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On Topology of Polynomial Type Sequences with Bounded Integer Coefficients

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Abstract. In this work, we examine the topology of special type of sequences in which each candidate is a polynomial with its coefficients are of integers taken from a discrete set. The paper develops a different perspective with regard to the recent progress in the theory of polynomials with bounded integer coefficients. In the context of discrete control systems, we apply the theory to one dimensional reachability problem of a convex polyhedra with finite number of controls acting on the system and prove that the reachable set of the discrete model is dense in the set of real numbers under the suitable finite control set.

1. Introduction

Much attention has recently been attracted to the topological structure of sets of polynomials with bounded integer coefficients with applications in analysis and theory of numbers [2, 12, 13]. Among these sets, the one that holds its importance in this context, is the following spectrum

$$X_m(\lambda) = \left\{c_0 + c_1\lambda + \cdots + c_k\lambda^k : c_i \in \{0, 1, \cdots, m\}, k \in \mathbb{N}\right\},$$

which has been introduced by Erdös et al in [10] for the case $1 < \lambda < 2$ and $m = 1$. The topology of the set $X_m(\lambda)$ was examined and results were obtained about the behaviour of differences between consecutive terms of the set for cases of all transcendental and Pisot numbers (see also [9]). Recall that a Pisot number is an algebraic integer which is greater than one and all of its conjugates have modulus less than one. There are huge literature for application of Pisot numbers in theory of numbers, we refer to [2, 13, 14, 16]. Recent progress was done in [12] where answers were given for perspectives and open problem raised by Erdös et al in [10]. In [12], the following set of polynomials was considered

$$Y_m(\lambda) = \left\{c_0 + c_1\lambda + \cdots + c_k\lambda^k : c_i \in \{0, \pm 1, \pm 2, \cdots, \pm m\}, k \in \mathbb{N}\right\},$$

for real number $\lambda > 1$, and m integer. It was shown that $Y_m(\lambda)$ is dense in $\mathbb{R}$ if and only if $\lambda < m + 1$ and λ is not a Pisot number. It should be noted that $Y_m(\lambda)$ is not dense in $\mathbb{R}$ whenever λ is a Pisot number(see

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[14]), or $\lambda \geq m+1$ (see [11]). In addition to that, $Y_m(\lambda)$ has a finite accumulation point in $\mathbb{R}$ if and only if $\lambda < m+1$ and λ is not a Pisot number (see [2]).

The present paper is devoted to the study of topological structure of the set $X_m(\lambda)$ for the case $m = \lfloor \lambda \rfloor$, i.e., integer part of λ and $\lambda > 1$. Here, we analyse the behaviour of differences of consecutive terms of the set and obtain related results in the general setting. Moreover, the work is extended to the new class of $X := X_{\lfloor \lambda \rfloor}(\lambda)$ as the set of pairwise combination of terms from the set $X_m(\lambda)$ with a transcendental number $\gamma \in (0,1)$ in the following sense

$$Y := \{\alpha_i + \gamma \alpha_j | \ \alpha_i, \alpha_j \in X \ \text{ for } i, j \in \mathbb{N}\}.$$

Here, we study the topology of the set Y and prove our main result that the gap between each consecutive terms of the set tends to zero. As an application of the main result, we concentrate on the controllability problem of a convex polyhedra rolling in $\mathbb{R}$. More precisely, the problem is a discrete model of the form

$$x_{n+1} = \lambda x_n + \eta, \quad x_0 = 0, \tag{1}$$

where $\lambda > 1$ and η is ranging from finite control set. Our motivation comes from here that each discrete points of the system can be represented as polynomial in λ with bounded coefficients emerged from the finite set. In literature, the problem is related to robotics and we refer to the papers [4, 6, 7] for analysing reachability property of the system under finite control set. Those works are answering the problem whether it is possible to design a finite control set in such a way that the reachable set of a convex polyhedra with those controls acting on the system is dense in the space. Several techniques applied to the papers [18], [26] to obtain denseness criteria of a vehicle's reachable set in some certain spaces and to the paper [20] for a model of a robot's finger in the framework of the theory of expansions in non-integer bases where the density of its reachability set was studied. In addition, an iterative algorithm was developed in [5] to interpolate graph signals from only a partial set of samples which results a better performance in computational efficiency. There are other estimates and iterations technique applied to different types of discrete systems which were considered in the papers [3, 8, 27]. Moreover, we refer to the paper [1] for finding sufficient conditions that assure existence and uniqueness of approximate fixed point of a mapping with best proximity point results.

In the monograph [25] and papers [22–24] consider different problems of optimal control theory with higher order linear/semilinear discrete and continuous systems. Necessary and sufficient optimality conditions are derived incorporating the Euler-Lagrange and Hamiltonian type inclusions/equations. Moreover, we refer to the books [15, 19] for several basic notions and results on topology, measure theory and functional analysis.

The paper is organized as follows.

In Section 2, we give preliminaries about some basic concepts, theorem and lemma. In this part, we mention about Kronecker's density theorem which simply asserts that given any irrational number ξ, the set of fractional part of $\{i\xi\}$ for $i \in \mathbb{Z}$ is dense in open interval $(0,1)$ (see [17]) and provide one application. Moreover, we consider certain subsets of X and Y, namely sets of all finite sum of even powers of λ terms in X and Y respectively. In addition, γ is defined as a transcendental number which is bounded by constants depending on λ for later purposes.

In Section 3, we provide the main results of the paper. Proof techniques applied in this section are adopted from the papers [9, 10] and without utilizing Pisot number analysis, topological properties of the sets X and Y are derived and at the end, it is concluded that differences of successive terms of Y approach to zero. The paper develops a different point of view in regard to the recent progress done in the work [12].

