Filomat 34:13 (2020), 4533–4553
https://doi.org/10.2298/FIL2013533D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Optimality Conditions for Higher Order Polyhedral Discrete and Differential Inclusions

Sevilay Demir Sağlam[a], Elimhan N. Mahmudov[b]

[a]*Istanbul University, Department of Mathematics, Istanbul, Turkey.*
[b]*Istanbul Technical University, Department of Mathematics, Istanbul, Turkey,*
Azerbaijan National Academy of Sciences, Institute of Control Systems, Baku, Azerbaijan.

Abstract. The problems considered in this paper are described in polyhedral multi-valued mappings for higher order(s-th) discrete (*PDSIs*) and differential inclusions (*PDFIs*). The present paper focuses on the necessary and sufficient conditions of optimality for optimization of these problems. By converting the *PDSIs* problem into a geometric constraint problem, we formulate the necessary and sufficient conditions of optimality for a convex minimization problem with linear inequality constraints. Then, in terms of the Euler-Lagrange type *PDSIs* and the specially formulated transversality conditions, we are able to obtain conditions of optimality for the *PDSIs*. In order to obtain the necessary and sufficient conditions of optimality for the discrete-approximation problem *PDSIs*, we reduce this problem to the form of a problem with higher order discrete inclusions. Finally, by formally passing to the limit, we establish the sufficient conditions of optimality for the problem with higher order *PDFIs*. Numerical approach is developed to solve a polyhedral problem with second order polyhedral discrete inclusions.

1. Introduction

This paper is generally concerned with discrete and differential optimization of polyhedral inclusions. In our first approach, we study the following problem labeled (*PD*) for higher order(s-th) polyhedral discrete inclusions (*PDSIs*)

$$\text{minimize} \sum_{t=s}^{T-1} f(x_t, t), \tag{1}$$

(*PD*) subject to

$$x_{t+s} \in F(x_t, x_{t+1}, \dots, x_{t+s-1}) \quad t = 0, 1, \dots, T-s, \tag{2}$$
$$x_k = \tilde{\theta}_k, \quad k = 0, 1, \dots, s-1,$$

$$F(x, v_1, v_2, \dots, v_{s-1}) = \{v_s : P_0 x + \sum_{r=1}^{s-1} P_r v_r - Q v_s \le d\}, \tag{3}$$

2010 *Mathematics Subject Classification*. Primary 49K15; Secondary 49J52, 49M25, 90C25, 34A60
Keywords. Polyhedral, Higher-order, Differential inclusions, Euler-Lagrange, Transversality.
Received: 23 January 2020; Revised: 31 March 2020; Accepted: 02 April 2020
Communicated by Calogero Vetro
Email addresses: sevilay.demir@istanbul.edu.tr (Sevilay Demir Sağlam), elimhan22@yahoo.com (Elimhan N. Mahmudov)

where $F : (\mathbb{R}^n)^s \to P(\mathbb{R}^n)$ is a polyhedral multi-valued mapping, $P(\mathbb{R}^n)$ is a set of all non-empty subsets of $\mathbb{R}^n$. In addition, $P_0, P_1, \dots, P_{s-1}$ are $m \times n$ dimensional matrices with rows $P_0^i, P_1^i, \dots, P_{s-1}^i$ $(i = 1, 2, \dots, m)$ and Q is $m \times n$ dimensional matrix with rows Q_i, $(i = 1, 2, \dots, m)$. Let d be a m-dimensional column vector with d_i, $i = 1, 2, \dots, m$ and $\tilde{\theta}_k$, $k = 0, 1, \dots, s-1$ are fixed vectors. Moreover $f(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ is a polyhedral function, i.e., epi $f(\cdot, t) \subset \mathbb{R}^{n+1}$ is a polyhedral set.

The problem is to find a sequence of vectors $\{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T\} \equiv \{\tilde{x}_t\}_{t=0}^T$ of problem (1) − (3) that minimizes $\sum_{t=s}^{T-1} f(x_t, t)$. To this end, we will deduce necessary and sufficient conditions of optimality for the solution $\{\tilde{x}_t\}_{t=0}^T$ to the problem (*PD*).

In the second part of the paper the problem (*PC*) given by s-th order polyhedral differential inclusions (*PDFIs*) is studied:

$$\text{minimize} \quad J(x(\cdot)) = \int_0^1 f(x(t), t)dt + \varphi_0\Big(x(1), x'(1), \dots, x^{(s-1)}(1)\Big), \tag{4}$$

(*PC*) subject to

$$x^{(s)}(t) \in F\Big(x(t), x'(t), \dots, x^{(s-1)}(t)\Big) \quad \text{a.e.} \quad t \in [0, 1], \tag{5}$$

$$x^{(k)}(0) = \theta_k, \quad k = 0, 1, \dots, s-1. \tag{6}$$

Here F is a multi-valued mapping, $f(\cdot, t)$ and $\varphi_0 : \mathbb{R}^{sn} \to \mathbb{R}$ are polyhedral functions.

