Filomat 34:13 (2020), 4555–4566
https://doi.org/10.2298/FIL2013555N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Inequalities for P-Class Functions

Ismail Nikoufar[a], Davuod Saeedi[a]

[a]*Department of Mathematics, Payame Noor University, P.O. Box 19395-3697 Tehran, Iran.*

Abstract. In this paper, we provide some inequalities for P-class functions and self-adjoint operators on a Hilbert space including an operator version of the Jensen's inequality and the Hermite-Hadamard's type inequality. We improve the Hölder-MacCarthy inequality by providing an upper bound. Some refinements of the Jensen type inequality for P-class functions will be of interest.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We say that an operator A in $B(\mathcal{H})$ is positive and write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The spectrum of an operator $A \in B(\mathcal{H})$ is denoted by $Sp(A)$. A function $f : \mathbb{I} \to \mathbb{R}$ is a P-class function on $\mathbb{I}$ if

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y), \tag{1}$$

where $x, y \in \mathbb{I}$ and $\lambda \in [0, 1]$. Some properties of P-class functions can be found in [2, 3]. The set of all P-class functions contains the set of all convex functions and the set of all nonnegative monotone functions. Every non-zero P-class function is nonnegative valued. Indeed, choose $\lambda = 0$ and fix $y_0 \in \mathbb{I}$ in (1). Hence,

$$f(y_0) \leq f(x) + f(y_0),$$

where $x \in \mathbb{I}$. Thus, $f(x) \geq 0$ for all $x \in \mathbb{I}$.

Jensen's inequality for convex functions is one of the most important result in the theory of inequalities due to the fact that many other famous inequalities are particular cases of this for appropriate choices of the function involved. Mond and Pečarić established an operator version of the Jensen inequality for a convex function in [5] (see also [4]) as follows:

Theorem 1.1. *Let $f : [m, M] \to \mathbb{R}$ be a continuous convex function. If $x \in \mathcal{H}$, $\langle x, x\rangle = 1$, then for every self-adjoint operator C such that $mI \leq C \leq MI$,*

$$f(\langle Cx, x\rangle) \leq \langle f(C)x, x\rangle. \tag{2}$$

for each $x \in \mathcal{H}$ with $\langle x, x\rangle = 1$.

2010 *Mathematics Subject Classification*. Primary 47A63, Secondary 47A60, 26D15.
Keywords. P-class function, Jensen's inequality, Hermite-Hadamard's inequality, Hölder-MacCarthy inequality.
Received: 23 January 2020; Accepted: 18 May 2020
Communicated by Fuad Kittaneh
Email addresses: nikoufar@pnu.ac.ir (Ismail Nikoufar), dsaeedi3961@gmail.com (Davuod Saeedi)

As a special case of Theorem 1.1 we have the following Hölder-MacCarthy inequality.

Theorem 1.2. *[1, Theorem 2] Let C be a self-adjoint positive operator on a Hilbert space $\mathcal{H}$. Then*

(i) $\langle C^r x, x\rangle \geq \langle Cx, x\rangle^r$ *for all* $r > 1$ *and* $x \in \mathcal{H}$ *with* $\langle x, x\rangle = 1$*;*

(ii) $\langle C^r x, x\rangle \leq \langle Cx, x\rangle^r$ *for all* $0 < r < 1$ *and* $x \in \mathcal{H}$ *with* $\langle x, x\rangle = 1$*;*

(i) If C is invertible, then $\langle C^r x, x\rangle \geq \langle Cx, x\rangle^r$ *for all* $r < 0$ *and* $x \in \mathcal{H}$ *with* $\langle x, x\rangle = 1$.

In this paper, we show that many general inequalities can be given for P-class functions and self-adjoint operators on a Hilbert space including an operator version of the Jensen's inequality and the Hermite-Hadamard's type inequality for P-class functions. We improve the Hölder-MacCarthy inequality by providing an upper bound.

2. Mond and Pečarić inequality for P-class functions and its application

Taking into account Theorem 1.1 and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of P-class functions and its special cases.

Theorem 2.1. *Let C be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $Sp(C) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous P-class function on $[m, M]$, then*

$$f(\langle Cx, x\rangle) \leq 2\langle f(C)x, x\rangle \tag{3}$$

for each $x \in \mathcal{H}$ with $\|x\| = 1$.

Proof. Since f is P-class,

$$f(\lambda x + (1-\lambda)y) - f(y) \leq f(x) \tag{4}$$

for every $x, y \in [m, M]$, and $\lambda \in (0, 1)$. Consider

$$\alpha := \min_{y \in [m,M]} \frac{f(\lambda x + (1-\lambda)y) - f(y)}{\lambda(x-y)}. \tag{5}$$

It follows from (4) that $\alpha\lambda(x-y) \leq f(x)$ and so $\alpha(x-y) \leq \frac{1}{\lambda} f(x)$. Notice that $l(x) := \alpha(x-y)$ is a linear equation and $l(x) \leq \frac{1}{\lambda} f(x)$ for every $x \in [m, M]$. By assumption, $m \leq \bar{g} \leq M$ where $\bar{g} = \langle Cx, x\rangle$. Consider the straight line $l'(x) := \alpha(x - \bar{g}) + f(\bar{g})$ passing through the point $(\bar{g}, f(\bar{g}))$ and parallel to the line l. By continuity of f, we get

$$l'(\bar{g}) \geq f(\bar{g}) - \epsilon \tag{6}$$

for arbitrary $\epsilon > 0$. We realize two cases:

(i) Let $l'(x) \leq \frac{1}{\lambda} f(x)$ for every $x \in [m, M]$. Then, $l'(C) \leq \frac{1}{\lambda} f(C)$. Hence,

$$\langle l'(C)x, x\rangle \leq \frac{1}{\lambda}\langle f(C)x, x\rangle. \tag{7}$$

By using (6), (7) and linearity of l', we observe that

$$f(\langle Cx, x\rangle) - \epsilon \leq l'(\langle Cx, x\rangle) = \langle l'(C)x, x\rangle \leq \frac{1}{\lambda}\langle f(C)x, x\rangle.$$

Since ϵ is arbitrary, we deduce

$$f(\langle Cx,x\rangle)\le\frac{1}{\lambda}\langle f(C)x,x\rangle. \tag{8}$$

(ii) There exits some points $x\in[m,M]$ such that $l'(x)>\frac{1}{\lambda}f(x)$. Let

$$A:=\{x\in[m,\bar{g}]:l'(x)>\frac{1}{\lambda}f(x)\},$$

$$B:=\{x\in[\bar{g},M]:l'(x)>\frac{1}{\lambda}f(x)\}.$$

Consider $x_A:=\max\{x:x\in A\}$ and $x_B:=\min\{x:x\in B\}$. Let l_A be the line passing through the points $(x_A,0)$ and $(\bar{g},f(\bar{g}))$ and l_B the line passing through the points $(x_B,0)$ and $(\bar{g},f(\bar{g}))$. Define

$$L(x):=\begin{cases} l_A(x), x\in[m,\bar{g}],\\ l_B(x), x\in[\bar{g},M].\end{cases}$$

We show that $L(x)\le\frac{1}{\lambda}f(x)$ for every $x\in[m,M]$. We consider the partition $\{m,x_A,\bar{g},x_B,M\}$ for the closed interval $[m,M]$. Note that $l_A(x)\le 0$ for every $x\in[m,x_A]$ and since $f(x)\ge 0$, we reach $l_A(x)\le\frac{1}{\lambda}f(x)$ for every $x\in[m,x_A]$. On the other hand, one clearly has

$$l'(x)\le\frac{1}{\lambda}f(x) \tag{9}$$

for every $x\in(x_A,\bar{g}]$, otherwise, there exists $x_0\in(x_A,\bar{g}]$ such that $l'(x_0)>\frac{1}{\lambda}f(x_0)$. This infers $x_0\in A$ and so $x_0<x_A$, which is a contradiction. So, by letting x tends to x_A from right in (9), one can deduce $l'(x_A)\le\frac{1}{\lambda}f(x_A)$. Moreover, since $x_A\in\bar{A}$, $l'(x_A)\ge\frac{1}{\lambda}f(x_A)$ and hence $l'(x_A)=\frac{1}{\lambda}f(x_A)$. It follows that l' is the line passing through the points $(x_A,\frac{1}{\lambda}f(x_A))$ and $(\bar{g},f(\bar{g}))$ and the slope of l' is $\alpha=\frac{f(\bar{g})-\frac{1}{\lambda}f(x_A)}{\bar{g}-x_A}$, where the slope of l_A is $\alpha'=\frac{f(\bar{g})}{\bar{g}-x_A}$. By the inequality (9) we have

$$l_A(x)=\alpha'(x-\bar{g})+f(\bar{g})\le\alpha(x-\bar{g})+f(\bar{g})=l'(x)\le\frac{1}{\lambda}f(x)$$

for every $x\in(x_A,\bar{g}]$. So, $L(x)=l_A(x)\le\frac{1}{\lambda}f(x)$ for every $x\in[m,\bar{g}]$.

By the same way, one has $L(x)=l_B(x)\le\frac{1}{\lambda}f(x)$ for every $x\in[\bar{g},M]$. Note that $l_A(\bar{g})=l_B(\bar{g})$ and since f is continuous,

$$l_A(\bar{g})\ge f(\bar{g})-\epsilon \tag{10}$$

for arbitrary $\epsilon>0$. For the case where $\mathrm{sp}(C)\subseteq[m,\bar{g}]$,

$$f(\langle Cx,x\rangle)-\epsilon\le l_A(\langle Cx,x\rangle)=\langle l_A(C)x,x\rangle\le\frac{1}{\lambda}\langle f(C)x,x\rangle.$$

Moreover, when $\mathrm{sp}(C)\subseteq[\bar{g},M]$, we have

$$f(\langle Cx,x\rangle)-\epsilon\le l_A(\langle Cx,x\rangle)=l_B(\langle Cx,x\rangle)=\langle l_B(C)x,x\rangle\le\frac{1}{\lambda}\langle f(C)x,x\rangle$$

and so we obtain (8). According to (8) and for $\lambda=\frac{1}{2}$ we deduce (3). We claim that $\frac{1}{2}$ is the best possible for λ in (8).