In Section 4, we conclude the paper by mentioning about some perspectives related to the research. Here, we consider a problem of controllability in robotics. In this part, we analyse the interaction in between displacements of a convex polyhedra by means of finite controls in one dimensional space. The movement of a convex polyhedra for each step can be described as the first order recurrence relation. The paper answers density property of the reachable set of the system for $\lambda > \mathcal{A} := \frac{1+\sqrt{5}}{2}$. The case $\lambda = 1$, is considered separately and proven that reachable set is dense in $\mathbb{R}$ as a consequence of Kronecker's theorem.

2. Preliminaries

In this section, we provide some definitions and principles together with theorems. Firstly, we start with definition of algebraic and transcendental number.

Definition 2.1. *An algebraic number is any real number which is a root of a non-zero polynomial in one variable with rational coefficients. If a number is not algebraic then it is called a transcendental number.*

Pigeonhole Principle: for $n+1$ different real numbers $x_0, x_1, \cdots, x_n$ in closed interval $[0,1]$. Then, one can say that there exists two numbers x_i, x_j with $i \neq j$ satisfying

$$|x_i - x_j| \leq \frac{1}{n}.$$

Now, we give Kronecker's Theorem which relies on estimating any real number by means of given irrational number (see [17]).

Theorem 2.2. *If ξ is irrational, then the set of points $(\{n\xi\} : n \in \mathbb{N})$ is dense in the interval $(0,1)$.*

As an application of Theorem 2.2, we provide the following useful result.

Lemma 2.3. *Let ξ and η be rationally independent positive real numbers, i.e., $\frac{\xi}{\eta} \in \mathbb{I}$. Then, the set*

$$K := \{\, j\eta + i\xi \mid i, j \in \mathbb{Z}\},$$

is dense in $\mathbb{R}$.

The proof of Lemma 2.3 is straightforward. Namely, for given ξ and η be rationally independent numbers, we know that $\frac{\xi}{\eta} \in \mathbb{I}$. Then for any $\epsilon > 0$, we have from Theorem 2.2 that for any real number $\tau \in \mathbb{R}$,

$$|\{\kappa\} - N - M\frac{\xi}{\eta}| < \frac{\epsilon}{\eta},$$

where $\kappa = \dfrac{\tau}{\eta}$ and for some $M, N \in \mathbb{Z}$. Hence, we conclude that

$$\left|\tau - \Big(\big(\lfloor\kappa\rfloor + N\big)\eta + M\xi\Big)\right| < \epsilon. \tag{2}$$

Observe that both $\lfloor\kappa\rfloor + N$ and M are integers in (2).

Let $\lambda > 1$ be a real number and for $n \in \mathbb{N}$, we consider the unique $(\lfloor\lambda\rfloor + 1)$ expansion of n as the following way

$$n = c_0 + c_1(\lfloor\lambda\rfloor + 1) + c_2(\lfloor\lambda\rfloor + 1)^2 + \cdots + c_k(\lfloor\lambda\rfloor + 1)^k,$$

where $c_i \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$, and set the sequence of the following form

$$\alpha_n = c_0 + c_1\lambda + \cdots + c_k\lambda^k. \tag{3}$$

Define the set X as all collection of (3) and let the sequence $\{\beta_i\}$ be its increasing rearrangement. Let $\gamma \in (0,1)$ be a transcendental number such that $\dfrac{\lambda}{\gamma} \in \mathbb{I}$ and satisfying the following inequality

$$\max\Big\{\frac{1}{\lambda^2}, \frac{\lambda - 1}{\lambda\lfloor\lambda\rfloor}\Big\} < \gamma < \frac{1}{\lambda}. \tag{4}$$

Define X_e as the set of all finite sum of even powers of λ terms in X. More precisely,

$$X_e := \Big\{ \sum_{i=0}^{k} c_i \lambda^{2i} \mid k \in \mathbb{N}, c_i \in \{0, 1, \cdots, \lfloor \lambda \rfloor\} \Big\}. \tag{5}$$

Now, define the following set

$$Y_e := \{\alpha_i + \gamma \alpha_j | \ \alpha_i, \alpha_j \in X_e \text{ for } \ i, j \in \mathbb{N}\}. \tag{6}$$

Let $\{\zeta_i\}$ and $\{\tau_i\}$ be the increasing rearrangement of Y_e and Y respectively, i.e.,

$$\zeta_0 < \zeta_1 < \zeta_2 \cdots \quad \text{and} \quad \tau_0 < \tau_1 < \tau_2 \cdots$$

Observe that $\zeta_0 = \tau_0 = 0$, $\zeta_1 = \tau_1 = \gamma$ and $\zeta_2 = \tau_2 = 1$. Moreover, define the following limits

$$\mathcal{I}(\lambda) = \liminf(\zeta_{i+1} - \zeta_i) \quad \text{and} \quad \mathcal{S}(\lambda) = \limsup(\tau_{i+1} - \tau_i). \tag{7}$$

Lemma 2.4. *For any $k \in \mathbb{N}$, the following inequalities*

$$\gamma \lambda^{2k+2} < 1 + \lfloor \lambda \rfloor \Big(1 + \gamma\Big) \sum_{i=0}^{k} \lambda^{2i}, \tag{8}$$

and

$$\lambda^{2k+2} < 1 + \gamma \lfloor \lambda \rfloor \lambda^{2k+2} + \lfloor \lambda \rfloor \Big(1 + \gamma\Big) \sum_{i=0}^{k} \lambda^{2i}, \tag{9}$$

hold.

