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Some Topology on Zero-Dimensional Subrings of Product of Rings

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Abstract. Let R be a ring and $\{R_i\}_{i\in I}$ a family of zero-dimensional rings. We define the Zariski topology on $\mathcal{Z}(R, \prod R_i)$ and study their basic properties. Moreover, we define a topology on $\mathcal{Z}(R, \prod R_i)$ by using ultrafilters; it is called the ultrafilter topology and we demonstrate that this topology is finer than the Zariski topology. We show that the ultrafilter limit point of a collections of subrings of $\mathcal{Z}(R, \prod R_i)$ is a zero-dimensional ring. Its relationship with $\mathcal{F} - \lim$ and the direct limit of a family of rings are studied.

1. Introduction

All rings considered in this paper are assumed to be commutative, and have identity element. Let S be a ring, we will denote by $\mathcal{Z}(S), \mathcal{V}(S), \mathcal{A}(S)$, respectively, the sets of zero-dimensional, von Neumann regular, artinian subrings of S. The study of zero-dimensionality in commutative rings has been widely treated in the literature with a purely algebraic approach, (see [6, 7]). Our purpose here is to give a new characterization of zero-dimensionality notion by using the ultrafilters topology and the $\mathcal{F}-$ limit, where $\mathcal{F}$ is an ultrafilter.

Let R be a subring of a ring S, we denote by $\mathcal{Z}(R, S)$ the set of intermediate zero-dimensional subrings of the pair (R, S). We define Zarisky topology on $\mathcal{Z}(R, S)$, more precisely on $\mathcal{Z}(R, \prod R_i)$, where $\{R_i\}_{i\in I}$ is a family of zero dimensional rings, whose open sets are of the form $\mathcal{Z}(R[x], \prod R_i)$ such that $x \in \prod R_i$. Moreover, we define a topology on $\mathcal{Z}(R, \prod R_i)$ by using ultrafilters it's called the ultrafilter topology and we demonstrate that this topology is finer than the Zariski topology. Based on the notion of the $\mathcal{F} - \lim$ one gives new rings can be expressible as direct union of artinian rings or zero dimensional rings with finite spectrum. In the second section, we define the Zariski topology on $\mathcal{Z}(R, \prod R_i)$ and study their basic properties, The ultrafilter topology is studied in the third section . In the fourth section we define the $\mathcal{F}$-limit of a collection of zero-dimensional rings and we give his relationship with the direct union of subrings of $\prod R_i$.

2. Preliminaries

In this paper, we focus on intermediate zero-dimensional rings of a pair of rings. En particularly, we characterize these families by using special topologies. Now, let I be a set and $\mathcal{F}$ be a collection of subsets of I, we define a ultrafilters topology on I when $\mathcal{F}$ is a collection of clopen sets. We begin by giving some notations. Let R be a subring of a ring S, we denote by $\mathcal{V}(S, R)$ and $\mathcal{A}(S, R)$, respectively, the set of von

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Neumann regular and artinian subrings of S that contain R. Thus, the following question arises naturally. Under what condition is a specified one of two sets nonempty?

Clearly $\mathcal{Z}(S,R)$ is nonempty if $\mathcal{V}(S,R)$ or $\mathcal{A}(S,R)$ is nonempty. On the other hand, suppose R is a subring of the ring S, the following conditions are equivalent:

1. $\mathcal{Z}(R,S) \neq \emptyset$.
2. The power of the ideal xS is idempotent for each x in R.
3. For each finitely generated ideal I, the set $\{Ann_R(I^j)_{j=1}^{\infty}\}$ stabilizes for some $m \in \mathbf{N}$.

For proof see [3, Proposition 1] and [6, Theorem 1.6]

Now, assume that $\mathcal{Z}(R,S) \neq \emptyset$, is that $\mathcal{Z}(R,S)$ closed under arbitrary intersection?

