Filomat 34:13 (2020), 4567–4574
https://doi.org/10.2298/FIL2013567A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Multiple Weak Solutions for a Kind of Time-Dependent Equation Involving Singularity

F. Abdolrazaghi[a], A. Razani[a], R. Mirzaei[a]

[a]*Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University Postal code: 34149-16818, Qazvin, Iran*

Abstract. The existence of at least three weak solutions for a kind of nonlinear time-dependent equation is studied. In fact, we consider the case that the source function has singularity at origin. To this aim, the variational methods and the well-known critical points theorem are main tools.

1. Introduction

The linear Sobolev equations have a real physical background (see [5, 35, 37]) and are studied in [11, 15]. Because of their complexity, they haven't exact solutions (except some very especial cases [3]). There are different methods to study the solution of these problems. One of the standard methods is the fixed point theory that investigate the existence of solutions of nonlinear boundary value problems [2, 6, 12–14, 16, 29, 31, 34, 38]. The calculus of variation is another impressive technique and for using this technique, one needs to show that the given boundary value problem should possess a variational structure on some convenient spaces [1, 4, 9, 10, 17–28, 30, 32, 33, 36].

In the present paper, we study the weak solutions of

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial(\triangle u)}{\partial t} = \mu f(x,t,u) \ \ in\ \Omega, \\ u = 0 \ \ on\ \partial\Omega, \\ u(x,0) = g(x) \ \ x \in \Omega, \end{cases} \tag{1}$$

where Ω is a non-empty bounded open subset of $\mathbb{R}^N$ with $\partial\Omega \in C^1$, μ is a positive parameter, $f : \Omega\times\mathbb{R}^+\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and has a singularity at the origin with respect to the time variable and $g : \Omega \to \mathbb{R}$ vanishes on $\partial\Omega$.
The aim of this paper is to find an interval for μ for which the problem (1) admits at least three distinct weak solutions.

By integrating the first equation of (1) we get

$$\int_0^t \frac{\partial u(x,s)}{\partial s} ds - \int_0^t \frac{\partial \Delta u(x,s)}{\partial s} ds = \int_0^t \mu f(x,s,u) ds, \tag{2}$$

2010 *Mathematics Subject Classification*. 35J20; 34B15
Keywords. Sobolev equation, Weak solution, Critical point theory, Variational method, Singularity
Received: 29 October 2019; Accepted: 24 November 2019
Communicated by Maria Alessandra Ragusa
Email addresses: f.abdolrazaghi@edu.ikiu.ac.ir (F. Abdolrazaghi), razani@sci.ikiu.ac.ir (A. Razani), r.mirzaei@sci.ikiu.ac.ir (R. Mirzaei)

or

$$-\Delta u(x,t) = \mu F(x,t,u) - u(x,t) + g(x) - \Delta g(x), \tag{3}$$

where

$$F(x,t,u) = \int_0^t f(x,s,u)ds. \tag{4}$$

The equation (3) is a time-dependent elliptic equation.

Definition 1.1. *A function $u : \Omega \to \mathbb{R}$ is called a weak solution of the problem* (1) *if $u \in H_0^1$ and*

$$\begin{aligned} \int_\Omega \nabla u(x,t)\cdot\nabla v(x)dx \quad & -\mu \int_\Omega F(x,t,u(x))v(x)dx + \int_\Omega u(x,t)v(x)dx \\ & - \int_\Omega g(x)v(x)dx + \int_\Omega \Delta g(x)v(x)dx = 0, \end{aligned} \tag{5}$$

for all $v \in H_0^1$ and $t \geq 0$.

Definition 1.2. *Define the functionals $\varphi, \vartheta : H_0^1 \to \mathbb{R}$ by $\varphi(u) := \frac{1}{2}\|u\|^2$ and*

$$\begin{aligned} \vartheta(u) := \quad & \int_\Omega \widetilde{F}(x,t,u)dx - \tfrac{1}{2\mu}\int_\Omega (u(x,t))^2\, dx + \tfrac{1}{\mu}\int_\Omega g(x)u(x,t)dx \\ & -\tfrac{1}{\mu}\int_\Omega \Delta g(x)u(x,t)dx, \end{aligned}$$

respective, where $\widetilde{F}(x,t,\eta) := \int_0^\eta F(x,t,s)ds$.