Proof. Firstly, we prove the inequality (8). By using (4),

$$\begin{aligned} 1 + \lfloor \lambda \rfloor \Big(1 + \gamma\Big) \sum_{i=0}^{k} \lambda^{2i} &> 1 + \Big(\lfloor \lambda \rfloor + \frac{\lambda - 1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} \\ &= 1 + \Big(1 + \lfloor \lambda \rfloor - \frac{1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} \\ &> 1 + \Big(\lambda - \frac{1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} = 1 + \lambda^{2k+1} - \frac{1}{\lambda} > \gamma \lambda^{2k+2}. \end{aligned}$$

Namely, we obtain the first inequality. For the second one, we use (4), i.e.,

$$\begin{aligned} 1 + \gamma \lfloor \lambda \rfloor \lambda^{2k+2} + \lfloor \lambda \rfloor \Big(1 + \gamma\Big) \sum_{i=0}^{k} \lambda^{2i} &> 1 + \frac{\lambda - 1}{\lambda} \lambda^{2k+2} + \Big(\lfloor \lambda \rfloor + \frac{\lambda - 1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} \\ &> 1 + \lambda^{2k+2} - \lambda^{2k+1} + \Big(\lambda - \frac{1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} > \lambda^{2k+2}. \end{aligned}$$

Hence, we have the second inequality. □

Lemma 2.5. *For $\lambda > \mathcal{A}$, the following inequality*

$$\lambda^{2n+2} > \lfloor\lambda\rfloor \sum_{i=0}^{n} \lambda^{2i},$$

holds for all $n \in \mathbb{N}$.

Proof. Since $\lambda > \mathcal{A}$, we have that $\lambda^2 - \lambda - 1 > 0$. Here, we would prove that for any $n \in \mathbb{N}$,

$$\lambda^{2n+1} > \sum_{i=0}^{n} \lambda^{2i}, \tag{10}$$

holds by induction. At first,

$$\lambda^2 - \lambda - 1 > 0 \iff \lambda^3 - \lambda^2 - \lambda > 0 \implies \lambda^3 - \lambda^2 - 1 > 0.$$

Hence, we have that $\lambda^3 > \lambda^2 + 1$, i.e., it is true for $n = 1$. Assume that (10) is true for n. Here, we observe that

$$1 + \lambda^2 + \cdots + \lambda^{2n} + \lambda^{2n+2} < \lambda^{2n+1} + \lambda^{2n+2} < \lambda^{2n+3},$$

which implies the case for $n + 1$. Therefore, we have (10) by induction.

As a result,

$$\lambda^{2n+1} > \sum_{i=0}^{n} \lambda^{2i} \implies \lambda^{2n+2} > \lfloor\lambda\rfloor \sum_{i=0}^{n} \lambda^{2i}.$$

□

3. Main Results

Theorem 3.1. *For each $n \in \mathbb{N}$, the following inequality*

$$\beta_{n+1} - \beta_n \leq 1,$$

holds.

Proof. To prove the theorem, we use induction by n. Here, for case $n = 1$, recall that $\beta_0 = 0$ and $\beta_1 = 1$, i.e., $\beta_1 - \beta_0 = 1$ which is true.

For induction hypothesis, assume that it is true for case n, i.e., $\beta_{i+1} - \beta_i \leq 1$ for $i = 1, 2, \cdots, n$. We prove that it is true for the case $n + 1$, i.e., $\beta_{n+2} - \beta_{n+1} \leq 1$.

Let $\beta_{n+1} = \xi_0 + \xi_1\lambda + \cdots + \xi_s\lambda^s$, for $\xi_i \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$. Here, ξ_0 is either $\lfloor\lambda\rfloor$ or different from $\lfloor\lambda\rfloor$.

If we have the case $\xi_0 \neq \lfloor\lambda\rfloor$, then ξ_0 would be in $\{0, 1, \cdots, \lfloor\lambda\rfloor - 1\}$, i.e., $\beta_{n+1} + 1 \in X$. This implies that $\beta_{n+1} \leq \beta_{n+2} \leq \beta_{n+1} + 1$ which means $\beta_{n+2} - \beta_{n+1} \leq 1$.

Now, assume that $\xi_0 = \lfloor\lambda\rfloor$. Let k be the largest integer such that $\xi_0 = \xi_1 = \cdots = \xi_k = \lfloor\lambda\rfloor$. Then, $\xi_{k+1} \in \{0, 1, \cdots, \lfloor\lambda\rfloor - 1\}$, i.e.,

$$\beta_{n+1} = \lfloor\lambda\rfloor(1 + \lambda + \cdots + \lambda^k) + \xi_{k+1}\lambda^{k+1} + \sum_{i=k+2}^{s} \xi_i\lambda^i. \tag{11}$$

Notice from (11) that $\beta_{n+1} + \lambda^{k+1} \in X$. Hence, if one can have $c_0, c_1, \cdots, c_k \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$ such that

$$\lfloor\lambda\rfloor(1 + \lambda + \cdots + \lambda^k) < c_0 + c_1\lambda + \cdots + c_k\lambda^k + \lambda^{k+1} \leq 1 + \lfloor\lambda\rfloor(1 + \lambda + \cdots + \lambda^k), \tag{12}$$

holds, then one can have the following

$$\beta_{n+2} \le c_0 + c_1\lambda + \cdots + c_k\lambda^k + \lambda^{k+1} + \sum_{i=k+1}^{s} \xi_i \lambda^i \le 1 + \beta_{n+1},$$

which proves the case $n+1$. Therefore, it suffices to prove the existence of such $\{c_i\}_{i=0}^k$ satisfying (12).