Theorem 2.1. ([6, Theorem 2.1]) *Suppose $\{R_a\}_{a\in A}$ is a nonempty family of zero-dimensional subrings of the ring S. Then $\bigcap_{a\in A} R_a$ is zero-dimensional subring of S*

Remark 2.2. Suppose R is subring of the ring S. If $\mathcal{Z}(R,S) \neq \emptyset$, then Theorem 2.1 shows that $\mathcal{Z}(R,S) \neq \emptyset$ has a unique minimal element. We denote this element by R^0, and call it the minimal zero-dimensional extension of R in S. Then for each x in R, assume that $x^{m(x)}S$ is idempotent, and let s_x be the pointwise inverse of $x^{m(x)}$ in S. By [6, Theorem 2.5] we have that $R^0 = R[\{s_x : x \in R\}]$.

We will work in at least ZFC, that is, Zermelo-Frankel set theory with the axiom of choice. We will in certain case use additional axioms. We recall that $\mathcal{F}$ is a filter on set I if it is a subset of the power set of I that satisfies the following conditions:

1. $\emptyset \notin \mathcal{F}$ and $I \in \mathcal{F}$;
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. If $A \in \mathcal{F}$ and $A \subset A' \subset I$, then $A' \in \mathcal{F}$.

A filter $\mathcal{F}$ on I is called an ultrafilter if $\mathcal{F}$ is maximal with respect to being a filter, or equivalently, if whenever $A \subset I$, then either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$. An ultrafilter $\mathcal{F}$ is called principal if there exists an element $i_0 \in I$ such that $\mathcal{F}$ consist of all subsets of I that contain i_0. Other ultrafilters are called non-principal. We denote the collection of all ultrafilters on a set I by $\beta(I)$.

Definition 2.3. Let R be a subring of the ring S and $S(R,S)$ be the set of all the rings between R and S. Let Y be a subset of $S(R,S)$ and $\mathcal{F}$ be an ultrafilter on Y. Set $Y_{\mathcal{F}} := \{a \in S : U_a \cap Y \in \mathcal{F}\}$. We call $Y_{\mathcal{F}}$ an ultrafilter limit point of Y with $U_a := \{C \in S(R,S) : a \in C\}$.

Lemma 2.4. *The set $Y_{\mathcal{F}}$ is a subring of S contains R.*

Proof. Let $x, y \in Y_{\mathcal{F}}$ then each of the sets U_{xy} and U_{x-y} contain $U_x \cap U_y \in \mathcal{F}$, from definition of an ultrafilter, we have $U_{xy}, U_{x-y} \in \mathcal{F}$, then $x - y, xy \in Y_{\mathcal{F}}$. Moreover, for each $a \in R$, $U_a = S(R,S) \in \mathcal{F}$, then $Y_{\mathcal{U}}$ contains R. Hence $Y_{\mathcal{F}}$ is a subring of S. □

A nonempty subset Y of $S(R,S)$ is an ultrafilter closed if, for any ultrafilter $\mathcal{U}$ on Y, we have $Y_{\mathcal{U}} \in Y$. Then the ultrafilter closed sets of $S(R,S)$ are closed sets for a topology, called the ultrafilter topology (see [2]).

Definition 2.5. Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. For each $Y \subseteq X$ and each ultrafilter $\mathcal{U}$ on Y, we define

$$Y_{(X,\mathcal{F})}(\mathcal{U}) := \{x \in X : [\forall F \in \mathcal{F}, x \in F \Leftrightarrow F \cap Y \in \mathcal{U}]\}.$$

We will denote the set $Y_{(X,\mathcal{F})}(\mathcal{U})$ simply by $Y(\mathcal{U})$, when no confusion can arise.