(1) Let $0<\lambda\le\frac{1}{2}$. So, $\frac{1}{\lambda}\ge 2$ and consequently by (3), we deduce

$$f(\langle Cx,x\rangle)\le 2\langle f(C)x,x\rangle<\frac{1}{\lambda}\langle f(C)x,x\rangle. \tag{11}$$

(2) Let $\frac{1}{2} < \lambda < 1$ and note that the function $g(t) = \frac{2-t^2}{\alpha}$, $t \in [-1, 1]$, is a P-class function for every $\alpha \geq 1$. Consider $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $x = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then, $g(\langle Cx, x\rangle) = g(0) = \frac{2}{\alpha}$ and $\langle g(C)x, x\rangle = \frac{1}{\alpha}$. Since g is P-class, by (8), we have $g(\langle Cx, x\rangle) \leq \frac{1}{\lambda}\langle g(C)x, x\rangle$ and so $\lambda \leq \frac{1}{2}$ which is a contradiction. □

Corollary 2.2. *Under the hypotheses of Theorem 2.1, if $x \in \mathcal{H}$, $\|x\| \neq 1$, then*

$$f\Big(\frac{\langle Cx, x\rangle}{\langle x, x\rangle}\Big) \leq \frac{2\langle f(C)x, x\rangle}{\langle x, x\rangle}. \tag{12}$$

Proof. Let $y := \frac{x}{\sqrt{\langle x,x\rangle}}$ and apply Theorem 2.1. □

Lemma 2.3. *Let f be a continuous P-class function and $\lambda < 0$. If f is decreasing, then*

$$f((1-\lambda)x + \lambda y) \geq f(x) - f(y) \tag{13}$$

for every $x, y \in [m, M]$ with $x < y$.

Proof. We have $(1-\lambda)x + \lambda y = x + \lambda(y - x) \leq x$. Since f is decreasing,

$$f((1-\lambda)x + \lambda y) \geq f(x) \geq f(x) - f(y).$$

□

Lemma 2.4. *Let f be a continuous P-class function and $\lambda > 1$. If f is increasing, then* (13) *holds.*

Proof. We have $(1-\lambda)x > (1-\lambda)y$ and so $(1-\lambda)x + \lambda y \geq y$. Since f is increasing, we obtain

$$f((1-\lambda)x + \lambda y) \geq f(y) \geq f(x) \geq f(x) - f(y).$$

□

Theorem 2.5. *Let $f : [m, M] \to \mathbb{R}$ be a continuous decreasing P-class function and let the self-adjoint operator C satisfies $mI \leq C \leq MI$. If $0 < \langle x, x\rangle < u$, $x \in \mathcal{H}$, $a \in [m, M]$, and $\frac{ua-\langle Cx,x\rangle}{u-\langle x,x\rangle} \in [m, M]$, then*

$$f\Big(\frac{ua - \langle Cx, x\rangle}{u - \langle x, x\rangle}\Big) \geq f(a) - \frac{2\langle f(C)x, x\rangle}{\langle x, x\rangle}. \tag{14}$$

Proof. Applying Lemma 2.3 with $\lambda = -\frac{\langle x,x\rangle}{u-\langle x,x\rangle} < 0$, $x = a$, $y = \frac{\langle Cx,x\rangle}{\langle x,x\rangle}$, and Corollary 2.2, we find that

$$\begin{aligned} f\Big(\frac{ua - \langle Cx, x\rangle}{u - \langle x, x\rangle}\Big) &= f\Big(\frac{u}{u - \langle x, x\rangle}a - \frac{\langle x, x\rangle}{u - \langle x, x\rangle}\frac{\langle Cx, x\rangle}{\langle x, x\rangle}\Big) \\ &\geq f(a) - f\Big(\frac{\langle Cx, x\rangle}{\langle x, x\rangle}\Big) \\ &\geq f(a) - \frac{2\langle f(C)x, x\rangle}{\langle x, x\rangle}. \end{aligned} \tag{15}$$

□

Corollary 2.6. *Under the hypotheses of Theorem 2.5, if f is increasing, then*

$$f\Big(\frac{ua - \langle Cx, x\rangle}{u - \langle x, x\rangle}\Big) \geq f\Big(\frac{\langle Cx, x\rangle}{\langle x, x\rangle}\Big) - f(a). \tag{16}$$

Proof. Applying Lemma 2.4 with $\lambda = \frac{u}{u-\langle x,x\rangle} > 1$, $x = \frac{\langle Cx,x\rangle}{\langle x,x\rangle}$, $y = a$, and Corollary 2.2, we obtain the result. □

Theorem 2.7. *Let the conditions of Theorem 2.1 be satisfied. Then*

$$\langle f(C)x, x\rangle \leq f(m) + f(M). \tag{17}$$

Proof. Let $u \in [m, M]$. Then $u = \frac{M-u}{M-m}m + \frac{u-m}{M-m}M$. The function f is P-class, so $f(u) \leq f(m) + f(M)$. The operator $f(m) + f(M) - f(C)$ is positive, and hence, (17) follows. □