Note firstly that $(\lfloor\lambda\rfloor + 1 - \lambda)(1 + \lambda + \cdots + \lambda^k) > 0$ which implies

$$\lambda^{k+1} < 1 + \lfloor\lambda\rfloor (1 + \lambda + \cdots + \lambda^k). \tag{13}$$

From (13), if we have $\lambda^{k+1} > \lfloor\lambda\rfloor (1 + \lambda + \cdots + \lambda^k)$, then we obtain (12) by simply choosing $c_0 = \cdots = c_k = 0$. Otherwise, assume that $\lambda^{k+1} \le \lfloor\lambda\rfloor (1 + \lambda + \cdots + \lambda^k)$. Let $\beta_\eta = \lfloor\lambda\rfloor (1 + \lambda + \cdots + \lambda^k)$ which is a term in X. Here, $\eta \le n+1$ and the next term would be of the form

$$\beta_{\eta+1} = c_0 + c_1\lambda + \cdots + c_k\lambda^k + \lambda^{k+1},$$

for some $c_0, c_1, \cdots, c_k \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$. Since there would be no term of X in between β_η and $\beta_{\eta+1}$, we conclude that

$$X \cap (\beta_\eta - \lambda^{k+1}, \beta_{\eta+1} - \lambda^{k+1}) = \emptyset. \tag{14}$$

Observe that $c_0 + c_1\lambda + \cdots + c_k\lambda^k \in X$ and $\beta_\eta - \lambda^{k+1} + 1 < \beta_\eta$. Therefore, by using the induction hypothesis together with (14), one can have the following

$$\beta_\eta - \lambda^{k+1} < c_0 + c_1\lambda + \cdots + c_k\lambda^k \le \beta_\eta - \lambda^{k+1} + 1, \tag{15}$$

which concludes the proof of (12). □

Theorem 3.2. *For each $n \in \mathbb{N}$, the following inequality*

$$\zeta_{n+1} - \zeta_n \le 1$$

holds.

Proof. We apply proof by induction. Here, for case $n = 1$ recall that $\zeta_0 = 0$ and $\zeta_1 = \gamma$, i.e., $\zeta_1 - \zeta_0 = \gamma < 1$ which is true.

For induction hypothesis, assume that it is true for case n, i.e., $\zeta_{i+1} - \zeta_i \le 1$ for $i = 1, 2, \cdots, n$. We prove that it is true for the case $n+1$, i.e., $\zeta_{n+2} - \zeta_{n+1} \le 1$. Let $\zeta_{n+1} = \xi_0 + \xi_1\lambda^2 + \cdots + \xi_s\lambda^{2s}$, where $\xi_i \in \{i + j\gamma \mid i, j \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\}$. As it is examined in Theorem 3.1, either $\xi_0 \ne (1+\gamma)\lfloor\lambda\rfloor$ or $\xi_0 = (1+\gamma)\lfloor\lambda\rfloor$.

For the first case, one can say either $\zeta_{n+1} + 1 \in Y_e$ or $\zeta_{n+1} + \gamma \in Y_e$ or both of them holds. Hence, we have either

$$\zeta_{n+2} \le \zeta_{n+1} + 1 \quad \text{or} \quad \zeta_{n+2} \le \zeta_{n+1} + \gamma,$$

which proves the case for $n+1$.

For the second case, we assume that $\xi_0 = (1+\gamma)\lfloor\lambda\rfloor$. Let k be the largest integer such that $\xi_0 = \xi_1 = \cdots = \xi_k = (1+\gamma)\lfloor\lambda\rfloor$. More precisely,

$$\zeta_{n+1} = \lfloor\lambda\rfloor \big(1+\gamma\big) \sum_{i=0}^{k} \lambda^{2i} + \xi_{k+1}\lambda^{2k+2} + \sum_{i=k+2}^{s} \xi_i \lambda^{2i}.$$

Here, we examine two cases separately.

Case 1. Let us assume that $\xi_{k+1} = i + \lfloor\lambda\rfloor\gamma$, where $i \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$. It can be seen from assumption on k that $i \neq \lfloor\lambda\rfloor$, which implies $\zeta_{n+1} + \lambda^{2k+2} \in Y_e$. More precisely,

$$\zeta_{n+1} - \lfloor\lambda\rfloor\left(1+\gamma\right)\sum_{i=0}^{k}\lambda^{2i} - \lfloor\lambda\rfloor\gamma\lambda^{2k+2} + \lambda^{2k+2} \in Y_e. \tag{16}$$

If we have

$$\lambda^{2k+2} > \lfloor\lambda\rfloor\gamma\lambda^{2k+2} + \lfloor\lambda\rfloor\left(1+\gamma\right)\sum_{i=0}^{k}\lambda^{2i}, \tag{17}$$

then, from Lemma 2.4, we conclude that

$$\zeta_{n+2} \leq \zeta_{n+1} - \lfloor\lambda\rfloor\left(1+\gamma\right)\sum_{i=0}^{k}\lambda^{2i} - \lfloor\lambda\rfloor\gamma\lambda^{2k+2} + \lambda^{2k+2} < 1 + \zeta_{n+1}.$$