It is necessary to find the solution $\tilde{x}(t)$ of Cauchy problem (*PC*) for *PDFIs* satisfying (5) almost everywhere (a.e.) on $[0, 1]$ and the initial conditions (6) which minimize the Bolza functional $J(x(\cdot))$. A feasible trajectory $x(t)$, $t \in [0, 1]$ is an absolutely continuous function together with the $s-1$ order derivatives for which $x^{(s)}(t) \in L_1^n([0, 1])$. Clearly such a class of functions is Banach space, endowed with different equivalent norms.

Convex optimization has a wide range of applications in many areas, such as combinatorial optimization and global optimization, where it is used to find bounds on optimal value as well as approximate solutions. However, it is commonly used in the fields of economy and engineering, electronic process automation, automatic control systems and optimum design problems in electrical, chemical, mechanical and aerospace engineering [1], [7], [8], [10], [12], [30]-[35].

In the study of the so-called Von Neumann economic dynamics model, the graph of which is a polyhedral cone, is the main application of mathematical methods to economic problems [14]. The emphasis of studies related to the Von Neumann model has recently been noted that the stochastic version of the Von Neumann system can be applied successfully to the study of fundamental problems in mathematical finance. In addition, the problems (*PD*) and (*PC*) can be extended to the linear discrete or linear differential optimal control problem where the control domain is a polyhedral set.

Optimal control theory is considered to be one of the key areas for the application of differential inclusions [2], [4], [6], [9], [13], [28], [29]. Essentially, the optimal control problems with ordinary and partial differential inclusions consist of intensive areas of development in the applied mathematical theory of analysis. Note that the reader can consult on the various problems described in the multi-valued mappings [15]-[27], [29], [**?**], [37].

The problem with *PDSIs* and *PDFIs* considered in this paper is more complicated due to the higher order discrete approximation problem associated with s-th order difference operators expressed by binomial coefficient. As a result, it is very difficult to establish an adjoint *PDSIs* and *PDFIs*. The discretization method is important in what follows to avoid this difficulty in polyhedral optimization problems with higher order

derivatives. Optimization of higher order discrete and differential inclusions were first developed by Mahmudov [20], [26] and [24]. To the best of our knowledge, there is no paper that considers the conditions of optimality for these problems but only the qualitative problems of second order differential inclusions. Most of them have been the subject of different mathematical competitions, for example many of the second order papers *PDFIs* concern the existence or viability of the results over the last few years (see [3], [5], [11], [38] and references therein).

Therefore, this paper discusses a specific type of optimization problem in which the constraints are defined by the *PDSI* and *PDFI*. Conditionally, the paper can be divided into three parts; in the first part, the optimization of the s-th order *PDSIs* is investigated; in the second part, the optimization of the s-th order *PDFIs* is studied. The third part of the paper deals with the s-th order discrete approximation problem, which allows us to bridge the gap between *PDSI* and *PDFI* problems.

The paper is organized as follows:

In Section 2, by reducing s-th order discrete polyhedral optimization problem (*PD*) into a problem with geometric constraints and by applying Farkas Theorem 1.13 [15, p.22], we formulate the necessary and sufficient conditions of optimality for a convex minimization problem with linear inequality constraints. In addition, in terms of the Euler-Lagrange polyhedral discrete inclusions (*ELPDSIs*) and the derived transversality conditions, we conclude conditions of optimality for the s-th order *PDSIs*.

In Section 3 at first using the discretization method, i.e., s-th order difference operators expressed by binomial coefficients, we define s-th order discrete-approximation problem (*PDA*) associated with s-th order polyhedral optimization problem (*PC*). Then by applying the Theorem 2.2 to the (*PDA*) problem and converting this problem to the form of (*PD*), we derive the necessary and sufficient conditions of optimality for the s-th order discrete-approximation problem (*PDA*). Note that the special proven equivalence Theorem 3.1 for subdifferential inclusions, which plays an important role in constructing conditions of optimality for the (*PDA*) problem, is necessary for the transition to the (*PDA*) problem.

In Section 4, by passing the limit procedure as a discrete step tends to be zero, we establish sufficient conditions of optimality for the *PDFIs*.

Some interesting application of Theorem 2.2 is described in Section 5. Namely, the necessary and sufficient conditions of optimality for second order polyhedral discrete inclusions are derived. In particular, it has been shown that this method can also play an important role in numerical procedures for computing the numerical solution.

2. Convex Mathematical Programming and The Problem With PDSIs

In this section, based on convex mathematical programming, we study the optimization of the s-th order *PDSIs* problem. In what follows, the following lemma plays a key role in the optimization of the s-th order problem with *PDSIs*.