Notice that φ and ϑ are well-defined and C^1, $\varphi', \vartheta' \in X^*$, $\varphi'(u)(v) = \int_\Omega \nabla u(x)\cdot\nabla v(x)dx$ and

$$\begin{aligned} \vartheta'(u)(v) = \quad & \int_\Omega F(x,t,u(x))v(x)dx - \tfrac{1}{\mu}\int_\Omega u(x,t)v(x)dx \\ & +\tfrac{1}{\mu}\int_\Omega g(x)v(x)dx - \tfrac{1}{\mu}\int_\Omega \Delta g(x)v(x)dx. \end{aligned}$$

Remark 1.3. *A critical point of $I_\mu := \varphi - \mu\vartheta$ is exactly a weak solution of* (1).

Fix $q \in [1, 2^*[$, Embedding Theorem [7] shows $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$, i.e. there exists $c_q > 0$ such that for all $u \in H_0^1(\Omega)$

$$\|u\|_{L^q(\Omega)} \leq c_q\|u\|, \tag{6}$$

where

$$c_q \leq \frac{meas(\Omega)^{\frac{2^*-q}{2^*q}}}{\sqrt{N(N-2)\pi}}\left(\frac{N!}{2\Gamma(N/2+1)}\right)^{\frac{1}{N}}, \tag{7}$$

Γ is the Gamma function, $2^* = 2N/(N-2)$ and $meas(\Omega)$ denotes the Lebesgue measure of Ω.

2. Three weak solutions

In this section the existence of at least three weak solutions for the problem (1) is proved. Due to do this, we apply [8, Theorem 3.6] which is given below

Theorem 2.1. *(see [8], Theorem 3.6). let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable functional whose Gateaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that:*

1. $\frac{\sup_{\Phi(x)\leq r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;
2. *for each* $\lambda \in \Lambda_r :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$*, the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X*.*

Set

$$\begin{aligned} &D := \sup_{x\in\Omega} dist(x, \partial\Omega), \quad \kappa := \frac{D\sqrt{2}}{2\pi^{N/4}}\left(\frac{\Gamma(N/2+1)}{D^N-(D/2)^N}\right)^{\frac{1}{2}}, \\ &K_1 := \frac{2\sqrt{2}c_1(2^N-1)}{D^2}, \qquad K_2 := \frac{2^{\frac{q+2}{2}}c_q^q(2^N-1)}{qD^2}. \end{aligned} \tag{8}$$

Now, we can state the main result.

Theorem 2.2. *Let* $f : \Omega\times\mathbb{R}^+\times\mathbb{R} \to \mathbb{R}$ *be a Carathéodory function and* $g : \Omega \to \mathbb{R}$ *vanishes on* $\partial\Omega$*. Assume*

(1) *There exist non-negative constants* m_1,m_2 *and* $q \in]1, \frac{2N}{N-2}[$ *such that*

$$F(x,t,s) \leq m_1 + m_2 \mid s \mid^{q-1} + \frac{1}{\mu}(s - g(x) + \Delta g(x))$$

for all $(x,t,s) \in \Omega\times\mathbb{R}^+\times\mathbb{R}$.

(2) $\widetilde{F}(x,t,\eta) \geq \frac{1}{\mu}\left(\frac{1}{2}\eta^2 - \eta g(x) + \eta\Delta g(x)\right)$ *for every* $(x,t,\eta) \in \Omega\times\mathbb{R}^+\times\mathbb{R}$.