However, if (17) is not satisfied, i.e.,

$$\lambda^{2k+2} \leq \lfloor\lambda\rfloor\gamma\lambda^{2k+2} + \lfloor\lambda\rfloor\left(1+\gamma\right)\sum_{i=0}^{k}\lambda^{2i},$$

then define $\zeta_\nu = \lfloor\lambda\rfloor\gamma\lambda^{2k+2} + \lfloor\lambda\rfloor\left(1+\gamma\right)\sum_{i=0}^{k}\lambda^{2i}$ which is an element of Y_e, and of course $\nu \leq n+1$. The next term is $\zeta_{\nu+1}$, and it would be in the following form

$$\zeta_{\nu+1} = c_0 + c_1\lambda^2 + \cdots + c_k\lambda^{2k} + \lambda^{2k+2},$$

for some $c_m \in \{i + j\gamma \mid i, j \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\}$ and $m = 0, 1, \cdots, k$. Consider the open interval

$$I_\nu = (\zeta_\nu - \lambda^{2k+2}, \zeta_{\nu+1} - \lambda^{2k+2}),$$

for some $c_i \in \{i + j\gamma \mid i, j \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\}$. Note that

$$(\zeta_\nu, \zeta_{\nu+1}) \cap Y_e = \emptyset. \tag{18}$$

Moreover, (4) and (18) imply $I_\nu \cap Y_e = \emptyset$. Since, $\zeta_\nu - \lambda^{2k+2} + 1 < \zeta_\nu$, we apply induction hypothesis that

$$\zeta_\nu - \lambda^{2k+2} < c_0 + c_1\lambda^2 + \cdots + c_k\lambda^{2k} \leq \zeta_\nu - \lambda^{2k+2} + 1,$$

which concludes

$$\zeta_{n+2} \leq \zeta_{n+1} - \zeta_\nu + c_0 + c_1\lambda^2 + \cdots + c_k\lambda^{2k} + \lambda^{2k+2} \leq 1 + \zeta_{n+1}.$$

Case 2. In this case, we suppose that $\xi_{k+1} = i + j\gamma$, where $i \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$ and $j \in \{0, 1, \cdots, (\lfloor\lambda\rfloor - 1)\}$. Therefore, $\zeta_{n+1} + \gamma\lambda^{2k+2} \in Y_e$. Namely,

$$\zeta_{n+1} - \lfloor\lambda\rfloor\left(1+\gamma\right)\sum_{i=0}^{k}\lambda^{2i} + \gamma\lambda^{2k+2} \in Y_e. \tag{19}$$

If we have

$$\gamma\lambda^{2k+2} > \lfloor\lambda\rfloor\left(1+\gamma\right)\sum_{i=0}^{k}\lambda^{2i}, \tag{20}$$

then from Lemma (2.4), we conclude that

$$\xi_{k+2} \leq \xi_{k+1} - \lfloor\lambda\rfloor\Big(1+\gamma\Big)\sum_{i=0}^{k}\lambda^{2i} + \gamma\lambda^{2k+2} < 1 + \xi_{k+1}.$$

However, if (17) is not satisfied, i.e.,

$$\gamma\lambda^{2k+2} \leq \lfloor\lambda\rfloor\Big(1+\gamma\Big)\sum_{i=0}^{k}\lambda^{2i}.$$

Let us define $\zeta_\mu = \lfloor\lambda\rfloor\Big(1+\gamma\Big)\sum_{i=0}^{k}\lambda^{2i}$ which is an element of Y_e, and clearly $\mu \leq n+1$. By using (4), the next term would be in the following form

$$\zeta_{\mu+1} = \ell_0 + \ell_1\lambda^2 + \cdots + \ell_k\lambda^{2k} + \gamma\lambda^{2k+2},$$

for some $\ell_m \in \{i + j\gamma \mid i, j \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\}$ and $m = 0, 1, \cdots, k$. Define the following open interval

$$I_\mu = (\zeta_\mu - \gamma\lambda^{2k+2}, \zeta_{\mu+1} - \gamma\lambda^{2k+2}).$$

From $(\zeta_\mu, \zeta_{\mu+1}) \cap Y_e = \emptyset$ and (4), we have $I_\mu \cap Y_e = \emptyset$. As it is discussed previously, by utilizing induction hypothesis, we obtain

$$\zeta_\mu - \gamma\lambda^{2k+2} < \ell_0 + \ell_1\lambda^2 + \cdots + \ell_k\lambda^{2k} \leq \zeta_\mu - \gamma\lambda^{2k+2} + 1,$$

which concludes

$$\zeta_{n+2} \leq \zeta_{n+1} - \zeta_\mu + \ell_0 + \ell_1\lambda^2 + \cdots + \ell_k\lambda^{2k} + \gamma\lambda^{2k+2} \leq 1 + \zeta_{n+1}.$$

Hence, for both possible cases, we prove that the hypothesis is true for the case $n+1$. As a result, we obtain that $\zeta_{n+1} - \zeta_n \leq 1$ for $n \in \mathbb{N}$.
□

Theorem 3.3. *For $\lambda > \mathcal{A}$, the following*

$$\mathcal{I}(\lambda) = 0,$$

satisfies.

Proof. For $n \in \mathbb{N}$, we consider the following close interval of the form

$$\mathcal{J}_n := \Big[0, (1+\gamma)\lfloor\lambda\rfloor(1 + \lambda^2 + \cdots + \lambda^{2n})\Big]. \tag{21}$$

Coefficients of each powers of λ can be any value from $\{0, 1, \cdots, \lfloor\lambda\rfloor\}$, i.e., there are $\lfloor\lambda\rfloor + 1$ different possible values. Hence, the set $\mathcal{J}_n \cap Y_e$ contains $(\lfloor\lambda\rfloor + 1)^{2n+2}$ number of ζ_i's. Here, we consider two cases: whether λ is a transcendental or an algebraic number.

