+], \\ &= S_{i+1}(y,z) + (\Psi_i(y,z) - S_{i+1}(y,z))_+. \end{aligned} \tag{7}$$

From (7), $\Psi_i(y,z)$ is a completely continuous mapping for all $i = 1,2,...,m$.

One can observe that $\Psi(y,z) = \Psi_m(y,z)$. The complete continuity of $\Psi(z,y)$ and $\Psi(z,y) = \Psi_m(z,y)$ are also drawn analogously. Since, $S_m(S \times S) + C \subset S$, from (7), we get

$$\Psi(S \times S) \subset S.$$

The complete continuity of $\Psi(y,z) \subset S$ and $\Psi(z,y) \subset S$ suggest that $G(y,z) = (\Psi(y,z), \Psi(z,y))$ is also completely continuous and $G(S \times S) \subset S \times S$.

Now, define the mapping $H : (X \times X) \times [0,1] \to X \times X$ as following:

$$H((y,z),t) = (1-t)G(y,z). \tag{8}$$

Clearly, H is a completely continuous mapping.
From (8), we deduce

$$H((y,z),0) = G(y,z)$$

and

$$H((y,z),1) = \theta$$

$\forall\ x, y \in S$ (where θ is the zero vector of $X \times X$).
Consider the mapping $h : (S \times S) \times [0,1] \to X \times X$ defined as:

$$\begin{aligned} h((y,z),t) &= (y,z) - H((y,z),t), \\ &= t(y,z) + (1-t)((y,z) - G(y,z)). \end{aligned} \tag{9}$$

From (9), we have

$$h((y,z),0) = (y,z) - G(y,z) \text{ and } h((y,z),1) = (y,z). \tag{10}$$

So, $h : (S \times S) \times [0,1] \to X \times X$ is a homotopy between $I - G$ and the identity mapping ($I : X \times X \to X \times X$).
For each $r > 0$, consider the following sets:

$$S_r = \{(y,z) \in S \times S : \|(y,z)\| < r\}, \ \ \bar{S_r} = \{(y,z) \in S \times S : \|(y,z)\| \leq r\}$$

and

$$\partial S_r = \{(y,z) \in S \times S : \|(y,z)\| = r\}.$$

One can observe that the homotopy mapping h is a single valued mapping. So, the mapping h may or may not have a fixed point depending upon the mapping G, which may or may not have a fixed point.

For each $r > 0$, we have the following two cases.
Case-1
If

$$\begin{aligned} &\theta \notin h((y,z),t) \ \ \forall \ \ (y,z) \in \partial S_r \text{ and } t \in [0,1], \\ \text{i.e,}\ \ &\theta \notin t(y,z) + (1-t)((y,z) - G(y,z)). \end{aligned}$$

By Theorem 2.5, $\deg(h((y,z),t), S_r, \theta)$ is a constant integer. Hence, we achieve

$$\deg(h((y,z),0), S_r, \theta) = \deg(h((y,z),1), S_r, \theta),$$

and from (10),

$$\deg((y,z) - G(y,z), S_r, \theta) = \deg(I, S_r, \theta). \tag{11}$$

But, from Theorem 2.7, we obtain $\deg(I, S_r, \theta) = 1$. Therefore, from (11), we get

$$\deg((y,z) - G(y,z), S_r, \theta) = 1.$$

Thus, by Theorem 2.6, $\exists\ (y_*, z_*) \in S_r$ such that

$$\begin{aligned} &(y_*, z_*) = G(y_*, z_*), \\ &(y_*, z_*) = (x_* - f_1(y_*, z_*) \wedge f_2(y_*, z_*) \wedge ... \wedge f_m(y_*, z_*), y_* - f_1(z_*, y_*) \wedge f_2(z_*, y_*) \wedge ... \wedge f_m(z_*, y_*)). \end{aligned} \tag{12}$$

From (12), we get

$$f_1(y_*, z_*) \wedge f_2(y_*, z_*) \wedge ... \wedge f_m(y_*, z_*) = 0$$

and

$$f_1(z_*, y_*) \wedge f_2(z_*, y_*) \wedge ... \wedge f_m(z_*, y_*) = 0,$$

which concerns the solvability of SIGOCP ($\{f_i\}_{i=1}^m, S$).