(3) *There exist positive constants* a *and* $b < 2$ *such that*

$$\widetilde{F}(x,t,\eta) \leq a(1 + |\eta|^b) + \frac{1}{\mu}\left(\frac{1}{2}\eta^2 - \eta g(x) + \eta\Delta g(x)\right).$$

(4) *There exist positive constants* α, β *with* $\beta > \alpha\kappa$ *such that*

$$\frac{\inf_{x\in\Omega}\left(\widetilde{F}(x,t,\beta) - \frac{1}{\mu}\left(\frac{1}{2}\beta^2 - \beta g(x) + \beta\Delta g(x)\right)\right)}{\beta^2} > m_1\frac{K_1}{\alpha} + m_2K_2\alpha^{q-2},$$

where κ, K_1, K_2 *are given by* (8).

Then the problem (1) *has at least three weak solutions in* $H_0^1(\Omega)$*, for each parameter* μ *belonging to* $\Lambda(\alpha,\beta) := \frac{2(2^N-1)}{D^2}\times(\delta_1,\delta_2)$*, where*
$\delta_1 := \frac{\beta^2}{\inf_{x\in\Omega}\left(\widetilde{F}(x,t,\beta)-\frac{1}{\mu}\left(\frac{1}{2}\beta^2-\beta g(x)+\beta\Delta g(x)\right)\right)}$ *and* $\delta_2 := \frac{1}{m_1\frac{K_1}{\alpha}+m_2K_2\alpha^{q-2}}$.

Proof. Set $X := H_0^1(\Omega)$ and define the functionals $\varphi(u)$ and $\vartheta(u)$ by Definition 1.2. Clearly, ϑ and φ satisfy the assumptions of [8, Theorem 3.6]. By (1)

$$\widetilde{F}(x,t,\eta) \leq \frac{1}{\mu}\left(\frac{1}{2}\eta^2 - \eta g(x) + \eta\Delta g(x)\right) + m_1|\eta| + m_2\frac{|\eta|^q}{q} \tag{9}$$

for every $(x,t,\eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}$. Thus

$$\begin{aligned}\vartheta(u) :=& \int_\Omega \widetilde{F}(x,t,u)dx - \frac{1}{2\mu}\int_\Omega (u(x,t))^2\,dx + \frac{1}{\mu}\int_\Omega g(x)u(x,t)dx \\ & -\frac{1}{\mu}\int_\Omega \Delta g(x)u(x,t)dx \\ \leq & \frac{1}{\mu}\int_\Omega \left(\frac{1}{2}(u(x,t))^2 - u(x,t)g(x) + u(x,t)\Delta g(x)\right)dx \\ & + \int_\Omega \left(m_1|u(x,t)| + m_2\frac{|u(x,t)|^q}{q}\right)dx - \frac{1}{2\mu}\int_\Omega (u(x,t))^2\,dx \\ & + \frac{1}{\mu}\int_\Omega g(x)u(x,t)dx - \frac{1}{\mu}\int_\Omega \Delta g(x)u(x,t)dx \\ \leq & m_1 \parallel u \parallel_{L^1(\Omega)} + \frac{m_2}{q} \parallel u \parallel^q_{L^q(\Omega)}.\end{aligned}$$

Let $r \in]0,+\infty[$ such that $\varphi(u) \leq r$. By (6),

$$\vartheta(u) \leq \left(\sqrt{2r}c_1 m_1 + \frac{2^{\frac{q}{2}} c_q^q m_2}{q} r^{\frac{q}{2}}\right).$$

Set $\chi(r) := \frac{\sup_{u\in\varphi^{-1}]-\infty,r[}\vartheta(u)}{r}$. Consequently

$$\chi(r) \leq \left(\sqrt{\frac{2}{r}}c_1 m_1 + \frac{2^{\frac{q}{2}} c_q^q m_2}{q} r^{\frac{q}{2}-1}\right), \tag{10}$$

for every $r > 0$.