Theorem 2.8. *Let the conditions of Theorem 2.1 be satisfied. Let J be an interval such that $f([m, M]) \subset J$. If $F(u, v)$ is a real function defined on $J \times J$ and non–decreasing in u, then*

$$\begin{aligned} F(2\langle f(C)x, x\rangle, f(\langle Cx, x\rangle)) &\leq \max_{t\in[m,M]} F(2(f(m) + f(M)), f(t)) \\ &= \max_{\theta\in[0,1]} F(2(f(m) + f(M)), f(\theta m + (1-\theta)M)). \end{aligned} \tag{18}$$

Proof. According to the non-decreasing character of F and Theorem 2.7, we deduce

$$\begin{aligned} F(2\langle f(C)x, x\rangle, f(\langle Cx, x\rangle)) &\leq F(2(f(m) + f(M)), f(\bar{g})) \\ &\leq \max_{t\in[m,M]} F(2(f(m) + f(M)), f(t)) \end{aligned}$$

since $\bar{g} = \langle Cx, x\rangle \in [m, M]$. The second form of the right side of (18) follows at once from the change of variable $\theta = \frac{M-t}{M-m}$, so that $t = \theta m + (1-\theta)M$, with $0 \leq \theta \leq 1$. □

In the same way (or more simply just by replacing F by $-F$ in the above theorem) we can prove the following:

Corollary 2.9. *Under the same hypotheses as Theorem 2.8, except that F is non–increasing in its first variable, we have*

$$\begin{aligned} F(2\langle f(C)x, x\rangle, f(\langle Cx, x\rangle)) &\geq \min_{t\in[m,M]} F(2(f(m) + f(M), f(t))) \\ &= \min_{\theta\in[0,1]} F(2(f(m) + f(M)), f(\theta m + (1-\theta)M)). \end{aligned}$$

Corollary 2.10. *Let the conditions of Theorem 2.1 be satisfied. Then,*

(i) $2\langle f(C)x, x\rangle \leq \lambda f(\langle Cx, x\rangle)$ *for some* $\lambda > 0$,

(ii) $2\langle f(C)x, x\rangle \leq \lambda + f(\langle Cx, x\rangle)$ *for some* $\lambda \in \mathbb{R}$.

Proof. (i) Consider $F(u, v) = \frac{u}{v}$, $\varphi(t) = \frac{2(f(m)+f(M))}{f(t)}$, and $J = (0, \infty)$. So, F is non-decreasing on its first variable and by Theorem 2.8 we have

$$\frac{2\langle f(C)x, x\rangle}{f(\langle Cx, x\rangle)} \leq \max_{t\in[m,M]} \varphi(t) = \frac{2(f(m) + f(M))}{\min_{t\in[m,M]} f(t)}.$$

The function φ essentially attains its maximum value when the function f attains its minimum value on $[m, M]$ by continuity of f. Hence, by letting $\lambda = \frac{2(f(m)+f(M))}{\min_{t\in[m,M]} f(t)}$, we find the result.

(ii) Consider $F(u, v) = u - v$, $\varphi(t) = 2(f(m) + f(M)) - f(t)$, and $J = \mathbb{R}$. So, F is non-decreasing on its first variable and Theorem 2.8 leads

$$2\langle f(C)x, x\rangle - f(\langle Cx, x\rangle) \leq \max_{t\in[m,M]} \varphi(t) = 2(f(m) + f(M)) - \min_{t\in[m,M]} f(t).$$

The function f attains its minimum value by continuity of f. Hence, it suffices to let $\lambda = 2(f(m) + f(M)) - \min_{t\in[m,M]} f(t)$. □

Combining Theorem 2.1 and Corollary 2.10 we identify the following result.

Corollary 2.11. *Let the conditions of Theorem 2.1 be satisfied. Then*

(i) $\frac{2}{\lambda}\langle f(C)x,x\rangle \leq f(\langle Cx,x\rangle) \leq 2\langle f(C)x,x\rangle$ *for some* $\lambda > 0$,

(ii) $0 \leq 2\langle f(C)x,x\rangle - f(\langle Cx,x\rangle) \leq \lambda$ *for some* $\lambda \in \mathbb{R}$.

For instance, when $f(t) = t^r$, $0 < r < 1$ and $t \in [m,M]$, we obtain

$$0 \leq 2\langle C^r x,x\rangle - \langle Cx,x\rangle^r \leq 2M^r + m^r$$

and when $f(t) = \ln t$, $t \in [m,M] \subseteq [1,\infty)$, f is P-class and we have

$$\frac{\ln m}{\ln M + \ln m}\langle \ln(C)x,x\rangle \leq \ln(\langle Cx,x\rangle) \leq 2\langle \ln(C)x,x\rangle,$$

$$0 \leq 2\langle \ln(C)x,x\rangle - \ln(\langle Cx,x\rangle) \leq 2\ln(M) + \ln(m).$$

As a consequence of the definition of a P-class function one can verify that if f is a continuous increasing P-class function and g is a convex function, then $f \circ g$ is a P-class function. Remember that f is homogeneous, whenever, $f(\lambda A) = \lambda f(A)$ for $\lambda > 0$. We have the following simple corollary.

Corollary 2.12. *Let the conditions of Theorem 2.1 be satisfied and let f be a non-decreasing function and $n \geq 1$.*

(i) If f is homogeneous, then $f^n(\langle Cx,x\rangle) \leq 2^n\langle f^n(C)x,x\rangle$.