Lemma 2.1. *Let M_t be a polyhedral set that is defined as*

$$M_t = \Big\{w = (x_0, \dots, x_T) : \sum_{k=0}^{s-1} P_k x_{t+k} - Q x_{t+s} \le d\Big\},\ t = 0, \dots, T-s.$$

Then

$$K^*_{M_t}(\tilde{w}) = \Big\{w^*(t) : x^*_{t+k}(t) = -P^*_k \lambda_t,\ k = 0, 1, \dots, s-1,\ \lambda_t \ge 0,\ \lambda_t \in \mathbb{R}^m, x^*_{t+s}(t) = Q^* \lambda_t,$$

$$x_l^* = 0\ ,\ l \neq t, t+1, \dots, t+s, \quad \Big\langle \sum_{k=0}^{s-1} P_k \widetilde{x}_{t+k} - Q\widetilde{x}_{t+s} - d\ ,\ \lambda_t \Big\rangle = 0, \quad t = 0, 1, \dots, T-s \Big\},$$

where $\tilde{w} = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$ *and* $K^*_{M_t}(\tilde{w})$ *is the dual cone of tangent directions.*

Proof. By the definition of the tangent directions, we obtain

$$K_{M_t}(\tilde{w}) = \Big\{ \overline{w} : \sum_{k=0}^{s-1} P_k(\tilde{x}_{t+k} + \mu\overline{x}_{t+k}) - Q(\tilde{x}_{t+s} + \mu\overline{x}_{t+s}) \leq d \text{ for a small } \mu > 0 \Big\},$$

$t = 0, 1, \dots, T-s$, where $\widetilde{x}_{t+s} \in F(\widetilde{x}_t, \widetilde{x}_{t+1}, \dots, \widetilde{x}_{t+s-1})$ is satisfied and follows from this formula that the following inequalities hold

$$\sum_{k=0}^{s-1} P_k^i(\tilde{x}_{t+k} + \mu\overline{x}_{t+k}) - Q_i(\tilde{x}_{t+s} + \mu\overline{x}_{t+s}) \leq d_i\ ,\ t = 0, \dots, T-s$$

as

$$\sum_{k=0}^{s-1} P_k^i \overline{x}_{t+k} - Q_i \overline{x}_{t+s} \leq 0\ , \quad i \in I(\tilde{w}) = \Big\{ i : \sum_{k=0}^{s-1} P_k^i \tilde{x}_{t+k} - Q_i \tilde{x}_{t+s} = d_i\ ,\ i = 1, \dots, m \Big\}. \tag{7}$$

It is easy to see that the inequalities before (7) hold strongly for small μ, regardless of choosing $(\overline{x}_t, \overline{x}_{t+1}, \dots, \overline{x}_{t+s})$ if i is not active indices, i.e., $i \notin I(\tilde{w})$. Then, because of the arbitrary nature of $\overline{x}_l$, $l \neq t, t+1, t+2, \dots, t+s$, applying the Farkas Theorem 1.13 (see, for example, [15, p.22]) it follows from the inequalities (7) that $w^*(t) = (x_0^*(t), x_1^*(t), \dots, x_T^*(t)) \in K^*_{M_t}(\tilde{w})$ if and only if

$$x^*_{t+k}(t) = -\sum_{i \in I(\tilde{w})} P_k^{i*} \lambda_i^t, \quad k = 0, 1, \dots, s-1, \quad x^*_{t+s}(t) = \sum_{i \in I(\tilde{w})} Q_i^* \lambda_i^t, \quad \lambda_i^t \geq 0, \tag{8}$$

where P_k^{i*}, Q_i^* are transposed vectors of $P_{k'}^i Q_i, k = 0, 1, \dots, s-1$ respectively. Finally, taking $\lambda_i^t = 0$ for $i \notin I(\tilde{w})$ and denoting λ_t for a vector with λ_i^t components, we have the desired result. Only that can be taken into account here

$$\Big\langle \sum_{k=0}^{s-1} P_k \widetilde{x}_{t+k} - Q\widetilde{x}_{t+s} - d\ ,\ \lambda_t \Big\rangle = 0\ , \quad t = 0, \dots, T-s. \qquad \square$$

Now let's convert the (PD) problem to a convex mathematical problem. Suppose A is partitioned into submatrices $P_0, P_1, \dots, P_{s-1}, -Q$ and $m \times n$ zero matrices 0 and D is $m(T-s+1)$ dimensional column vector. Obviously A is a matrix with a size of $m(T-s+1) \times n(T+1)$, i.e.,

$$A = \begin{pmatrix} P_0 & P_1 & \cdots & P_{s-1} & -Q & 0 & 0 & \cdots & \cdots & 0 \\ 0 & P_0 & P_1 & \cdots & P_{s-1} & -Q & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & P_0 & P_1 & \cdots & P_{s-1} & -Q \end{pmatrix}, \quad D = \begin{pmatrix} d \\ \vdots \\ \vdots \\ d \end{pmatrix}.$$

In addition M and N_k are defined as follows: $M = \{w = (x_0, \dots, x_T) : Aw \leq D\}$ and $N_k = \{w = (x_0, \dots, x_T) : x_k = \tilde{\theta}_k\}$, $k = 0, 1, \dots, s-1$.