By (8), there is $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$. Set

$$u_\beta(x,t) := \begin{cases} 0 & x \in \Omega\backslash B(x_0,D), \\ \frac{2\beta}{D}(D - |x - x_0|) & x \in B(x_0,D)\backslash B(x_0,D/2), \\ \beta & x \in B(x_0,D/2). \end{cases} \tag{11}$$

Thus $u_\beta \in H^1_0(\Omega)$. So

$$\begin{aligned}\varphi(u_\beta) &= \frac{1}{2}\int_\Omega |\nabla u_\beta(x,t)|^2 dx \\ &= \frac{1}{2}\int_{B(x_0,D)\backslash B(x_0,D/2)} \frac{(2\beta)^2}{D^2}dx \\ &= \frac{1}{2}\frac{(2\beta)^2}{D^2}(meas(B(x_0,D)) - meas(B(x_0,D/2))) \\ &= \frac{1}{2}\frac{(2\beta)^2}{D^2}\frac{\pi^{N/2}}{\Gamma(N/2+1)}\left(D^N - (D/2)^N\right).\end{aligned} \tag{12}$$

If we force $\beta > \alpha\kappa$, by (4), $\alpha^2 < \varphi(u_\beta)$ because $\alpha^2 < \frac{\beta^2}{\kappa^2}$. Also by assumption (2),

$$\begin{aligned} \vartheta(u_\beta) := & \int_\Omega \widetilde{F}(x,t,u_\beta)dx - \frac{1}{2\mu}\int_\Omega \left(u_\beta(x,t)\right)^2 dx + \frac{1}{\mu}\int_\Omega g(x)u_\beta(x,t)dx \\ & -\frac{1}{\mu}\int_\Omega \Delta g(x)u_\beta(x,t)dx \\ = & \int_\Omega \left[\widetilde{F}(x,t,u_\beta) - \frac{1}{\mu}\left(\frac{1}{2}u_\beta(x,t)^2 - g(x)u_\beta(x,t) + \Delta g(x)u_\beta(x,t)\right)\right]dx \\ \geq & \int_{B(x_0,D/2)} \left[\widetilde{F}(x,t,u_\beta) - \frac{1}{\mu}\left(\frac{1}{2}u_\beta(x,t)^2 - g(x)u_\beta(x,t) + \Delta g(x)u_\beta(x,t)\right)\right]dx \\ \geq & \inf_{x\in\Omega}\left(\widetilde{F}(x,t,\beta) - \frac{1}{\mu}\left(\frac{1}{2}\beta^2 - \beta g(x) + \beta\Delta g(x)\right)\right)\frac{\pi^{N/2}}{\Gamma(N/2+1)}\frac{D^N}{2^N}. \end{aligned} \tag{13}$$

Next by dividing (12) on (13), we have

$$\frac{\vartheta(u_\beta)}{\varphi(u_\beta)} \geq \frac{D^2}{2(2^N-1)}\frac{\inf_{x\in\Omega}\left(\widetilde{F}(x,t,\beta) - \frac{1}{\mu}\left(\frac{1}{2}\beta^2 - \beta g(x) + \beta\Delta g(x)\right)\right)}{\beta^2}. \tag{14}$$

Using (10), assumption (4) implies

$$\begin{aligned} \chi(\alpha^2) \leq & \left(\frac{\sqrt{2}c_1 m_1}{\alpha} + \frac{2^{\frac{q}{2}}c_q^q m_2 \alpha^{q-2}}{q}\right) \\ = & \frac{D^2}{2(2^N-1)}\left(m_1\frac{K_1}{\alpha} + m_2 K_2 \alpha^{q-2}\right) \\ < & \frac{D^2}{2(2^N-1)}\frac{\inf_{x\in\Omega}(\widetilde{F}(x,t,\beta) - U(x,t) - G(x) - \overset{\Delta}{G}(x))}{\beta^2} \\ \leq & \frac{\vartheta(u_\beta)}{\varphi(u_\beta)}. \end{aligned}$$