(ii) If f is subadditive, then f^n is P-class and $f^n(\langle Cx,x\rangle) \leq 2\langle f^n(C)x,x\rangle$.

In the next corollary, we obtain the Hermite-Hadamard's type inequality for P-class functions.

Corollary 2.13. *Let the conditions of Theorem 2.1 be satisfied and let p and q be nonnegative numbers, with $p+q > 0$, for which*

$$\langle Cx,x\rangle = \frac{pm+qM}{p+q}.$$

Then

$$\frac{1}{2}f\Big(\frac{pm+qM}{p+q}\Big) \leq \langle f(C)x,x\rangle \leq f(m) + f(M).$$

Proof. By virtue of Theorem 2.1 and 2.7 we reach

$$f\Big(\frac{pm+qM}{p+q}\Big) = f(\langle Cx,x\rangle) \leq 2\langle f(C)x,x\rangle) \leq 2(f(m) + f(M)).$$

□

We can improve the Hölder-MacCarthy inequality by providing an upper bound. We use the fact that the function t^r, $0 < r < 1$, is P-class, in addition to being concave.

Lemma 2.14. *Let $\alpha,\beta > 0$ and $0 < r < 1$. Then,* $(\alpha+\beta)^r \leq \alpha^r + \beta^r$.

Proof. Define $f_r(t) = (1+t)^r - t^r$, $t > 0$ and note that $f_r'(t) < 0$. So, f_r is decreasing and the result follows from the fact that $f_r(\frac{\alpha}{\beta}) \leq f_r(0)$. □

Corollary 2.15. *Let C be a self-adjoint positive operator on a Hilbert space $\mathcal{H}$. Then* *(i) for all* $0 < r < 1$ *and* $x \in \mathcal{H}$ *with* $\|x\| = 1$,

$$\langle C^r x, x\rangle \le \langle Cx, x\rangle^r \le 2\langle C^r x, x\rangle, \tag{19}$$

(ii) for all $r > 1$ *and* $x \in \mathcal{H}$ *with* $\|x\| = 1$,

$$\langle Cx, x\rangle^r \le \langle C^r x, x\rangle \le 2^r\langle Cx, x\rangle^r. \tag{20}$$

Proof. (i) The first inequality is Hölder-MacCarthy inequality for the case where $0 < r < 1$. Let $0 < a < b$ and $0 < \lambda < 1$. In view of Lemma 2.14 we get

$$(\lambda a + (1-\lambda)b)^r \le (\lambda a)^r + ((1-\lambda)b)^r \le a^r + b^r.$$

This ensures the function t^r is P-class and hence using Theorem 2.1 we reach the second inequality.
(ii) By applying $\frac{1}{r} < 1$ in part (i) we have

$$\langle C^{1/r}x, x\rangle \le \langle Cx, x\rangle^{1/r} \le 2\langle C^{1/r}x, x\rangle. \tag{21}$$

Replacing C^r with C in (21) we deduce

$$\langle Cx, x\rangle \le \langle C^r x, x\rangle^{1/r} \le 2\langle Cx, x\rangle,$$

which implies the result. □

Let w_i, x_i be positive numbers with $\sum_{i=1}^n w_i = 1$. Then the weighted power means are defined by

$$M_n^{[r]}(x; w) = \Big(\sum_{i=1}^n w_i x_i^r\Big)^{1/r}, \quad r \ne 0$$

and

$$M_n^{[0]}(x; w) = \prod_{i=1}^n x_i^{w_i}$$

is called weighted geometric mean and denoted by G_w. It is well-known that if $s \le r$, then

$$M_n^{[s]}(x; w) \le M_n^{[r]}(x; w). \tag{22}$$

The weighted arithmetic mean of a non-empty sequence of data $\{x_1, x_2, ..., x_n\}$ and corresponding non-negative weights $\{w_1, w_2, ..., w_n\}$ with $\sum_{i=1}^n w_i = 1$ is defined by

$$A_w = \sum_{i=1}^n w_i x_i$$

and the weighted harmonic mean of them is defined by

$$H_w = \Big(\sum_{i=1}^n w_i x_i^{-1}\Big)^{-1}.$$

The arithmetic-geometric-harmonic mean inequality is a well-known inequality as follows:

$$H_w \le G_w \le A_w.$$

According to improved Hölder-MacCarthy inequality we identify the following relation between the weighted arithmetic mean and the weighted power mean.

Corollary 2.16. *Let w_i, x_i be positive numbers with $\sum_{i=1}^{n} w_i = 1$. Then*

(i) for all $0 < r < 1$,

$$M_n^{[r]}(x;w) \leq A_w \leq 2^{1/r} M_n^{[r]}(x;w),$$

(ii) for all $r > 1$,

$$A_w \leq M_n^{[r]}(x;w) \leq 2A_w.$$

Proof. (i) Consider

$$C = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} \sqrt{w_1} \\ \vdots \\ \sqrt{w_n} \end{pmatrix}.$$

Clearly, we have $\langle Cx, x\rangle^r = (\sum_{i=1}^{n} w_i x_i)^r$ and $\langle C^r x, x\rangle = \sum_{i=1}^{n} w_i x_i^r$. In view of (19), we obtain the desired result.
(ii) By considering C and x as above and applying (20) we get the result. □

Some refinements of the arithmetic-geometric-harmonic mean inequality are of interest.