Then it's not hard to see that the problem posed by (PD) can be transformed into a convex mathematical programming problem:

$$\text{minimize} \quad g(w) = \sum_{t=s}^{T-1} f(x_t, t)\ ,\ w = (x_0, x_1, \dots, x_T) \in \mathbb{R}^{n(T+1)} \tag{9}$$

$$\text{subject to} \quad w \in M \cap N_k\ , k = 0, 1, \dots, s-1, \quad M = \bigcap_{t=0}^{T-s} M_t.$$

This transformation allows us to rigorously prove that if $\{\tilde{x}_t\}_{t=0}^{T}$ is the optimal solution to the problem (1)-(3), then $\widetilde{w}$ is the solution to the problem (9).

Let $K_M^*(\tilde{w}) = \{w^* : \langle \overline{w}, w^* \rangle \geq 0, \ \forall \ \overline{w} \in K_M(\tilde{w})\}$ is the dual cone to the cone of tangent directions $K_M(\tilde{w}) = \{\overline{w} = (\overline{x}_0, \ldots, \overline{x}_T) : \ \tilde{w} + \mu\overline{w} \in M \ , \ \mu \geq 0\}$. Cone of tangent directions $K_{M_t}(\tilde{w})$, $t = 0, \ldots, T-s$, are polyhedral cones and so by Lemma 1.22 [15, p.23], we have $K_M^*(\tilde{w}) = \sum_{t=0}^{T-s} K_{M_t}^*(\tilde{w})$. By Theorem 3.4 [15, p.99] there exist vectors $w_g^* \in \partial_w g(\tilde{w})$, $w_M^* \in K_M^*(\tilde{w})$ and $w_k^* \in K_{N_k}^*(\tilde{w})$, $k = 0, 1, \ldots, s-1$ such that

$$w_g^* = \sum_{k=0}^{s-1} w_k^* + \sum_{t=0}^{T-s} w_{M_t}^*, \quad w_{M_t}^* \in K_{M_t}^*(\tilde{w}). \tag{10}$$

This means that the $\tilde{w}$ solution of the problem (9) has a representation (10) and vice versa. Clearly $w_g^* = (x_{g0}^*, x_{g1}^*, \ldots, x_{gT}^*) \in \partial_w g(\tilde{w})$ implies that $x_{gt}^* \in \partial_x f(\tilde{x}_t, t)$, $t = s, \ldots, T$.
Besides the definition of the tangent direction cone $K_{N_k}(w) = \{\overline{w} = (\overline{x}_0, \ldots, \overline{x}_T) : \overline{x}_k = 0\}$ and we have

$$K_{N_k}^*(w) = \left\{w^* = (x_0^*, \ldots, x_T^*) : \ x_t^* = 0, t \neq k\right\}, \ k = 0, 1, \ldots, s-1. \tag{11}$$

The main effort in this section is to formulate the conditions of optimality for the (*PD*) problem. We give the following valuable theorem to achieve this goal.

Theorem 2.2. *For optimality of the trajectory $\{\tilde{x}(t)\}_{t=0}^{1}$ in the problem (PD) with PDSIs, it is necessary and sufficient that there are vectors x_t^*, $t = 0, \ldots, T-1$ not all equal to zero satisfying the ELPDSIs*

$$\begin{gathered}
x_t^* = \sum_{k=0}^{s-1} P_k^* \lambda_{t-k} + u_t^* \ , \quad u_t^* \in \partial f(\tilde{x}_t, t), \ \ t = 0, \ldots, T-s, \\
\partial f(\tilde{x}_k, k) = \{0\}, \ k = 0, 1, \ldots, s-1, \\
x_{t+s}^* = Q^* \lambda_t, \ \lambda_t \geq 0, \ \lambda_{-r} \equiv 0 \ , \ r = 1, 2, \ldots, s-1, \\
\left\langle \sum_{k=0}^{s-1} P_k \tilde{x}_{t+k} - Q\tilde{x}_{t+s} - d \ , \ \lambda_t \right\rangle = 0,
\end{gathered}$$

and transversality conditions

$$\begin{gathered}
x_{T-s+r}^* - \sum_{j=0}^{s-1-r} P_{r+j}^* \lambda_{T-s-j} \ \in \ \partial f(\tilde{x}_{T-s+r} \ , \ T-s+r), \ r = 1, 2, \ldots, s-1, \\
x_T^* = 0.
\end{gathered}$$

Proof. According to the formula (11) and Proposition 2.1, we can write

$$w_0^* = (x_{00}^*, 0, \ldots, 0), \ w_1^* = (0, x_{11}^*, 0 \ldots, 0) \ , \ldots, \ w_{s-1}^* = (0, 0, \ldots, x_{(s-1)(s-1)}^*, 0, \ \ldots \ , 0),$$

$$w^*(t) = (0, \ldots, 0, x_t^*(t), x_{t+1}^*(t), \ldots, x_{t+s}^*(t), 0, \ldots, 0), \ t = 0, \ldots, T-s.$$