Assuming $b < 2$ and considering $|u|^b \in L^{\frac{2}{s}}(\Omega)$ for all $u \in X$, Hölder's inequality for $u \in X$ implies $\int_\Omega |u(x,t)|^b dx \leq \| u \|^b_{L^2(\Omega)} (meas(\Omega))^{\frac{2-b}{2}}$. Therefore equation (6) shows for all $u \in X$

$$\int_\Omega |u(x,t)|^b dx \leq c_2^b \| u \|^b (meas(\Omega))^{\frac{2-b}{2}},$$

and by assumption (3),

$$\begin{aligned} I_\mu(u) = & \varphi(u) - \mu\vartheta(u) \\ = & \frac{\|u\|^2}{2} - \mu\int_\Omega \widetilde{F}(x,t,u)dx + \frac{1}{2}\int_\Omega (u(x,t))^2 dx \\ & - \int_\Omega g(x)u(x,t)dx + \int_\Omega \Delta g(x)u(x,t)dx \\ \geq & \frac{\|u\|^2}{2} - \mu\int_\Omega a\left(1 + |u(x,t)|^b\right)dx \\ \geq & \frac{\|u\|^2}{2} - \mu a c_2^b (meas(\Omega))^{\frac{2-b}{2}} \| u \|^b - a\mu meas(\Omega). \end{aligned}$$

This means for every $\mu \in \Lambda(\alpha,\beta) \subseteq \left]\frac{\vartheta(u_\beta)}{\varphi(u_\beta)}, \frac{\alpha^2}{\sup_{\varphi(u)\leq\alpha^2}\vartheta(u)}\right[$, I_μ is coercive. Therefore by Theorem 2.1 for each $\mu \in \Lambda(\alpha,\beta)$ the functional I_μ has at least three distinct critical points that they are weak solutions of the problem (1). □

3. Numerical Experiment

Now, we present an example.

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial(\Delta u)}{\partial t} = \frac{1}{100}\frac{99}{100t}\left(1 + \frac{\exp(-t)}{99}\right)(8 + 100u + u^2) \ \in \Omega, u\mid_{\partial\Omega} = 0, \\ u(x,0) = \frac{1}{1000}\left(\frac{1}{100} - \left(x_1^2 + x_2^2 + x_3^2\right)\right) \ \ x \in \Omega, \end{cases} \tag{15}$$

where

$$\Omega := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3, x_1^2 + x_2^2 + x_3^2 \le 0.1\right\},$$

then $\mu = 0.01, N = 3, D = r = 0.1, 2^* = 6$, $g(x) = 0.001\left(0.01 - \left(x_1^2 + x_2^2 + x_3^2\right)\right)$, $\Delta g(x) = -0.006$ and $f(x,t,u) = \frac{99}{100t}\left(1 + \frac{\exp(-t)}{99}\right)(8 + 100u + u^2)$. Now, setting $q = 3$, then

$$c_1 \le 0.00445759, \quad c_q \le 0.171543,$$
$$\kappa = 1.16798, \quad K_1 \le 8.82557, \quad K_2 \le 6.66307.$$

Clearly $F(x,t,s) = \frac{99}{100}\left(1 + \frac{\exp(-t)}{99}\right)\left(8 + 100s + s^2\right)$, suppose $m_1 = 9$ and $m_2 = 1$, then the assumption (1) of the Theorem 2.2 is satisfied, i.e.

$$\frac{99}{100}\left(1 + \frac{\exp(-t)}{99}\right)\left(8 + 100s + s^2\right) \le$$
$$9 + s^2 + \frac{1}{0.01}\left(s - 0.001\left(0.01 - \left(x_1^2 + x_2^2 + x_3^2\right)\right) - 0.006\right),$$

for all $(x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}$.
Obviously $\widetilde{F}(x,t,\eta) = \frac{99}{100}\left(1 + \frac{\exp(-t)}{99}\right)\left(8\eta + 50\eta^2 + \frac{\eta^3}{3}\right)$, then it can be easily verified that the assumption (2) of the Theorem 2.2 holds, i.e. for all $(x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}$

$$\frac{99}{100}\left(1 + \frac{\exp(-t)}{99}\right)\left(8\eta + 50\eta^2 + \frac{\eta^3}{3}\right) \ge$$
$$\frac{1}{0.01}\left(\frac{1}{2}\eta^2 - 0.001\eta\left(0.01 - \left(x_1^2 + x_2^2 + x_3^2\right)\right) - 0.006\eta\right).$$