Lemma 2.6. ([1, Lemma 2.5]) *Let X be a set, $\mathcal{F}$ be a given nonempty collection of subsets of X and $Y \subseteq Z \subseteq X$. Let $\mathcal{U}$ be an ultrafilter on Y , $T \in \mathcal{U}$ and let $\mathcal{U}_T$ and $\mathcal{U}^Z$,respectively, the ultrafilter defined by*

$$\mathcal{U}_T := \{U \cap T : U \in \mathcal{U}\} \quad \mathcal{U}^Z := \{Z' \subseteq Z : Z' \cap Y \in \mathcal{U}\}.$$

Then we have

$$Y(\mathcal{U}) = T(\mathcal{U}_T) = Z(\mathcal{U}^Z).$$

Remark 2.7. Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X that is closed under complements. Then, for any subset Y of X and any ultrafilter $\mathcal{U}$ on Y, we have

$$Y_{(X,\mathcal{F})}(\mathcal{U}) = \bigcap \{F \in \mathcal{F} : F \cap Y \in \mathcal{U}\}.$$

Definition 2.8. Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then, we say that a subset Y of X is $\mathcal{F}-$stable under ultrafilter if $Y(\mathcal{U}) \subseteq Y$, for each ultrafilter $\mathcal{U}$ on Y.

Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then the family of all the subsets of X that are $\mathcal{F}-$stable under ultrafilter is the collection of the closed sets form a topology on X. We will call it the $\mathcal{F}-$ultarfilter topology on X, and denote by $X^{\mathcal{F}-ultra}$ the set X endowed with the $\mathcal{F}-$ultrafilter topology.

Proposition 2.9. ([1, Proposition 2.13]) *Let X be a set, $\mathcal{F}$ be a nonempty collection of subsets of X. Then, for each subspace Y of $X^{\mathcal{F}-ultra}$, we have*

$$Ad(Y) = \bigcup \{Y(\mathcal{U}) : \mathcal{U} \in \beta(Y)\}.$$

Remark 2.10. If $\mathcal{F} \subseteq C$ are collections of subsets of X, then the $C-$ultrafilter topology is finer than or equal to the $\mathcal{F}-$ultrafilter topology. In fact, for each subset Y of X and each ultrafilter $\mathcal{U}$ on Y, we have $Y_{\mathcal{F}}(\mathcal{U}) \subseteq Y_C(\mathcal{U})$.

In the following example we give some relation between the $\mathcal{F}-$ultrafilter topology and ultrafilter topology for particular cases.

Example 2.11. 1. Let A be a ring and $\mathcal{P}$ be the collection of all the principal open subsets of $X := Spec(A)$. Then, the $\mathcal{P}-$ultrafilter topology of X is equal to the ultrafilter topology.

2. Let K be a field, A be a subring of K and C be the natural basis of open sets for the Zariski topology on the spectral space $Zar(K|A)$ of all the valuation domains of a field K containing a fixed subring A of K. Then, the $C-$ultrafilter topology is equal to to the ultrafilter topology on $Zar(K|A)$.

Now, we are interested in the topological structure on $\mathcal{Z}(R, \prod R_i)$. Let R be a ring and $\{R_i\}_{i\in I}$ be a family of zero-dimensional rings such that R is imbeddable into $\prod R_i$. The set $\mathcal{Z}(R, \prod R_i)$ endowed with a topological structure defined by taking, as a basis for the open sets, the subsets

$$B_S := \{T \in \mathcal{Z}(R, \prod R_i) \backslash S \subseteq T\}.$$

For S varying in $B_{fin}(\prod R_i)$. This topology is called the Zariski topology on $\mathcal{Z}(R, \prod R_i)$.

Remark 2.12. If $S := \{x_1, x_2, ..., x_n\}$ with $x_j \in \prod R_i$ for each $j \in \{1, ..., n\}$, then

$$B_S := \mathcal{Z}(R[x_1, x_2, ..., x_n], \prod R_i).$$

Therefore the collection of subsets $\mathcal{B} := \{\mathcal{Z}(R[x], \prod R_i) : x \in \prod R_i\}$ is a base for the Zariski topology on $\mathcal{Z}(R, \prod R_i)$.

As a simple consequence of the previous remark, if $\mathcal{Z}(R, \prod R_i) \neq \emptyset$, then for each $x \in R$, there exists $m(x) \in \mathbf{N}$ such that $x^{m(x)} \prod R_i$ is idempotent, let s_x be the pointwise inverse of $x^{m(x)}$ in $\prod R_i$, then $R^0 = R[\{s_x : x \in R\}]$. with R^0 is a unique minimal zero-dimensional in $\mathcal{Z}(R, \prod R_i)$.