Now, using the latter relationship from (10) we have

$$\begin{gathered}
x_{00}^* + x_0^*(0) = 0, \\
x_{11}^* + x_1^*(0) + x_1^*(1) = 0, \\
\ldots\ldots\ldots\ldots\ldots \\
x_{(s-1)(s-1)}^* + x_{s-1}^*(0) + x_{s-1}^*(1) + \cdots + x_{s-1}^*(s-1) = 0,
\end{gathered} \tag{12}$$

$$x^*_{gt} = x^*_t(t-s) + x^*_t(t-(s-1)) + x^*_t(t-(s-2)) + \cdots + x^*_t(t)\ , \quad t = s, \ldots, T-s, \tag{13}$$

$$\sum_{j=0}^{s-r} x^*_{T-s+r}(T-s-j) = \ x^*_{g(T-s+r)}\ , \quad r = 1, 2, \ldots, s-1,$$

$$x^*_T(T-s) \ = \ 0. \tag{14}$$

For convenience, by denoting $x^*_{t+s}(t) \equiv x^*_{t+s},\ t = 1, \ldots, T-s,\ x^*_{gt} \equiv u^*_t$ and taking into account the formula (13) and the Proposition 2.1, we deduce the ELPDSIs of theorem

$$x^*_t = \sum_{k=0}^{s-1} P^*_k \lambda_{t-k} + u^*_t, \quad u^*_t \in \partial f(\tilde{x}_t, t),\ x^*_{t+s} = Q^* \lambda_t \quad ,\ \lambda_t \geq 0\ , \quad t = 0, \ldots, T-s,$$

under the condition

$$\Big\langle \sum_{k=0}^{s-1} P_k \tilde{x}_{t+k} - Q\tilde{x}_{t+s} - d,\ \lambda_t \Big\rangle = 0.$$

In addition, by virtue of (12) and setting $f(\tilde{x}_k, k) \equiv 0,\ x^*_{kk} \equiv x^*_k,\ k = 0, 1, \ldots, s-1,\ \lambda_{-r} \equiv 0,\ r = 1, 2, \ldots, s-1$, the formula (13) remains true for $t = 0, 1, \ldots, s-1$. On the other hand, using the first relationship of (14) we have

$$x^*_{T-s+r} - \sum_{j=0}^{s-1-r} P^*_{r+j} \lambda_{T-s-j} \ \in \ \partial f\Big(\tilde{x}_{T-s+r}, T-s+r\Big),\ r = 1, 2, \ldots, s-1.$$

Since $\partial f(\tilde{x}_T, T) \equiv \{0\}$ indicates that $x^*_T = 0$. □

3. Necessary and Sufficient Conditions of Optimality For Higher Order Polyhedral Discrete-Approximation Problem

Let us introduce, first of all, the following s-th order difference operators

$$\Delta^s x(t) = \frac{1}{\delta^s}\Big(\sum_{j=0}^{s} (-1)^j \ \mathrm{C}^j_s \ x(t + (s-j)\delta)\Big), \quad t = 0, \delta, \ldots, 1-\delta,$$

where δ is a step on the t-axis, $x(t)$ is a grid function on $[0, 1]$ and $\mathrm{C}^j_s = \binom{s}{j} = \dfrac{s!}{j!(s-j)!}$ is a binomial coefficient.

Let us explain the main method that we use to obtain the sufficient conditions of optimality for the (PC) problem. This is a direct method based on discrete approximations. Therefore, the basic idea is to substitute the continuous problem (PC) with a discrete-approximation problem that can be effectively tested. Then, by formally passing the limit on the discrete-approximation problem, we formulate sufficient conditions of optimality for the original problem with s-th order derivatives. As a result, according to the problem (PC), we associate the following s-th order discrete approximation problem (PDA):

$$\text{minimize } \sum_{t=0}^{1-s\delta} \delta f(x(t), t) + \varphi_0\Big(x(1-(s-1)\delta), \Delta x(1-(s-1)\delta), \ldots, \Delta^{s-1} x(1-(s-1)\delta)\Big),$$

subject to

$$(PDA) \qquad \sum_{k=0}^{s-1} P_k \Delta^k x(t) - Q\Delta^s x(t) \leq d, \quad t = 0, \delta, \ldots, 1-s\delta,$$

$$\Delta^k x(0) = \theta_k\ ,\ k = 0, 1, \ldots, s-1. \tag{15}$$

The method used in this paper requires some special equivalence theorem, which allows us to bridge the gap between (*PD*) and (*PC*).

Theorem 3.1. *Suppose* $\phi : (\mathbb{R}^n)^s \to \mathbb{R}$ *is a function defined by the relationship* $\phi(x, v_1, \dots, v_{s-1}) \equiv \varphi_0\Big(x, \eta_1, \dots, \eta_{s-1}\Big)$ *where* $\eta_r = \frac{1}{\delta^r}\Big[\sum_{j=0}^{r-1}(-1)^j\binom{r}{j}v_{r-j} + (-1)^r x\Big]$, $r = 1, 2, \dots, s-1$, $(x^0, v_1^0, \dots, v_{s-1}^0) \in dom\phi$, $(x^0, \eta_1^0, \dots, \eta_{s-1}^0) \in dom\varphi_0$. *The following subdifferential inclusions are equivalent:*

$$(i) \qquad \Big(\overline{x}^*,\ \overline{v}_1^*, \dots,\ \overline{v}_{s-1}^*\Big) \in \partial\phi\Big(x^0, v_1^0, \dots, v_{s-1}^0\Big)$$

$$(ii) \qquad \Big(\overline{x}^* + \sum_{j=1}^{s-1}\overline{v}_j{}^*,\ v_1^*, v_2^*, \dots, v_{s-1}^*\Big) \in \partial\varphi_0\Big(x^0, \eta_1^0, \dots, \eta_{s-1}^0\Big)$$

where $v_r^* = \delta^r\Big[\sum_{j=0}^{s-(r+1)}\binom{j+r}{j}\overline{v}_{r+j}^*\Big]$, $r = 1, 2, \dots, s-1$.