Also, by choosing $a = b = 10$, the assumption (3) of the Theorem 2.2 is satisfied, i.e. for all $(x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}$

$$\frac{99}{100}\left(1 + \frac{\exp(-t)}{99}\right)\left(8\eta + 50\eta^2 + \frac{\eta^3}{3}\right) \le$$
$$10(1 + \eta^{10}) + \frac{1}{0.01}\left(\frac{1}{2}\eta^2 - 0.001\eta\left(0.01 - \left(x_1^2 + x_2^2 + x_3^2\right)\right) - 0.006\eta\right).$$

More, set $\alpha = 1$ and $\beta = 500 > \alpha\kappa$ hence, for all $t \ge 0$, it is not difficult to see that

$$\begin{aligned} 162.872 &= \frac{\inf_{x\in\Omega}\left\{\left(\begin{array}{c} \frac{99}{100}\left(1 + \frac{\exp(-t)}{99}\right)\left(8\eta + 50\eta^2 + \frac{\eta^3}{3}\right) - \\ \frac{1}{0.01}\left(\frac{1}{2}\eta^2 - 0.001\eta\left(0.01 - \left(x_1^2 + x_2^2 + x_3^2\right)\right) - 0.006\eta\right) \end{array}\right)\right\}}{\beta^2} \\ &> m_1K_1 + m_2K_2 = 86.0932. \end{aligned}$$

Furthermore, it is observed that $\mu = 0.01 \in \left]\frac{1}{162.872}, \frac{1}{86.0932}\right[$, therefore the problem (15) admits at least three week solutions in according to the Theorem 2.2.

Acknowledgements

The authors are very grateful to anonymous reviewers for carefully reading the paper and for their comments and suggestions which have improved the paper very much.