Moreover, the collection of subsets $\mathcal{B} := \{\mathcal{Z}(R[x], \prod R_i) : x \in \prod R_i\}$ is a base for the Zariski topology, as $R[\{s_x : x \in R\}] \in \mathcal{Z}(R, \prod R_i)$, then there is an element $t \in \prod R_i$ such that $R[\{s_x : x \in R\}] \in \mathcal{Z}(R[t], \prod R_i)$,then $R[t] \subseteq R[\{s_x : x \in R\}]$, and as $R[\{s_x : x \in R\}]$ is an unique minimal element, then $R[t] = R[\{s_x : x \in R\}]$ or $dimR[t] \neq 0$.

3. The Ultrafilter Topology on $\mathcal{Z}(R, \prod R_i)$

It is worth reminding that not all rings admit a zero-dimensional subring. Particularly, infinite product of rings. Now, let R be a ring and $\{R_i\}_{i\in I}$ a family of zero-dimensional rings such that R is imbeddable into $\prod R_i$. The main goal is to study the behavior of $\mathcal{Z}(R, \prod R_i)$ with respect to the ultrafilter topology and will compare it with the Zariski topology.

We start with following properties.

Theorem 3.1. *Let R be a ring and $\{R_i\}_{i\in I}$ a family of zero-dimensional rings such that R is imbeddable into $\prod R_i$ with $\mathcal{Z}(R, \prod R_i) \neq \emptyset$, if Y is a nonempty subset of $\mathcal{Z}(R, \prod R_i)$ and $\mathcal{U}$ is an ultrafilter on Y, then:*

1. $R_{\mathcal{U}} := \{x \in \prod R_i \ / \ B_{\{x\}} \bigcap Y \in \mathcal{U}\} \ \in \ \mathcal{Z}(R, \prod R_i)$.
2. *The collection of all subsets Y of $\mathcal{Z}(R, \prod R_i)$ stable for ultrafilters (i.e for each $\mathcal{U} \in \beta(Y)$, $R_{\mathcal{U}} \in Y$) is the family of closed sets for a topology on $\mathcal{Z}(R, \prod R_i)$ called the ultrafilter topology of $Z(R, \prod R_i)$.*

Proof. 1. Let $\mathcal{C} := \{B_S \ : \ S \in B_{fin}(\prod R_i)\}$ be the natural basis of open sets of the Zariski topology of $\mathcal{Z}(R, \prod R_i)$. If Y is a subset of $\mathcal{Z}(R, \prod R_i)$ and $\mathcal{U}$ is an ultrafilter on Y, we have:

$$x \in Y_{\mathcal{C}}(\mathcal{U}) \Leftrightarrow [\forall B_S \in \mathcal{C}, \, x \in B_S \Leftrightarrow B_S \cap Y \in \mathcal{U}]$$

$$\Leftrightarrow B_{\{x\}} \cap Y \in \mathcal{U}$$

$$\Leftrightarrow x \in R_{\mathcal{U}}.$$

Then $Y_{\mathcal{C}}(\mathcal{U}) = \{R_{\mathcal{U}}\}$, as $\mathcal{C}$ is closed under complements. According to Remark 2.7, we have that $R_{\mathcal{U}} = \bigcap\{F \in \mathcal{C} \ : \ F \cap Y \in \mathcal{U}\}$, then $R_{\mathcal{U}}$ is a zero-dimensional ring by Theorem 2.1. On the other hand, $R \subseteq R_{\mathcal{U}}$ because $B_{\{x\}} = \mathcal{Z}(R, \prod R_i)$ for each $x \in R$.