Case-2

If for each $r > 0$, $\exists\ (y_r, z_r) \in \partial S_r$ and $t_r \in [0, 1]$ such that

$$h((y_r, z_r), t_r) = \theta,$$

i.e,

$$h((y_r, z_r), t_r) = t_r(y_r, z_r) + (1 - t_r)((y_r, z_r) - G(y_r, z_r)) = \theta. \tag{13}$$

Depending upon the scalar $t_r \in [0, 1]$, we consider the following three cases for the homotopy h.

If $t_r = 0$, from (13), we have $(y_r, z_r) - G(y_r, z_r) = \theta$, which concerns the solvability of SIGOCP $(\{f_i\}_{i=1}^m, S)$. If $t_r = 1$, from (13), we have $(y_r, z_r) = \theta$, which arises a contradiction to $\|(y_r, z_r)\| = r$. So, we exclude the consideration for $t_r = 1$.

Now, for the case $0 < t_r < 1$, we have

$$\begin{aligned} \theta &= t_r(y_r, z_r) + (1 - t_r)((y_r, z_r) - G(y_r, z_r)), \\ -\frac{t_r}{1 - t_r}(y_r, z_r) &= (y_r, z_r) - G(y_r, z_r), \\ -\frac{t_r}{1 - t_r}(y_r, z_r) &= (y_r, z_r) - (\Psi(y_r, z_r), \Psi(z_r, y_r)). \end{aligned} \tag{14}$$

Let $\mu_r = \frac{t_r}{1-t_r}$. So, clearly $\mu_r > 0$. From (14), we have

$$\begin{aligned} -(\mu_r y_r, \mu_r z_r) &= (y_r, z_r) - (\Psi(y_r, z_r), \Psi(z_r, y_r)), \\ -(\mu_r y_r, \mu_r z_r) &= (f_1(y_r, z_r) \wedge f_2(y_r, z_r) \wedge ... \wedge f_m(y_r, z_r), f_1(z_r, y_r) \wedge f_2(z_r, y_r) \wedge ... \wedge f_m(z_r, y_r)). \end{aligned}$$

Therefore, we obtain

$$-\mu_r y_r = \inf\{f_1(y_r, z_r), f_2(y_r, z_r), ..., f_m(y_r, z_r)\}, \tag{15}$$

$$-\mu_r z_r = \inf\{f_1(z_r, y_r), f_2(z_r, y_r), ..., f_m(z_r, y_r)\}. \tag{16}$$

Denote

$$u_r^i = \mu_r y_r + f_i(y_r, z_r), i = 1, 2, ..., m,$$

and

$$v_r^i = \mu_r z_r + f_i(z_r, y_r), i = 1, 2, ..., m.$$

Then,

$$\begin{aligned} u_r^1 \wedge u_r^2 \wedge ... \wedge u_r^m &= \mu_r y_r + f_1(y_r, z_r) \wedge \mu_r y_r + f_2(y_r, z_r) \wedge ... \wedge \mu_r y_r + f_m(y_r, z_r), \\ &= \mu_r y_r + f_1(y_r, z_r) \wedge f_2(y_r, z_r) \wedge ... \wedge f_m(y_r, z_r). \end{aligned} \tag{17}$$

From (15) and (17), we obtain $u_r^1 \wedge u_r^2 \wedge ... \wedge u_r^m = 0$.
Also,

$$\begin{aligned} v_r^1 \wedge v_r^2 \wedge ... \wedge v_r^m &= \mu_r z_r + f_1(z_r, y_r) \wedge \mu_r z_r + f_2(z_r, y_r) \wedge ... \wedge \mu_r z_r + f_m(z_r, y_r), \\ &= \mu_r z_r + f_1(z_r, y_r) \wedge f_2(z_r, y_r) \wedge ... \wedge f_m(z_r, y_r). \end{aligned} \tag{18}$$

Combining (16) and (18), we obtain $v_r^1 \wedge v_r^2 \wedge ... \wedge v_r^m = 0$.