Proof. Clearly $\partial_z\phi(z_0)$ is a convex closed set and for $z_0 = (x^0, v_1^0, \dots, v_{s-1}^0) \in ri(dom\phi)$ is bounded. Let's denote $y_0 = (x^0, \eta_1^0, \eta_2^0, \dots, \eta_{s-1}^0)$, by the classical definition of subdifferential sets we get

$$\partial_z\phi(z_0) = \Big\{\Big(\overline{x}^*, \overline{v}_1^*, \dots, \overline{v}_{s-1}^*\Big) : \phi\Big(z\Big) - \phi\Big(z_0\Big) \geq \Big\langle\overline{x}^*, x - x^0\Big\rangle + \Big\langle\overline{v}_1{}^*, v_1 - v_1^0\Big\rangle$$

$$+ \cdots + \Big\langle\ \overline{v}_{s-1}^*, v_{s-1} - v_{s-1}^0\Big\rangle,\ \forall z = (x, v_1, \dots, v_{s-1}) \in \mathbb{R}^{sn}, z_0 \in dom\phi\ \Big\}, \tag{16}$$

$$\partial\varphi_0\Big(y_0\Big) = \Big\{\Big(x^*, v_1{}^*, \dots, v_{s-1}^*\Big) : \varphi_0\Big(y\Big) - \varphi_0\Big(y_0\Big) \geq \Big\langle x^*, x - x^0\Big\rangle$$

$$+ \sum_{j=1}^{s-1}\Big\langle v_j^*\ ,\ \frac{1}{\delta^j}\Big(\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}\Big(v_{j-i} - v_{j-i}^0\Big) + (-1)^j\Big(x - x^0\Big)\Big)\Big\rangle\ ,\forall\ y \in \mathbb{R}^{sn}\ \Big\}.$$

The last relation for $\partial\varphi_0\Big(y_0\Big)$ implies,

$$\partial\varphi_0\Big(y_0\Big) = \Big\{\Big(x^*, v_1{}^*, \dots, v_{s-1}^*\Big) : \varphi_0\Big(y\Big) - \varphi_0\Big(y_0\Big) \geq \Big\langle x^* + \sum_{j=1}^{s-1}\frac{(-1)^j}{\delta^j}v_j{}^*\ ,\ x - x^0\Big\rangle$$

$$+\Big\langle\sum_{j=0}^{s-2}\frac{(-1)^j}{\delta^{j+1}}\binom{j+1}{j}v_{j+1}^*\ ,\ v_1 - v_1^0\Big\rangle + \Big\langle\sum_{j=0}^{s-3}\frac{(-1)^j}{\delta^{j+2}}\binom{j+2}{j}v_{j+2}^*\ ,\ v_2 - v_2^0\Big\rangle$$

$$+ \cdots + \Big\langle\sum_{j=0}^{1}\frac{(-1)^j}{\delta^{j+s-2}}\binom{j+s-2}{j}v_{j+s-2}^*\ ,\ v_{s-2} - v_{s-2}^0\Big\rangle + \Big\langle\frac{v_{s-1}^*}{\delta^{s-1}}\ ,\ v_{s-1} - v_{s-1}^0\Big\rangle,\ \forall\ y \in \mathbb{R}^{sn}\ \Big\}.$$

Rewriting this inequality, we have

$$\partial\varphi_0\Big(y_0\Big) = \Big\{\Big(x^*, v_1{}^*, \dots, v_{s-1}^*\Big) : \varphi_0\Big(y\Big) - \varphi_0\Big(y_0\Big) \geq \Big\langle x^* + \sum_{j=1}^{s-1}\frac{(-1)^j}{\delta^j}v_j{}^*, x - x^0\Big\rangle$$

$$+ \sum_{r=1}^{s-1}\Big\langle\sum_{j=0}^{s-(r+1)}\frac{(-1)^j}{\delta^{j+r}}\binom{j+r}{j}v_{j+r}^*\ ,\ v_r - v_r^0\Big\rangle, \quad \forall y \in \mathbb{R}^{sn}\ \Big\}. \tag{17}$$

On the basis of (16) and (17), it can be claimed that

$$\overline{x}^* = x^* + \sum_{j=1}^{s-1} \frac{(-1)^j}{\delta^j} v_j{}^*, \quad \overline{v}_r{}^* = \sum_{j=0}^{s-(r+1)} \frac{(-1)^j}{\delta^{j+r}} \binom{j+r}{j} v^*_{j+r}, \quad r = 1, 2, \dots, s-1. \tag{18}$$