2. Suppose that A, B are ultrafilter closed of $Z(R, \prod R_i)$ (i.e closed set for an ultrafilter topology of $\mathcal{Z}(R, \prod R_i)$), and $\mathcal{U}$ be an ultrafilter on $Y = A \cup B$. Take into account the properties of ultrafilters, we can assume that $A \in \mathcal{U}$. According to proof of (1) and Definition 2.8 , A is $\mathcal{C}-$stable with $\mathcal{C} := \{B_S \ : \ S \in B_{fin}(\prod R_i)\}$, by Lemma 2.6, we have $\{R_{\mathcal{U}}\} = Y_{\mathcal{C}}(\mathcal{U}) = A_{\mathcal{C}}(\mathcal{U}_A) \subseteq A \subseteq Y$, then Y is ultrafilter closed.

Now, let $\mathcal{C}$ be a collection of ultrafilter closed on $\mathcal{Z}(R, \prod R_i)$. Let $\mathcal{U}$ be an ultrafilter on $X = \bigcap \mathcal{C} := \{C \ / \ C \in \mathcal{C}\}$. By Lemma 2.6, for each $C \in \mathcal{C}$, we have $C(\mathcal{U}^C) = X(\mathcal{U})$, and thus $X(\mathcal{U}) \subseteq X$. Then X is ultrafilter closed. □

Theorem 3.2. *Let R be a ring and $\{R_i\}_{i\in I}$ a family of zero-dimensional rings such that R is imbeddable into $\prod R_i$ with $\mathcal{Z}(R, \prod R_i) \neq \emptyset$.*

1. *The ultrafilter topology is finer than the Zariski topology on $\mathcal{Z}(R, \prod R_i)$.*
2. *The basic open sets of the Zariski topology on $\mathcal{Z}(R, \prod R_i)$ are both open and closed by the ultrafilter topology.*

Proof. 1. Since $\mathcal{C} := \{B_S \ : \ S \in B_{fin}(\prod R_i)\}$ is a natural basis of open sets on the Zariski topology of $\mathcal{Z}(R, \prod R_i)$, it is enough to prove that $O := \mathcal{Z}(R, \prod R_i) \backslash B_S$ is stable for ultrafilter. Assume, by contradiction, that there exists an ultrafilter $\mathcal{U}$ on O such that $R_{\mathcal{U}} \notin O$. It follows that $S \subseteq R_{\mathcal{U}}$, and then $B_x \cap O \in \mathcal{U}$, for every $x \in S$. Then $B_S \cap O \in \mathcal{U}$, because S is finite. This is a contradiction by the definition of O.

2. Direct consequence of the Theorem 3.1 and the Remark 2.12. □

Remark 3.3. Let R be a ring and $\{R_i\}_{i\in I}$ a family of zero-dimensional rings such that R is imbeddable into $\prod R_i$. According to Remark 2.7 and [6] the set $\mathcal{A}(R, \Pi R_i)$ is not $\mathcal{C}-$stable, where $\mathcal{C} := \{B_S \ : \ S \in B_{fin}(\prod R_i)\}$ is the basis of open sets of the Zariski topology on $\mathcal{Z}(R, \prod R_i)$.

4. The $\mathcal{F}-\lim$ of a Collection of Zero-Dimensional Rings

Let R be a subring of a ring S. The first goal of this section is to define the $\mathcal{F}-\lim$ of the set $\mathcal{Z}(R,S)$. Then we give a characterization of $\mathcal{Z}(R,S)$ by using the $\mathcal{F}-\lim$. Thereby, its relationship with ultrafilter limit and the direct limit of a family of rings.