Now, for the case $t_r \in (0, 1)$, we have $(y_r, z_r) \in \partial S_r$. So, for each r, we have $\|(y_r, z_r)\| = r$. Therefore, $\|(y_r, z_r)\| \to \infty$ $(r \to \infty)$. All the above calculations are well fitted to the Definition 3.1 for the case $t_r \in (0, 1)$. Hence, $\{(y_r, z_r)\}_{r>0}$ is an OEFE. □

Theorem 3.3. *Let the conditions of Theorem 3.2 be satisfied. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings with the form $f_i(y,z) = y - S_i(y,z)$, where $S_i : X \times X \to X$ is completely continuous for each i such that $S_m(S \times S) + C \subset S$. If the problem SIGOCP $(\{f_i\}_{i=1}^m, S)$ is without OEFE, then SIGOCP $(\{f_i\}_{i=1}^m, S)$ has a solution.*

From Theorem 3.3, it is clear that the nonexistence of OEFE is a sufficient condition for the solvability of SIGOCP $(\{f_i\}_{i=1}^m, S)$. Now, we present some remarkable sufficient conditions under which SIGOCP $(\{f_i\}_{i=1}^m, S)$ does not have OEFE.

Condition 3.4. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings with the form $f_i(y,z) = y - S_i(y,z)$. We say that the family of mappings $\{f_i\}_{i=1}^m$ satisfy the condition, if there is $\gamma > 0$ such that either one or both of the following conditions*

$$inf\{f_1(y,z), f_2(y,z), ..., f_m(y,z)\} \notin -S \setminus \{0\}, \tag{19}$$

$$inf\{f_1(z,y), f_2(z,y), ..., f_m(z,y)\} \notin -S \setminus \{0\} \tag{20}$$

are satisfied for all $(y,z) \in S \times S$ with $\|(y,z)\| > \gamma$.

Theorem 3.5. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings with the form $f_i(y,z) = y - S_i(y,z)$. If the family of mappings $\{f_i\}_{i=1}^m$ satisfy Condition 3.4. Then, SIGOCP $(\{f_i\}_{i=1}^m, S)$ is without OEFE.*

Proof. On the contrary, suppose that the family of mappings $\{f_i\}_{i=1}^m$ has an OEFE $\{(y_r, z_r)\}_{r>0}$ in the sense of Definition 3.1. So, there exists a sufficient large index $\tilde{r}$, such that $\|(y_r, z_r)\| > \gamma > 0$ for all $r > \tilde{r}$ with $u_r^1 \wedge u_r^2 \wedge ... \wedge u_r^m = 0$ and $v_r^1 \wedge v_r^2 \wedge ... \wedge v_r^m = 0$. Proceeding as in Theorem 3.2, we have

$$\begin{aligned} -\mu_r y_r &= f_1(y_r, z_r) \wedge f_2(y_r, z_r) \wedge ... \wedge f_m(y_r, z_r), \\ -\mu_r z_r &= f_1(z_r, y_r) \wedge f_2(z_r, y_r) \wedge ... \wedge f_m(z_r, y_r). \end{aligned} \tag{21}$$

For all $r > \tilde{r}$, if $y_r = 0$ (or respectively, $z_r = 0$) then (20) (or respectively, (19)) does not hold, which implies $f_1(z_r, y_r) \wedge f_2(z_r, y_r) \wedge ... \wedge f_m(z_r, y_r) \in -S \setminus \{0\}$, (or, respectively, $f_1(y_r, z_r) \wedge f_2(y_r, z_r) \wedge ... \wedge f_m(y_r, z_r) \in -S \setminus \{0\}$), which contradicts first part of Condition 3.4.