References

[1] F. Abdolrazaghi, A. Razani, On the weak solutions of an overdetermined system of nonlinear fractional partial integro-differential equations, Miskolc Mathematical Notes 20 (2019) 3–16.
[2] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Applicandae Mathematicae 109 (2010) 973–1033.
[3] A. I. Aristov, On exact solutions of a Sobolev equation, Mathematical Notes 101 (2017) 928–941.
[4] F. Bahboudi, A. Razani, Abdolrahman, Two weak solutions for a singular (p, q)-Laplacian problem, Filomat 33 (2019) 3399-3407.
[5] G.I Barenblatt, Iu.P Zheltov, I.N Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata], Journal of Applied Mathematics and Mechanics 24 (1960) 1286–1303.
[6] M. Benchohra, S. Hamani, S Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Analysis: Theory, Methods & Applications 71 (2009) 2391–2396.
[7] G. Bonanno, G. M. Bisci, Three weak solutions for elliptic Dirichlet problems, Journal of Mathematical Analysis and Applications 382 (2011) 1–8.
[8] G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, Applicable Analysis 89 (2010) 1–10.
[9] J. Chu, Sh. Heidarkhani, A. Salari, G. Caristi, Weak solutions and energy estimates for singular p-Laplacian-type equations, Journal of Dynamical and Control Systems (2017) 1–13.
[10] J.N. Corvellec, V.V. Motreanu, C. Saccon, Doubly resonant semilinear elliptic problems via nonsmooth critical point theory, Journal of Differential Equations 248 (2010) 2064–2091.
[11] Paul L. Davis, A quasilinear parabolic and a related third order problem, Journal of Mathematical Analysis and Applications 40 (1972) 327–335.
[12] A. Dinmohammadi, A. Razani, E. Shivanian, Analytical solution to the nonlinear singular boundary value problem arising in biology, Boundary Value Problems 2017, 63 (2017).
[13] A. Dinmohammadi, E. Shivanian, A. Razani, Existence and uniqueness of solutions for a class of singular nonlinear two-point boundary value problems with sign-Changing nonlinear terms, Numerical Functional Analysis and Optimization 38 (2017) 344–359.
[14] D. Ehsani, M.R. Mokhtarzadeh, A. Razani, Expansions of Dirichlet to Neumann operators under perturbations, Applicable Analysis 89 (2010) 789–799.
[15] Richard E. Ewing, A coupled non-linear hyperbolic-Sobolev system, Annali di Matematica Pura ed Applicata 114 (1977) 331–349.
[16] Z. Goodarzi, A. Razani, A periodic solution of the generalized forced Liénard equation, Abstract and Applied Analysis 2014 (2014) 1–5.
[17] H. Goudarzi, E. Shivanian, S.J. Hosseini Ghoncheh, Weak solutions to a system of nonlinear fractional boundary value problems via variational form, Bulletin of the Malaysian Mathematical Sciences Society (2019) 1–17.
[18] M.H. Heydari, H. Laeli Dastjerdi, M. Nili Ahmadabadi, An efficient method for the numerical solution of a class of nonlinear fractional fredholm integro-differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 19 (2018) 165–173.
[19] S.M. Khalkhali, S. Hidarkhani, A. Razani, Infinitely many solutions for a fourth-order boundary-value problem, Electronic Journal of Differential Equations 164 (2012) 1–14.
[20] S. M. Khalkhali, A. Razani, Multiple Solutions for a Quasilinear (p, q)-Elliptic System, Electronic Journal of Differential Equations 2013 (2013) 1–14.
[21] F. Li, Z. Liang, Q. Zhang Existence of solutions to a class of nonlinear second order two-point boundary value problems, Journal of Mathematical Analysis and Applications 312 (2005) 357–373.
[22] R. Mahdavi Khanghahi, A. Razani, Solutions for a singular elliptic problem involving the $p(x)$-Laplacian, Filomat 32 (2018) 4841–4850.
[23] M. Makvand Chaharlang, M. A. Ragusa, A. Razani, A sequence of radially symmetric weak solutions for some nonlocal elliptic problem in $\mathbb{R}^N$, Mediterranean Journal of Mathematics 17 (2020) 1–12.
[24] M. Makvand Chaharlang, A. Razani, Infinitely many solutions for a fourth order singular elliptic problem, Filomat 32 (2018) 5003–5010.
[25] M. Makvand Chaharlang, A. Razani, Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition, Communications of the Korean Mathematical Society 34 (2019) 155–167.
[26] M. Makvand Chaharlang, A. Razani, A fourth order singular elliptic problem involving p-biharmonic operator, Taiwanese Journal of Mathematics 23 (2019) 589–599.
[27] M. Makvand Chaharlang, A. Razani, Two weak solutions for some Kirchhoff-type problem with Neumann boundary condition, Georgian Mathematical Journal (2020) 10 pages.
[28] J. Mawhin, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, Berlin/New York, 1989.
[29] M.R. Mokhtarzadeh, M.R. Pournaki, A. Razani, An existence-uniqueness theorem for a class of boundary value problems, Fixed Point Theory 13 (2012) 583–592.
[30] J.J. Nieto, D. ORegan, Variational approach to impulsive differential equations, Nonlinear Analysis: Real World Applications 10 (2009) 680–690.
[31] M.R. Pournaki, A. Razani, On the existence of periodic solutions for a class of generalized forced Liénard equations, Applied Mathematics letters 20 (2007) 248–254.
[32] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, American Mathematical Soc., 1986.

[33] M. A. Ragusa, Cauchy-Dirichlet problem associated to divergence form parabolic equations, Communications in Contemporary Mathematics 6 (2004) 377–393.
[34] A. Razani, An existence theorem for ordinary differential equation in Menger probabilistic metric space, Miskolc Mathematical Notes 15 (2014) 711–716.
[35] D. Shi, On the initial boundary value problem of nonlinear the equation of the migration of the moisture in soil, Acta Mathematicae Applicatae Sinica 13 (1990) 31–38.
[36] C.L. Tang, X.P. Wu, Some critical point theorems and their applications to periodic solution for second order Hamiltonian systems, Journal of Differential Equations 284 (2010) 660–692.
[37] T.W. Ting, A cooling process according to two-temperature theory of heat conduction, Journal of Mathematical Analysis and Applications 45 (1974) 23–31.
[38] S. Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation, Computers & Mathematics with Applications 59 (2010) 1300–1309.