Definition 4.1. Let A be a set, $S(A)$ be the set of all subsets of A and let I be an infinite set. Let $\{S_i\}_{i\in I}$ be a family of $S(A)$, and let $\mathcal{F}$ be an ultrafilter on I, then we define the $\mathcal{F}-\lim$ of $\{S_i\}_{i\in I}$ by:

$$\mathcal{F}-\lim_{i\in I} S_i := \{a \in A : \{i \in I \,:\, a \in S_i\} \in \mathcal{F}\}.$$

We note that the set $\mathcal{F}-\lim_{i\in I} S_i$ is a subset of A and we have that:

$$\mathcal{F}-\lim_{i\in I} S_i = \bigcup_{X\in\mathcal{F}}(\bigcap_{i\in X} S_i).$$

Proposition 4.2. *Let R is subring of the ring S such that $\mathcal{Z}(R,S) \neq \emptyset$. Let $\{R_i\}_{i\in I} \in \mathcal{Z}(R,S)$, and $\mathcal{F}$ is an ultrafilter on I. Then the ring $\mathcal{F}-\lim_{i\in I} R_i$ is a direct union of zero-dimensional subrings of S.*

Proof. By Definition 4.1, we have that $\mathcal{F}-\lim_{i\in I} R_i = \bigcup_{A\in\mathcal{F}}(\bigcap_{i\in A} R_i)$, and according to Theorem 2.1 $\bigcap_{i\in A} R_i$ is a zero-dimensional ring for each A in $\mathcal{F}$. On the other hand, if $A \in \mathcal{F}$ and $A \subset A' \subset I$, then $A' \in \mathcal{F}$. Then the union is direct. □

Proposition 4.3. *Let $\{R_i\}_{i\in I}$ be a nonempty family of zero-dimensional subrings of a ring S and $\mathcal{F}$ be an ultrafilter on I. Let $S_X = \cap_{i\in X} R_i$ for each $X \in \mathcal{F}$, if for each X some R_i is an artinian reduced ring, $\mathcal{F}-\lim_{i\in I} R_i$ is a direct union of artinian rings.*

Proof. Let I be set and let $\mathcal{F}$ an ultrafilter on I. For each nonempty family $\{R_i\}_{i\in I}$ of zero-dimensional subrings of a ring S, let $S_X = \cap_{i\in X} R_i$ for each $X \in \mathcal{F}$. Assuming that some R_x in $\{R_i\}_{i\in X}$ is an artinian reduced ring then S_X is a zero-dimensional sub-ring of artinian reduced ring (because $S_X = \cap_{i\in X} R_i \subseteq R_x$). That means, S_X is a zero-dimensional reduced ring with only finitely many idempotents, therefore is an artinian ring. Then similar proof of the Lemma 4.2 may show that $\mathcal{F}-\lim_{i\in I} R_i$ is a direct union of artinian rings. □

Lemma 4.4. *Let $X \subseteq \mathcal{Z}(R,\Pi R_i)$ and $\mathcal{U}$ be an ultrafilter on X, then for each subset $\{S_j \,:\, j \in J\} \subseteq X$ and each ultrafilter $\mathcal{F}$ on J. We have $\mathcal{F}-\lim_{j\in J} S_i \in \mathcal{Z}(R,\Pi R_i)$, and $S_{\mathcal{U}} = \mathcal{F}-\lim_{j\in J} S_j$ with $S_{\mathcal{U}}$ is the ultrafilter limit of X.*

Proof. Let $\sigma : J \to X$ be a bijection, and let $\mathcal{F} = \{\sigma^{-1}(F) \,:\, F \in \mathcal{U}\}$. Then $\mathcal{F}$ is an ultrafilter on J. For each $j \in J$, we put $\sigma(j) = S_j$. Then,

$$a \in S_{\mathcal{U}} \Leftrightarrow B_{\{a\}} \cap X \in \mathcal{U} \Leftrightarrow \{j \in J \,:\, a \in S_j\} \in \mathcal{F} \Leftrightarrow a \in \mathcal{F}-\lim_{j\in J} S_j.$$

Therefore, $S_{\mathcal{U}} = \mathcal{F}-\lim_{j\in J} S_j$. On other hand, according to Theorem 3.1 $\mathcal{F}-\lim_{j\in J} S_i \in \mathcal{Z}(R,\Pi R_i)$. □

Definition 4.5. Let I be an arbitrary set and let $\mathcal{F}$ be any ultrafilter on I. Let $x_i \in X$ for $i \in I$, we say that x is an ultralimit of x_i with respect to $\mathcal{F}$ if only if for every open set O of X with $x \in X$ $\{i \in I\ x_i \in O\} \in \mathcal{F}$ when denoted by $lim_{\mathcal{F}}\ x_i = x$.