Now, starting with the last equation, by sequentially substituting in (18), we derive

$$x^* = \overline{x}^* + \sum_{j=1}^{s-1} \overline{v}_j{}^*, \quad v_r{}^* = \delta^r \Big[\sum_{j=0}^{s-(r+1)} \binom{j+r}{j} \overline{v}^*_{r+j} \Big], \quad r = 1, 2, \dots, s-1. \qquad \square$$

Theorem 3.2. *For optimality of the trajectory $\{\tilde{x}(t)\}_{t=0}^{1}$ in the problem (PDA), it is necessary and sufficient that there is an adjoint trajectory of vectors $\{x^*(t)\}_{t=0}^{1}$ simultaneously not all equal to zero satisfying the approximate ELPDSIs*

$$(1)\ \ (-1)^s \Delta^s x^*(t) \in \sum_{k=0}^{s-1} (-1)^k P_k^* \Delta^k \lambda(t - k\delta) + \partial f(\tilde{x}(t), t),$$

$$(2)\ \ \Big\langle \sum_{k=0}^{s-1} P_k \Delta^k \tilde{x}(t) - Q\Delta^s \tilde{x}(t) - d\ ,\ \lambda(t) \Big\rangle = 0, \quad \lambda(t) \geq 0, \quad t = s\delta, \dots, 1 - s\delta,$$

and transversality condition

$$(3)\ \ \Big(\xi_1, \xi_2, \dots, \xi_s\Big) \in \partial\varphi_0\Big(\tilde{x}(1-(s-1)\delta),\ \Delta\tilde{x}(1-(s-1)\delta), \dots,\ \Delta^{s-1}\tilde{x}(1-(s-1)\delta)\Big),$$

$$\xi_k = (-1)^{k+s} \Delta^{s-k} x^*(1-(s-k)\delta) + \sum_{j=1}^{s-k} (-1)^j P_j^* \Delta^{j-1} \lambda(1-(j+s-k)\delta)\ ,\ k = 1, 2, \dots, s.$$

Proof. We use the result of Theorem 2.2 to formulate conditions of optimality for problem (15), that's why we transform this problem into a (*PD*) form problem:

$$\text{minimize } \sum_{t=0}^{1-s\delta} \delta f\Big(x(t), t\Big) + \varphi_0\Big(\tilde{x}(1-(s-1)\delta),\ \Delta\tilde{x}(1-(s-1)\delta), \dots,\ \Delta^{s-1}\tilde{x}(1-(s-1)\delta)\Big),$$

subject to

$$\Big[\sum_{k=0}^{s-1} (-1)^k \delta^{s-k} P_k - (-1)^s Q\Big] x(t) + \Big[\sum_{k=1}^{s-1} (-1)^{k+1} \binom{k}{1} \delta^{s-k} P_k + (-1)^s \binom{s}{1} Q\Big] x(t+\delta)$$

$$+ \Big[\sum_{k=2}^{s-1} (-1)^k \binom{k}{2} \delta^{s-k} P_k - (-1)^s \binom{s}{2} Q\Big] x(t+2\delta) + \dots + \Big[\sum_{k=s-1}^{s-1} (-1)^{k+s+1} \binom{k}{s-1} \delta^{s-k} P_k$$

$$+ \binom{s}{s-1} Q\Big] x(t+(s-1)\delta) - Qx(t+s\delta) \leq \delta^s d\ ,\ \ t = 0, \delta, \dots, 1 - s\delta,$$

$$x(k\delta) = \sum_{j=0}^{k} \binom{k}{j} \delta^j \theta_j, \quad k = 0, 1, \dots, s-1. \tag{19}$$

Let $\{\tilde{x}(t)\},\ t = 0, \delta, \dots, 1$ be the optimal solution to the problem (19). It is not hard to see that the adjoint discrete inclusions for s-th order polyhedral problems have the forms

$$x^*(t) \in \Big[\sum_{k=0}^{s-1}(-1)^k\delta^{s-k}P_k - (-1)^sQ\Big]^*\lambda(t) + \Big[\sum_{k=1}^{s-1}(-1)^{k+1}\binom{k}{1}\delta^{s-k}P_k + (-1)^s\binom{s}{1}Q\Big]^*\lambda(t-\delta)$$

$$+\Big[\sum_{k=2}^{s-1}(-1)^k\binom{k}{2}\delta^{s-k}P_k - (-1)^s\binom{s}{2}Q\Big]^*\lambda(t-2\delta) + \cdots + \Big[\delta P_{s-1} + sQ\Big]^*\lambda(t-(s-1)\delta) + \delta\partial f(\tilde{x}(t),t),$$

$$t = 0, \delta, \dots, 1-\delta, \quad x^*(t+s\delta) = Q^*\lambda(t)\ ,\ \lambda(t) \geq 0, \tag{20}$$