For all $r > \tilde{r}$, if $y_r, z_r \neq 0$, then from (21), we get

$$f_1(y_r, z_r) \wedge f_2(y_r, z_r) \wedge ... \wedge f_m(y_r, z_r) \in -S \setminus \{0\},$$

$$f_1(z_r, y_r) \wedge f_2(z_r, y_r) \wedge ... \wedge f_m(z_r, y_r) \in -S \setminus \{0\},$$

which contradicts second part of Condition 3.4. Thus, $\{f_i\}_{i=1}^m$ is without OEFE. □

Theorem 3.6. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings with the form $f_i(y,z) = y - S_i(y,z)$, where $S_i : X \times X \to X$ is completely continuous for each i such that $S_m(S \times S) + C \subset S$. If the family of mappings $\{f_i\}_{i=1}^m$ satisfy Condition 3.4, then SIGOCP $(\{f_i\}_{i=1}^m, S)$ has a solution.*

Condition 3.7. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings of the form $f_i(y,z) = y - S_i(y,z)$. We say that the family of mappings $\{f_i\}_{i=1}^m$ satisfy the condition, if there is non zero element $(\hat{x}, \hat{y}) \in X \times X$, such that either (19) (or (20)) or both (19) and (20) are satisfied $\forall$ $(y,z) \in S \times S$ with $\|(y,z)\| > \|(\hat{y}, \hat{z})\|$.*

Remark 3.8. Setting $\|(\hat{y}, \hat{z})\| = \gamma$, Condition 3.7 $\Rightarrow$ Condition 3.4, i.e, Condition 3.7 is stronger.

Similar to Theorems 3.6, we state the following result by using Condition 3.7 whose proof is omitted.

Theorem 3.9. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings with the form $f_i(y,z) = y - S_i(y,z)$, where $S_i : X \times X \to X$ is completely continuous for each i such that $S_m(S \times S) + C \subset S$. If the family of mappings $\{f_i\}_{i=1}^m$ satisfy Condition 3.7, then SIGOCP $(\{f_i\}_{i=1}^m, S)$ has a solution.*

Theorem 3.10. *([23]) Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \to X$ are given mappings of the form $f_i(y) = y - S_i(y)$, where $S_i : X \to X$ is completely continuous for each i such that $S_m(S) + C \subset S$. Then, either OCP $(\{f_i\}_{i=1}^m, S)$ has a solution or an OEFE.*

Proof. Setting $S_i(y,z) = S_i(y)\ \forall\ y,z \in X$ in Theorem 3.2, the results follows. □

Theorem 3.11. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings of the form $f_i(y,z) = y - S_i(y,z)$, with $S_1(y,z) = 0$ where $S_i : X \times X \to X$ is completely continuous for each $i \neq 1$, such that $S_m(S \times S) + C \subset S$. Then either SGOCP $(\{f_i\}_{i=1}^m, S)$ has a solution or an OEFE.*

Proof. Since $S_1(y,z) = 0\ \forall\ y,z \in X$. The assertion follows from Theorem 3.2. □

Remark 3.12. Theorem 3.11 is weaker than Theorem 3.1 [24], which was studied by Zhao et al. (our definition of OEFE is different from [24]). Moreover, in [24] Theorem 3.1 required the condition $S_i(S \times S) + C \subset S$ for each i=1,2,...,m. (in our case, it requires only for the index m, but not for all $i \neq m$). Secondly, Theorem 3.1 [24] is demanded $S_i(S \times S) \subset S$ for each i, which is being dropped in our Theorem 3.11.

Theorem 3.13. *Let S be a nonempty closed convex set of the vector lattice $(X, \|.\|, C_{\leq})$. Suppose that $f_1, f_2, ..., f_m : X \times X \to X$ are given mappings with the form $f_i(y,z) = y - S_i(z)$, where $S_i : X \to X$ is completely continuous for each i such that $S_m(S) + C \subset S$ satisfying Condition 3.4 (or 3.7). Then, the problem 3 has a solution.*

Proof. Since $S_i(y,z) = S_i(y)$ for all $y,z \in X$. So, $f_i(y,z) = y - S_i(z)$ and $f_i(z,y) = z - S_i(y)$ for all $y,z \in X$. Now, by Theorem 3.2, we conclude that $\{f_i\}_{i=1}^m$ has either a solution or an OEFE. The rest of the proofs are drawn immediately from Theorem 3.6 (or Theorem 3.9), hence, it is omitted. □

Using Theorem 3.3 and Condition 3.4, we present the following example in finite dimensional framework for SIGOCP $(\{f_i\}_{i=1}^m, S)$.