We say that X is $\mathcal{F}-$complete if and only if for all choices of $x_i \in X$ there is a $x \in X$ such that $lim_{\mathcal{F}}\ x_i = x$. We recall that X is ultracomplete if for every set I, all sequences $x_i \in X$ for $i \in I$ and every ultrafilter $\mathcal{F}$ on I, there is an $x \in X$ with $lim_{\mathcal{F}}\ x_i = x$, this is the equivalent of saying that X is $\mathcal{F}-$complete for every I and every ultrafilter $\mathcal{F}$ on I.

Proposition 4.6. *Let $\{S_i\}_{i\in I} \in S(A)$ for each $i \in I$ and $\mathcal{F}$ be an ultrafilter on I. We have $lim_{\mathcal{F}}\ S_i = \mathcal{F}-\lim_{i\in I} S_i$.*

Proof. Let $S_i \subseteq A$ for some $i \in I$, we claim that $lim_{\mathcal{F}}\ S_i = S$ where $S = \mathcal{F} - \lim_{i\in I} S_i$. Suppose that O is an open set in $S(A)$ such that $S \in O$. On the other hand let $B_{F,G} = \{S \subseteq A\ :\ F \subseteq S \text{ and } S \cap G = \emptyset\}$ where F, G are finite subsets of A. Each $B_{F,G}$ is a basic open set of $S(A)$ such that $S \in B_{F,G} \subseteq O$. But this asserts that $F \subseteq S$. Let $g \in G$, $g \notin S$. So $\{i\ :\ F \subseteq S_i\} \in \mathcal{F}$ and also for each $g \in G$, $\{i\ :\ g \in S_i\} \notin \mathcal{F}$. It follows that for each $g \in G$, $\{i\ :\ g \notin S_i\} \in \mathcal{F}$. Thus , $\{i\ :\ F \subseteq S_i \text{ and for each } g \in G, g \notin S_i\} = \{i\ :\ S_i \in B_{F,G}\} \in \mathcal{F}$. Thus the claim. □

Proposition 4.7. ([8, Proposition 1.9]) *X is Hausdorff if and only if for all I, and all ultrafilters $\mathcal{F}$ on I with all sequences $x_i \in X$ for $i \in I$ and if $\lim_{\mathcal{F}}\ x_i$ exists, then this limit is unique.*

Before studying the relations between the notion of $\mathcal{F} - \lim$ and the direct union of the rings, give us a topological property of $\mathcal{Z}(R, \prod R_i)$ using the definition and the previous property.

Proposition 4.8. *Let R be a ring and $\{R_i\}_{i\in I}$ a family of zero-dimensional rings with $\mathcal{Z}(R, \prod R_i) \neq \emptyset$ then:*

1. *$\mathcal{Z}(R, \prod R_i)$ is Hausdorff.*
2. *$\mathcal{Z}(R, \prod R_i)$ is compact if and only if $R_{\mathcal{U}} \neq \emptyset$ for each ultrafilter $\mathcal{U}$ on $\mathcal{Z}(R, \prod R_i)$.*

Proof. 1. Let $X \subseteq \mathcal{Z}(R, \prod R_i)$ and $\mathcal{U}$ is an ultrafilter on X. Suppose that J is an indexed set, $\mathcal{F}$ is an ultrafilter on J, and $S_j \in X$ then by Lemma 4.4, $S_{\mathcal{U}} = \mathcal{F} - \lim_{j\in J} S_j$ with $S_{\mathcal{U}}$ is the ultrafilter limit of X, according to proof of Theorem 3.1 $\mathcal{F} - \lim_{j\in J} S_j$ is unique. On the other hand, by Proposition 4.6, Definition 4.5 and Proposition 4.7 we have that $\mathcal{Z}(R, \prod R_i)$ is a Hausdorff space.