Case 1. Assume that λ is a transcendental number. Then, there are $(\lfloor\lambda\rfloor + 1)^{2n+2}$ different elements of the set $\mathcal{J}_n \cap Y_e$. This comes from the fact that $\dfrac{\lambda}{\gamma} \in \mathbb{I}$ and a transcendental number cannot be expressed as a root of polynomial with integer coefficients. In addition to that, for given any $\epsilon > 0$, it is possible to find a large $m \in \mathbb{N}$ such that

$$(1+\gamma)\lfloor\lambda\rfloor(1 + \lambda^2 + \cdots + \lambda^{2m}) < \epsilon\Big((1 + \lfloor\lambda\rfloor)^{2m+2} - 1\Big). \tag{22}$$

Here, we obtain that $(\lfloor\lambda\rfloor+1)^{2m+2}-1$ number of intervals with length ϵ cover $\mathcal{J}_m$ and there are $(\lfloor\lambda\rfloor+1)^{2m+2}$ number of different ζ_i's in $\mathcal{J}_m$. Therefore, from pigeonhole principle, one interval contains at least two different values from $\mathcal{J}_m$, i.e., this is true for arbitrary small ϵ, implies $\mathcal{I}(\lambda)=0$.

Case 2. Let λ be an algebraic number. In this part, we prove that for any $k\in\mathbb{N}$, λ cannot be the root of following polynomial of even powers

$$P(x):=c_0+c_1x^2+\cdots+c_kx^{2k}$$

for $c_i\in\{0,\pm1,\cdots,\pm\lfloor\lambda\rfloor\}$ whenever $\lambda>\mathcal{A}$.

Assume the contrary, i.e., if such $k\in\mathbb{N}$ exists, then we have that $c_k\lambda^{2k}=-c_{k-1}\lambda^{2k-2}-\cdots-c_0$. Suppose without loss of generality that $c_k>0$, by using Lemma 2.5, we obtain

$$c_k\lambda^{2k}\geq\lambda^{2k}>\lfloor\lambda\rfloor\sum_{i=0}^{k-1}\lambda^{2i}\geq -c_{k-1}\lambda^{2k-2}-\cdots-c_0,$$

which contradicts with λ being the root of $P(x)$. This observation shows us that there would be no common terms of Y_e in $\mathcal{J}_n$. More precisely, there would be $(\lfloor\lambda\rfloor+1)^{2n+2}$ terms which cannot coincide in the set $\mathcal{J}_n\cap Y_e$. As it is discussed before, by applying same procedure, we obtain that $\mathcal{I}(\lambda)=0$. □

Theorem 3.4. *The following*

$$\liminf_{k\to\infty}(\beta_{k+1}-\beta_k)=0,$$

satisfies for all transcendental $\lambda>1$.

The proof of Theorem 3.4 is very similar to the proof of Theorem 3.3, so it is omitted.

Lemma 3.5. *For $\lambda>\mathcal{A}$ and given any $\epsilon>0$, one can find a subsequence $\{z_n\}$ of ζ_n such that the following two conditions are satisfied*

(i) $\sum_{i\in I}z_i\in Y_e$ *where* $I\subset\mathbb{N}$.

(ii) $\epsilon<z_{2k+1}-z_{2k}<(1+\lfloor\lambda\rfloor)^2\epsilon$ *for* $k\in\mathbb{N}$.

Proof. We give a proof by induction. Firstly, we prove that both cases hold for $n=1$, i.e., find $z_1,z_0\in Y_e$ such that both conditions satisfy. From Theorem 3.3, we have that $\mathcal{I}(\lambda)=0$ which means for an arbitrary $\epsilon>0$, it is possible to find $k\in\mathbb{N}$ such that $0<\zeta_{k+1}-\zeta_k<\epsilon$. More precisely, we have

$$0<\beta_{s_{k+1}}-\beta_{s_k}+\gamma(\beta_{r_{k+1}}-\beta_{r_k})<\epsilon$$

where $\zeta_{k+1}=\beta_{s_{k+1}}+\gamma\beta_{r_{k+1}}$ and $\zeta_k=\beta_{s_k}+\gamma\beta_{r_k}$.

If $\beta_{s_{k+1}}$ and β_{s_k} share the same term λ^{2n} for some $n\in\mathbb{N}$, with coefficient $\nu_{k+1},\nu_k\in\{0,1,\cdots,\lfloor\lambda\rfloor\}$ respectively. Without loss of generality, assume that $\nu_{k+1}\leq\nu_k$, then we establish new terms as

$$\hat{\beta}_{s_{k+1}}=\beta_{s_{k+1}}-\nu_{k+1}\lambda^{2n}\quad\text{and}\quad\hat{\beta}_{s_k}=\beta_{s_k}-\nu_{k+1}\lambda^{2n}.\tag{23}$$

It should be noted that there may be more terms λ^n such that both $\beta_{s_{k+1}}$ and β_{s_k} share, but doing the same process as it is done in (23), we can still have that $\hat{\beta}_{s_{k+1}}+\hat{\beta}_{s_k}\in X_e$ which means both terms share no common terms λ^n. Same observation is valid for $\beta_{r_{k+1}}$ and β_{r_k}, i.e., one can have $\hat{\beta}_{r_{k+1}}$ and $\hat{\beta}_{r_k}$ such that $\hat{\beta}_{r_{k+1}}-\hat{\beta}_{r_k}=\beta_{r_{k+1}}-\beta_{r_k}$ and $\hat{\beta}_{r_{k+1}}+\hat{\beta}_{r_k}\in X_e$. Hence, our new constructed terms would be the following

$$\hat{\zeta}_{k+1}=\hat{\beta}_{s_{k+1}}+\gamma\hat{\beta}_{r_{k+1}}\quad\text{and}\quad\hat{\zeta}_k=\hat{\beta}_{s_k}+\gamma\hat{\beta}_{r_k}$$

satisfying both $\hat{\zeta}_{k+1}+\hat{\zeta}_k\in Y_e$ and $0<\hat{\zeta}_{k+1}-\hat{\zeta}_k<\epsilon$. Let $m\in\mathbb{N}$ be the least even number satisfying the following relation

$$\lambda^m(\hat{\zeta}_{k+1}-\hat{\zeta}_k)<\epsilon<\lambda^{m+2}(\hat{\zeta}_{k+1}-\hat{\zeta}_k).\tag{24}$$