Remark 2.17. *Let w_i, x_i be positive numbers with $\sum_{i=1}^{n} w_i = 1$.*
(i) For all $0 < r < 1$,

$$\begin{aligned} 2^{-1/r} M_n^{[-r]}(x;w) &\leq H_w \leq M_n^{[-r]}(x;w) \\ &\leq G_w \leq M_n^{[r]}(x;w) \leq A_w \leq 2^{1/r} M_n^{[r]}(x;w). \end{aligned}$$

Replacing x_i^{-1} with x_i in Corollary 2.16(i) and applying the monotonically decreasing function t^{-1} to both sides of the inequalities we get the first and second inequalities. The third and forth inequalities obtain by (22)*. We deduce the last two inequalities by Corollary 2.16(i).*
(ii) For all $r > 1$,

$$\frac{1}{2} H_w \leq M_n^{[-r]}(x;w) \leq H_w \leq G_w \leq A_w \leq M_n^{[r]}(x;w) \leq 2A_w.$$

Similar to that of part (i) and Corollary 2.16(ii) we reach the first and second inequalities. The third and forth inequalities are well-known inequalities. The last two inequalities are obtained in Corollary 2.16(ii).

3. Multiple operator versions and its application

In this section, we investigate a multiple operator version of Theorem 2.1 and the corresponding applications for the P-class functions.

Theorem 3.1. *Let C_i be self-adjoint operators with $Sp(C_i) \subseteq [m, M]$ for some scalars $m < M$ and $x_i \in \mathcal{H}$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} \|x_i\|^2 = 1$. If f is a P-class function on $[m, M]$, then*

$$f\Big(\sum_{i=1}^{n} \langle C_i x_i, x_i\rangle\Big) \leq 2 \sum_{i=1}^{n} \langle f(C_i) x_i, x_i\rangle.$$

Proof. We consider

$$\tilde{C} = \begin{pmatrix} C_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_n \end{pmatrix} \quad \text{and} \quad \tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

By a simple verification we get $Sp(\tilde{C}) \subseteq [m, M]$ and $\|\tilde{x}\| = 1$. On the other hands,

$$f(\langle \tilde{C}\tilde{x}, \tilde{x}\rangle) = f\Big(\sum_{i=1}^{n} \langle C_i x_i, x_i\rangle\Big),$$

$$\langle f(\tilde{C})\tilde{x},\tilde{x}\rangle = \sum_{i=1}^{n}\langle f(C_i)x_i,x_i\rangle.$$

According to Theorem 2.1 we have $f(\langle\tilde{C}\tilde{x},\tilde{x}\rangle) \le 2\langle f(\tilde{C})\tilde{x},\tilde{x}\rangle$ and so we deduce the desired result. □

The following particular case is of interest.

Corollary 3.2. *Let C_i be self-adjoint operators with $Sp(C_i) \subseteq [m,M]$, $i \in \{1,...,n\}$ for some scalars $m < M$. If f is a P-class function on $[m,M]$ and $p_i \ge 0$ with $\sum_{i=1}^{n} p_i = 1$, then*

$$f\Big(\sum_{i=1}^{n} p_i\langle C_i x,x\rangle\Big) \le 2\sum_{i=1}^{n} p_i\langle f(C_i)x,x\rangle$$

for every $x \in \mathcal{H}$ with $\|x\| = 1$.

Proof. It follows from Theorem 3.1 by choosing $x_i = \sqrt{p_i}x$, $i \in \{1,...,n\}$, where $x \in \mathcal{H}$ with $\|x\| = 1$. □

The following corollary is also of interest.

Corollary 3.3. *Let f be a P-class function on $[m,M]$, C_i self-adjoint operators with $Sp(C_i) \subseteq [m,M]$, $i \in \{1,...,n\}$ and $p_i \ge 0$ with $\sum_{i=1}^{n} p_i = 1$. Assume that $I \subsetneq \{1,...,n\}$ and $I^c = \{1,...,n\}\backslash I$, $p_I = \sum_{i\in I} p_i$, $p_{I^c} = 1 - \sum_{i\in I} p_i$. Then for any $x \in \mathcal{H}$ with $\|x\| = 1$,*

$$\begin{aligned} f\Big(\sum_{i=1}^{n} p_i\langle C_i x,x\rangle\Big) &\le \Omega_1(f,I) \le \Omega_2(f,I) \\ &\le 2\sum_{i=1}^{n}\langle f(C_i)x,x\rangle, \end{aligned}$$

where

$$\begin{aligned} \Omega_1(f,I) &= f\Big(\sum_{i\in I}\frac{p_i}{p_I}\langle C_i x,x\rangle\Big) + f\Big(\frac{p_i}{p_{I^c}}\sum_{i\in I^c} p_i\langle C_i x,x\rangle\Big) \\ \Omega_2(f,I) &= 2\sum_{i\in I}\frac{p_i}{p_I}\langle f(C_i)x,x\rangle + 2\sum_{i\in I^c}\frac{p_i}{p_{I^c}}\langle f(C_i)x,x\rangle. \end{aligned}$$