2. Assume that $\mathcal{Z}(R, \prod R_i) \neq \emptyset$ then by [8, Theorem 1.6] we have that: $\mathcal{Z}(R, \prod R_i)$ is compact if and only if $\mathcal{Z}(R, \prod R_i)$ is ultracomplete. Moreover, by Definition 4.5, for every set J, all sequences $\{T_j\}_{j\in J} \in \mathcal{Z}(R, \prod R_i)$ for $j \in J$ there exists a $R_{\mathcal{U}} = \mathcal{F} - \lim_{j\in J} T_j$ for every ultrafilter $\mathcal{F}$ on J, and it is equivalent to $R_{\mathcal{U}} \neq \emptyset$ for each ultrafilter $\mathcal{U}$ in $\mathcal{Z}(R, \prod R_i)$. □

Corollary 4.9. *Let A be a nonempty subset of $\mathcal{A}(R, \Pi R_i)$ and $\mathcal{U}$ an ultrafilter in A, then $A_{\mathcal{U}}$ the ultrafilter limit point of A is a zero-dimensional ring.*

Proof. We know that every artinian ring is a zero-dimensional ring, then A is a nonempty subsets of $Z(R, \prod R_i)$, by Theorem 3.1, we have that $A_{\mathcal{U}}$ is a zero-dimensional ring. □

By Remark 3.3 the ring $A_{\mathcal{U}}$ is not necessary artinian.

Definition 4.10. Let R be a ring and S a ring containing R. An element $x \in S$ is said to be integral over R if there exists an integer n and elements $r_1, ..., r_n$ in R such that

$$x^n + r_1x^{n-1} + ... + r_{n-1}x + r^n = 0.$$

This equation is called an equation of integral dependence of x over R (of degree n). The set of all elements of S that are integral over R is called the integral closure of R in S. If every element of S is integral over R, we say that S is integral over R.

Lemma 4.11. *Let R be a noetherian ring and R_α be a family of zero-dimensional rings, and let $\mathcal{IZ}(R, \prod R_\alpha)$ be the set of all the rings $C \in \mathcal{Z}(A, \prod R_\alpha)$ such that C is integrally closed of R in $\prod R_\alpha$, then:*

1. *$\forall S_j \in \mathcal{IZ}(R, \prod R_\alpha)\ \exists T_i^j \in \mathcal{A}(R, \prod R_\alpha)$ such that*

$$S_j = \varinjlim_{i\in I} T_i^j.$$

2. *For each family $\{S_j\ :\ j \in J\} \subseteq \mathcal{IZ}(R, \prod R_\alpha)$ we have that $\{K_i\ :\ i \in I\} \subseteq \mathcal{A}(R, \prod R_\alpha)$ such that*

$$\mathcal{F} - \lim_{j\in J} S_j = \varinjlim_{i\in I} K_i.$$

Proof. 1. Let $\mathcal{IZ}(R, \prod R_\alpha)$ be the set of all the subrings $C \in \mathcal{Z}(R, \prod R_\alpha)$ such that C is integrally closed of A in $\prod R_\alpha$, and let $S_j \in \mathcal{IZ}(A, \prod R_\alpha)$, then by proof of [7, corollary 5.5] S_j is a direct union of artinian subrings, from where $\exists T_i^j \in \mathcal{A}(R, \prod R_\alpha)$ such that

$$S_j = \varinjlim_{i \in I} T_i^j.$$

with I is a direct set.

2. According to Lemma 4.4 and Theorem 3.1 and [1, Proposition 3.6] , we have that $\mathcal{F} - \lim_{j\in J} S_j$ is a zero-dimensional integral closed of R in $\prod R_\alpha$, then $\mathcal{F} - \lim_{j\in J} S_j \in \mathcal{IZ}(R, \prod R_\alpha)$ and by (1) we have that

$$\mathcal{F} - \lim_{j\in J} S_j = \varinjlim_{i \in I} K_i$$

with $\{K_i \ : \ i \in I\} \subseteq \mathcal{A}(A, \prod R_\alpha)$. □

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