Example 3.14. Let $X = (\mathbb{R}^n, \|.\|_p)$, where $p \in [1, \infty)$. Then, $X \times X = (\mathbb{R}^n \times \mathbb{R}^n, \|(.,.)\|_p)$, where $\|(y,z)\|_p$ is defined as $(\|x\|_p^p + \|y\|_p^p)^{\frac{1}{p}}$. Assume $S = C = \mathbb{R}^n_+$. Let $L_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $M_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be arbitrary continuous mappings.

Further, let $\lim_{\|(y,z)\| \to \infty} L_{ij}(y,z) \geq \tau_{ij}$ and $\lim_{\|(y,z)\| \to \infty} M_{ij}(y,z) \geq \varrho_{ij}$, where τ_{ij} and ϱ_{ij} are non-negative constants. Define $f_1, f_2, ..., f_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as:

$$f_i(y,z) = (L_{i1}(y,z)\|(y,z)\|_p^{M_{i1}(y,z)} + y_1, ..., L_{in}(y,z)\|(y,z)\|_p^{M_{in}(y,z)} + y_n)$$

for $i = 1, 2, .., m-1$, and

$$f_m(y,z) = (c_1(y,z))|y_1|, ..., c_n(y,z))|y_n|),$$

where $c_i : \mathbb{R}^n \times \mathbb{R}^n \to [0,1]$ are arbitrary continuous mappings for $i = 1, 2, ..., n$.

It is easy to calculate S_i as:
$S_i = (-L_{i1}(y,z))\|(y,z)\|_p^{M_{i1}(y,z)}, ..., -L_{1n}(y,z))\|(y,z)\|_p^{M_{in}(y,z)})$ for $i = 1, 2, ..., m-1$ and $S_m = (y_1 - c_1(y,z))|y_1|, ..., y_n - c_n(y,z))|y_n|)$.

Clearly, for each i, S_i is completely continuous and the condition $S_m(\mathbb{R}^n_+ \times \mathbb{R}^n_+) + \mathbb{R}^n_+ \subset \mathbb{R}^n_+$ is also satisfied.

Now, when $\|(y,z)\|_p$ is sufficiently large, it is not difficult to see $f_i(S \times S) \in \mathbb{R}^n_+$ for all $i = \{1, 2, ..., m\}$. Therefore,

$$f_1(y,z) \wedge f_2(y,z)\wedge, ..., f_m(y,z) \geq 0.$$

As $\|(y,z)\|_p = \|(z,y)\|_p$, we can proceed as in above and release the following :

$$f_1(z,y) \wedge f_2(z,y)\wedge, ..., f_m(z,y) \geq 0,$$

which implies

$$f_1(y_r, z_r) \wedge f_2(y_r, z_r) \wedge ... \wedge f_m(y_r, z_r) \notin -S \setminus \{0\},$$

and

$$f_1(z_r, y_r) \wedge f_2(z_r, y_r) \wedge ... \wedge f_m(z_r, y_r) \notin -S \setminus \{0\}.$$

Thus, all the hypotheses of Theorem 3.6 are satisfied. Hence, SIGOCP $(\{f_i\}_{i=1}^m, \mathbb{R}^n_+)$ has a solution.

4. Conclusions

In our work, we extend the notion of OEFE for SIGOCP $(\{f_i\}_{i=1}^m, S)$ and prove that the nonexistence of OEFE suffices the solution to SIGOCP $(\{f_i\}_{i=1}^m, S)$. We establish several new sufficient conditions for the nonexistence of OEFE. Finally, we deduce that our results generalize the results of Németh [23] and Zhao et al. [24] in stronger sense. We require weaker conditions to prove the results of Zhao et al. [24]. We also formulate and connect the existence results of Fang et al. [12] via OEFE. It would be interesting to study the notion of OEFE for the system of the set-valued version of SIGOCP, which leaves future research avenues.

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