Multiplying (24) by λ^2, gives the following

$$\epsilon < \lambda^{m+2}(\hat{\zeta}_{k+1} - \hat{\zeta}_k) < \lambda^2\epsilon < (\lfloor\lambda\rfloor + 1)^2\epsilon.$$

Now let $z_1 = \lambda^{m+2}\hat{\zeta}_{k+1}$ and $z_0 = \lambda^{m+2}\hat{\zeta}_k$, then we establish the first two terms satisfying both conditions. Now, let us assume that it is true for the case n, i.e., there exist $z_0, z_1, \cdots, z_{2n-1}$ such that both conditions (i) and (ii) hold. We prove that the case is true for $n+1$, i.e., we construct z_{2k} and z_{2k+1} in such a way that these together with $z_0, z_1, \cdots, z_{2n-1}$ satisfy (i) and (ii). Let p be the highest even power of λ in z_i for $i = 0, 1, \cdots, 2n-1$. Here, one can find $\eta \in \mathbb{N}$ such that

$$0 < \zeta_{\eta+1} - \zeta_\eta < \frac{\epsilon}{\lambda^p},$$

where $\zeta_{\eta+1} = \beta_{u_{\eta+1}} + \gamma\beta_{w_{\eta+1}}$ and $\zeta_\eta = \beta_{u_\eta} + \gamma\beta_{w_\eta}$. If $\beta_{u_{\eta+1}}$ and β_{u_η} share the same term λ^{2n} for some $n \in \mathbb{N}$, with coefficients $\nu_\eta, \nu_{\eta+1} \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$ respectively and satisfying $\nu_{\eta+1} \le \nu_\eta$, then simply choosing

$$\hat{\beta}_{u_{\eta+1}} = \beta_{u_{\eta+1}} - \nu_{\eta+1}\lambda^{2n} \quad \text{and} \quad \hat{\beta}_{u_\eta} = \beta_{u_\eta} - \nu_{\eta+1}\lambda^{2n},$$

we have that $\hat{\beta}_{u_{\eta+1}} + \hat{\beta}_{u_\eta} \in X_e$. Similarly, one can build $\hat{\beta}_{w_{\eta+1}}$ and $\hat{\beta}_{w_\eta}$ such that $\beta_{w_{\eta+1}} - \beta_{w_\eta} = \hat{\beta}_{w_{\eta+1}} - \hat{\beta}_{w_\eta}$ and $\hat{\beta}_{w_{\eta+1}} + \hat{\beta}_{w_\eta} \in X_e$. Hence, we construct the following terms

$$\hat{\zeta}_{\eta+1} = \hat{\beta}_{u_{\eta+1}} + \gamma\hat{\beta}_{w_{\eta+1}} \quad \text{and} \quad \hat{\zeta}_\eta = \hat{\beta}_{u_\eta} + \gamma\hat{\beta}_{w_\eta},$$

which satisfy $\hat{\zeta}_{\eta+1} + \hat{\zeta}_\eta \in Y_e$ and $0 < \hat{\zeta}_{\eta+1} - \hat{\zeta}_\eta < \frac{\epsilon}{\lambda^p}$. Let $\ell \in \mathbb{N}$ be an even number such that

$$\lambda^\ell(\hat{\zeta}_{\eta+1} - \hat{\zeta}_\eta) < \frac{\epsilon}{\lambda^p} < \lambda^{\ell+2}(\hat{\zeta}_{\eta+1} - \hat{\zeta}_\eta) \implies \frac{\epsilon}{\lambda^p} < \lambda^{\ell+2}(\hat{\zeta}_{\eta+1} - \hat{\zeta}_\eta) < (\lfloor\lambda\rfloor + 1)^2\frac{\epsilon}{\lambda^p}.$$

Now, we choose our next candidates as $z_{2n+1} = \lambda^{p+\ell+2}\hat{\zeta}_{\eta+1}$ and $z_{2n} = \lambda^{p+\ell+2}\hat{\zeta}_\eta$, which together with $z_0, z_1, \cdots, z_{2n-1}$ satisfy both conditions (i) and (ii). Hence, the proof is completed by induction. □

Lemma 3.6. *For $\lambda > \mathcal{A}$ and given any $\epsilon > 0$, it is possible to find a finite subsequence $\{\rho_i\}_{i=0}^k$ of $\{\zeta_n\}$ such that the following two conditions hold*

(i) $\rho_0 < \rho_1 < \cdots < \rho_k$ *and* $\rho_k - \rho_0 > \lambda$.

(ii) $\rho_i - \rho_{i-1} < (1 + \lfloor\lambda\rfloor)^2\epsilon$ *for* $i = 1, \cdots, k$.