Proof. By rearranging the terms in $f\Big(\sum_{i=1}^{n} p_i\langle C_i x,x\rangle\Big)$ we reach

$$\begin{aligned} f\Big(\sum_{i=1}^{n} p_i\langle C_i x,x\rangle\Big) &= f\Big(p_I(\frac{1}{p_I}\sum_{i\in p_I} p_i\langle C_i x,x\rangle) + p_{I^c}(\frac{1}{p_{I^c}}\sum_{i\in I^c} p_i\langle C_i x,x\rangle)\Big) \\ &\le f\Big(\frac{1}{p_I}\sum_{i\in I} p_i\langle C_i x,x\rangle\Big) + f\Big(\frac{1}{p_{I^c}}\sum_{i\in I^c} p_i\langle C_i x,x\rangle\Big) \\ &= \Omega_1(f,I). \end{aligned}$$

On the other hand, Corollary 3.2 infers

$$\begin{aligned}\Omega_1(f,I) &= f\Big(\sum_{i\in I}\frac{p_i}{p_I}\langle C_i x,x\rangle\Big)+f\Big(\sum_{i\in I^c}\frac{p_i}{p_{I^c}}\langle C_i x,x\rangle\Big)\\ &\le 2\sum_{i\in I}\frac{p_i}{p_I}\langle f(C_i)x,x\rangle+2\sum_{i\in I^c}\frac{p_i}{p_{I^c}}\langle f(C_i)x,x\rangle)\\ &= \Omega_2(f,I)\\ &\le 2\sum_{i\in I}\langle f(C_i)x,x\rangle+2\sum_{i\in I^c}\langle f(C_i)x,x\rangle\\ &= 2\sum_{i=1}^{n}\langle f(C_i)x,x\rangle.\end{aligned}$$

□

Corollary 3.4. *Let f be a non-decreasing P-class function on $[m,M]$ and let C_i, $p_i \ge 0$, I, I^c, p_I, and p_{I^c} be as in Corollary 3.3. Then*

$$\begin{aligned}f\Big(\Big\|\sum_{i=1}^{n}p_iC_i\Big\|\Big) &\le f\Big(\Big\|\sum_{i\in I}\frac{p_i}{p_I}C_i\Big\|\Big)+f\Big(\Big\|\sum_{i\in I^c}\frac{p_i}{p_{I^c}}C_i\Big\|\Big)\\ &\le 2\Big\|\sum_{i\in I}\frac{p_i}{p_I}f(C_i)\Big\|+2\Big\|\sum_{i\in I^c}\frac{p_i}{p_{I^c}}f(C_i)\Big\|\\ &\le 2\Big\|\sum_{i=1}^{n}f(C_i)\Big\|.\end{aligned}$$

Proof. We have

$$\begin{aligned}f\Big(\Big\|\sum_{i=1}^{n}p_iC_i\Big\|\Big) &= f\Big(\Big\|\sum_{i\in I}p_iC_i+\sum_{i\in I^c}p_iC_i\Big\|\Big)\\ &\le f\Big(\Big\|\sum_{i\in I}p_iC_i\Big\|+\Big\|\sum_{i\in I^c}p_iC_i\Big\|\Big)\\ &= f\Big(p_I\Big\|\frac{1}{p_I}\sum_{i\in I}p_iC_i\Big\|+p_{I^c}\Big\|\frac{1}{p_{I^c}}\sum_{i\in I^c}p_iC_i\Big\|\Big)\\ &\le f\Big(\frac{1}{p_I}\Big\|\sum_{i\in I}p_iC_i\Big\|\Big)+f\Big(\frac{1}{p_{I^c}}\Big\|\sum_{i\in I^c}p_iC_i\Big\|\Big).\end{aligned}$$

On the other hand and by virtue of Corollary 3.2 we get

$$
\begin{aligned}
& f\Big(\frac{1}{p_I}\Big\|\sum_{i\in I} p_iC_i\Big\|\Big) + f\Big(\frac{1}{p_{I^c}}\Big\|\sum_{i\in I^c} p_iC_i\Big\|\Big) \\
&= f\Big(\frac{1}{p_I}\sup_{\|x\|=1}\langle\sum_{i\in I} p_iC_ix, x\rangle\Big) + f\Big(\frac{1}{p_{I^c}}\sup_{\|x\|=1}\langle\sum_{i\in I^c} p_iC_ix, x\rangle\Big) \\
&= \sup_{\|x\|=1} f\Big(\frac{1}{p_I}\langle\sum_{i\in I} p_iC_ix, x\rangle\Big) + \sup_{\|x\|=1} f\Big(\frac{1}{p_{I^c}}\langle\sum_{i\in I^c} p_iC_ix, x\rangle\Big) \\
&\le 2\sup_{\|x\|=1}\sum_{i\in I}\frac{p_i}{p_I}\langle f(C_i)x, x\rangle + 2\sup_{\|x\|=1}\sum_{i\in I^c}\frac{p_i}{p_{I^c}}\langle f(C_i)x, x\rangle \\
&= 2\Big\|\sum_{i\in I}\frac{p_i}{p_I}f(C_i)\Big\| + 2\Big\|\sum_{i\in I^c}\frac{p_i}{p_{I^c}}f(C_i)\Big\| \\
&\le 2\Big\|\sum_{i\in I} f(C_i)\Big\| + 2\Big\|\sum_{i\in I^c} f(C_i)\Big\| \\
&= 2\Big\|\sum_{i=1}^{n} f(C_i)\Big\|.
\end{aligned}
$$