Proof. Let $\epsilon > 0$ given and $k > \frac{\lambda}{\epsilon}$. Now, we consider the sequence $\{z_i\}_{i=0}^{2k-1}$ defined in Lemma 3.5 and adopt the following construction for $\{\rho_i\}_{i=0}^k$

$$\rho_0 = \sum_{i=0}^{k-1} z_{2i} \quad \text{and} \quad \rho_i = \rho_{i-1} + z_{2i-1} - z_{2i-2} \ \text{ for } \ i = 1, 2, \cdots, k. \tag{25}$$

From first condition of Lemma 3.5, we obtain that $\rho_i \in Y_e$ for $i = 0, \cdots, k$. Moreover, from the second condition of Lemma 3.5, we have both cases (i) and (ii). □

Theorem 3.7. *The following*

$$\mathcal{S}(\lambda) = 0,$$

holds for $\lambda > \mathcal{A}$.

Proof. For given an arbitrary $\epsilon > 0$, construct the finite subsequence $\{\rho_i\}_{i=0}^{k}$ of $\{\zeta_n\}$ defined in Lemma 3.6. Namely, we have

$$\rho_0 < \rho_1 < \cdots < \rho_k \ \text{ with } \ \rho_i - \rho_{i-1} < (1 + \lfloor\lambda\rfloor)^2\epsilon \ \text{ and } \ \rho_k - \rho_0 > \lambda.$$

Define the set of the following form

$$Y_o := \lambda Y_e = \{\lambda\zeta_n\}_{n\in\mathbb{N}}.$$

which is the collection of all odd powers of λ, and so, $Y_o \cap Y_e = \emptyset$. Moreover, from Theorem 3.2, it can be concluded that for any interval of length λ contains at least one element from the set Y_o. Now, we consider the following open intervals

$$I_\tau := (\tau, \tau + (1 + \lfloor\lambda\rfloor)^2\epsilon) \subset (0, \infty). \tag{26}$$

for $\tau > \rho_0 + \lambda$, and show that $I_\tau \cap Y \neq \emptyset$. By using above argument, one can find at least one element

$$\nu \in [\tau - \rho_0 - \lambda, \tau - \rho_0] \cap Y_o. \tag{27}$$

Here, ν consists of only odd powers and from (27), we have that

$$\nu + \rho_0 \leq \tau \leq \nu + \rho_k,$$

which means the set I_τ would contain at least one number of the form $\nu + \rho_i$ for some $i = 0, 1, \cdots, k$. Since, $\nu + \rho_i \in Y$ for all i, we have $I_\tau \cap Y \neq \emptyset$.

As a result, we obtain that

$$\mathcal{S}(\lambda) \leq (1 + \lfloor\lambda\rfloor)^2\epsilon,$$

for arbitrary $\epsilon > 0$. Hence, we have $\mathcal{S}(\lambda) = 0$.

□

4. An Application

In this part, we consider the discrete model (1) with its controls taking from the set $\mathcal{I}$. Here, the aim is to build a finite control set $\mathcal{I}$ in such a way that the robot can reach nearly all points in the space. Let us design the following two sets

$$C_\lambda = \{0, \pm 1, \pm 2, \cdots, \pm\lfloor\lambda\rfloor\} \ \text{ and } \ C_{\lambda,\gamma} = \{i + j\gamma \mid i, j \in C_\lambda\}, \tag{28}$$

and reachable set for the model (1) with control set $\mathcal{I}$

$$\mathcal{R}(0, \mathcal{I}) = \Big\{ \sum_{i=0}^{n} c_i \lambda^i \Big| c_i \in \mathcal{I}, n \in \mathbb{N} \Big\}. \tag{29}$$

Firstly, let us consider the case $\lambda = 1$ for the model (1). If η is ranging in the set C_1, then the reachable set $\mathcal{R}(0, C_1)$ of (1) would be the set of integers which is not dense in $\mathbb{R}$. As a result of Lemma 2.3, for $\eta \in \mathcal{I} = \{\pm p, \pm q\}$ where p, q are rationally independent numbers, we have that the reachable set $\mathcal{R}(0, \mathcal{I})$ of (1) is dense in $\mathbb{R}$. Therefore, for density, the control set can be designed as $\mathcal{I} = \{\pm p, \pm q\}$ for the case $\lambda = 1$.

For case $\lambda > 1$, we provide the following result for density.

Theorem 4.1. *The reachable set $\mathcal{R}(0, C_{\lambda,\gamma})$ of (1) is dense in $\mathbb{R}$ for $\lambda > \mathscr{A}$.*

Proof. Let $v \in \mathbb{R}$ be a real number and by applying division algorithm, we would have its λ expansion as

$$v = \sum_{i=0}^{s} c_i \lambda^i + r_v, \quad \text{where } c_i \in C_\lambda \text{ and } 0 \le r_v < 1.$$

For $r_v = 0$, we have that $v \in \mathcal{R}(0, C_{\lambda,\gamma})$. Let us assume that $0 < r_v < 1$.

From Theorem 3.7, for $\epsilon = \frac{r_v}{\lambda^p} > 0$, where p is a large integer satisfying $p > s$, it is possible to find $k \in \mathbb{N}$ such that

$$\tau_k - \tau_{k-1} \approx \epsilon \quad \text{for} \quad \tau_k, \tau_{k-1} \in Y.$$

As a result,

$$v = \sum_{i=0}^{s} c_i \lambda^i + r_v \approx \sum_{i=0}^{s} c_i \lambda^i + \lambda^p \tau_k - \lambda^p \tau_{k-1}.$$

where coefficients of right hand side belong to the set $C_{\lambda,\gamma}$, which is an element of $\mathcal{R}(0, C_{\lambda,\gamma})$. Since, v is arbitrary here, we conclude that the reachable set $\mathcal{R}(0, C_{\lambda,\gamma})$ of (1) is dense in $\mathbb{R}$. □

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