□

Remark 3.5. *Let C_i, $p_i \ge 0$, I, I^c, p_I, and p_{I^c} be as in Corollary 3.3. Then*
(i) For $0 < r < 1$,

$$
\begin{aligned}
\Big\|\sum_{i=1}^{n} p_iC_i\Big\|^r &\le \Big\|\sum_{i\in I}\frac{p_i}{p_I}C_i\Big\|^r + \Big\|\sum_{i\in I^c}\frac{p_i}{p_{I^c}}C_i\Big\|^r \\
&\le 2\Big\|\sum_{i\in I}\frac{p_i}{p_I}C_i^r\Big\| + 2\Big\|\sum_{i\in I^c}\frac{p_i}{p_{I^c}}C_i^r\Big\| \\
&\le 2\Big\|\sum_{i=1}^{n} C_i^r\Big\|.
\end{aligned}
$$

(ii) For $r > 1$, and applying part (i) for $\frac{1}{r} < 1$ and replacing C_i^r with C_i we conclude

$$
\begin{aligned}
\Big\|\sum_{i=1}^{n} p_iC_i^r\Big\| &\le \Big(\Big\|\sum_{i\in I}\frac{p_i}{p_I}C_i^r\Big\|^{\frac{1}{r}} + \Big\|\sum_{i\in I^c}\frac{p_i}{p_{I^c}}C_i^r\Big\|^{\frac{1}{r}}\Big)^r \\
&\le 2^r\Big(\Big\|\sum_{i\in I}\frac{p_i}{p_I}C_i\Big\| + \Big\|\sum_{i\in I^c}\frac{p_i}{p_{I^c}}C_i\Big\|\Big)^r \\
&\le \Big(2\Big\|\sum_{i=1}^{n} C_i\Big\|\Big)^r.
\end{aligned}
$$

Theorem 3.6. *Let the conditions of Theorem 3.1 be satisfied. Then*

$$
\sum_{i=1}^{n}\langle f(C_i)x_i, x_i\rangle \le f(m) + f(M). \tag{23}
$$

Proof. Consider $\tilde{C}$ and $\tilde{x}$ as in the proof of Theorem 3.1 and apply Theorem 2.7. □

Theorem 3.7. *Let the conditions of Theorem 3.1 be satisfied. Let J be an interval such that $f([m, M]) \subset J$. If $F(u, v)$ is a real function defined on $J \times J$ and non–decreasing in u, then*

$$F\Big(2\sum_{i=1}^{n}\langle f(C_i)x_i, x_i\rangle, f\Big(\sum_{i=1}^{n}\langle C_i x_i, x_i\rangle\Big)\Big) \leq \max_{t\in[m,M]} F(2(f(m) + f(M)), f(t)). \tag{24}$$

Proof. Consider $\tilde{C}$ and $\tilde{x}$ as in the proof of Theorem 3.1 and apply Theorem 2.8. □

Corollary 3.8. *Let the conditions of Theorem 3.1 be satisfied. Then*

(i) the inequality

$$2\sum_{i=1}^{n}\langle f(C_i)x_i, x_i\rangle \leq \lambda f(\langle\sum_{i=1}^{n} C_i x_i, x_i\rangle) \tag{25}$$

holds for some $\lambda > 0$,

(ii) the inequality

$$2\sum_{i=1}^{n}\langle f(C_i)x_i, x_i\rangle \leq \lambda + f(\langle\sum_{i=1}^{n} C_i x_i, x_i\rangle) \tag{26}$$

holds for some $\lambda \in \mathbb{R}$.

Proof. Consider $\tilde{C}$ and $\tilde{x}$ as in the proof of Theorem 3.1.

(i) Apply Corollary 2.10 (i) and note that $\lambda = \frac{2(f(m)+f(M))}{\min_{t\in[m,M]} f(t)}$.

(ii) Apply Corollary 2.10 (ii) and note that $\lambda = 2(f(m) + f(M)) - \min_{t\in[m,M]} f(t)$. □

Acknowledgement: The first author was supported by Payame Noor University under grant number 46384.

References

[1] R. P. Agarwal and S. S. Dragomir, A survey of Jensen type inequalities for functions of self-adjoint operators in Hilbert spaces, Comput. Math. Appl. 59(12), 3785–3812 (2010).

[2] S. S. Dragomir, J. Pečarić, and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21(3), 335–341 (1995).

[3] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, Bull. Aust. Math. Soc. 57(3), 377-385 (1998).

[4] T. Furuta, J. Mićić, J. Pečarić, Y. Seo, Mond–Pečarić Method in Operator Inequalities, Inequalities for Bounded Self-adjoint Operators on a Hilbert Space, Element, Zagreb, 2005.

[5] B. Mond, J. Pečarić, Convex inequalities in Hilbert space, Houston J. Math. 19, 405–420 (1993).