$$\Big\langle \Big[\sum_{k=0}^{s-1}(-1)^k\delta^{s-k}P_k - (-1)^sQ\Big]\widetilde{x}(t) + \Big[\sum_{k=1}^{s-1}(-1)^{k+1}\binom{k}{1}\delta^{s-k}P_k + (-1)^s\binom{s}{1}Q\Big]\tilde{x}(t+\delta)$$

$$+\Big[\sum_{k=2}^{s-1}(-1)^k\binom{k}{2}\delta^{s-k}P_k - (-1)^s\binom{s}{2}Q\Big]\tilde{x}(t+2\delta) + \cdots + \Big[\delta P_{s-1} + sQ\Big]\widetilde{x}(t+(s-1)\delta)$$

$$-Q\tilde{x}(t+s\delta) - \delta^s d\ \ ,\ \lambda(t)\Big\rangle = 0. \tag{21}$$

Rewriting the inclusion (20), we have

$$\Big(x^*(t) + (-1)^sQ^*\lambda(t) - (-1)^s\binom{s}{1}Q^*\lambda(t-\delta) + (-1)^s\binom{s}{2}Q^*\lambda(t-2\delta) + \cdots + (-s)Q^*\lambda(t-(s-1)\delta)\Big)$$

$$\in \Big[\sum_{k=0}^{s-1}(-1)^k\delta^{s-k}P_k^*\Big]\lambda(t) + \Big[\sum_{k=1}^{s-1}(-1)^{k+1}\binom{k}{1}\delta^{s-k}P_k^*\Big]\lambda(t-\delta) + \cdots + \delta P_{s-1}^*\lambda(t-(s-1)\delta) + \delta\partial f(\tilde{x}(t),t).$$

Recall that $x^*(t+s\delta) = Q^*\lambda(t)$, then we deduce from the last inclusion that

$$\Big(x^*(t) + (-1)^sx^*(t+s\delta) - (-1)^s\binom{s}{1}x^*(t+(s-1)\delta) + (-1)^s\binom{s}{2}x^*(t+\delta(s-2)) + \cdots + (-s)x^*(t+\delta)\Big)$$

$$\in\ \delta^sP_0^*\lambda(t) + \delta^{s-1}\Big[-\lambda(t) + \lambda(t-\delta)\Big]P_1^* + \delta^{s-2}\Big[\lambda(t) - 2\lambda(t-\delta) + \lambda(t-2\delta)\Big]P_2^*$$

$$+\cdots + \delta\Big[(-1)^{s-1}\lambda(t) + (-1)^s\binom{s-1}{1}\lambda(t-\delta) - (-1)^s\binom{s-1}{2}\lambda(t-2\delta)$$

$$+\cdots + \lambda(t-(s-1)\delta)\Big]P_{s-1}^* + \delta\partial f(\tilde{x}(t),t).$$

Dividing the left hand side and the right hand side of this inclusion by δ^s (here $\delta^{s-1}x^*(t)$, $\delta^{s-1}\lambda(t)$ are again denoted by $x^*(t)$ and $\lambda(t)$ respectively), we obtain

$$(-1)^s\Delta^sx^*(t)\ \in\ \sum_{k=0}^{s-1}(-1)^kP_k^*\Delta^k\lambda(t-k\delta) + \partial f(\tilde{x}(t),t).$$

Similarly, the equation (21) implies

$$\left\langle \sum_{k=0}^{s-1} P_k \Delta^k \tilde{x}(t) - Q\Delta^s \tilde{x}(t) - d \ , \ \lambda(t) \right\rangle = 0 \ , \ \lambda(t) \geq 0, \ t = s\delta, \dots, 1 - s\delta.$$

Then, by virtue of Theorem 2.2, the transversality conditions of problem (19) are derived:

$$x^*(1-(s-1)\delta) - \sum_{j=0}^{s-2}\Big[\sum_{k=1+j}^{s-1}(-1)^{k+j+1}\binom{k}{1+j}\delta^{s-k}P_k^* + (-1)^{s-j}\binom{s}{1+j}Q^*\Big]\lambda(1-(s+j)\delta)$$

$$\in \ \delta\partial f(\tilde{x}(1-(s-1)\delta), 1-(s-1)\delta),$$

$$x^*(1-(s-2)\delta) - \sum_{j=0}^{s-3}\Big[\sum_{k=2+j}^{s-1}(-1)^{k+j+2}\binom{k}{2+j}\delta^{s-k}P_k^* + (-1)^{s-j-1}\binom{s}{2+j}Q^*\Big]\lambda(1-(s+j)\delta)$$

$$\in \ \delta\partial f(\tilde{x}(1-(s-2)\delta), 1-(s-2)\delta),$$

$$\dots\dots\dots\dots\dots$$

$$x^*(1-2\delta) - \sum_{j=0}^{1}\Big[\sum_{k=s-2+j}^{s-1}(-1)^{k+j+s}\binom{k}{s-2+j}\delta^{s-k}P_k^* + (-1)^{1-j}\binom{s}{s-2+j}Q^*\Big]\lambda(1-(s+j)\delta)$$

$$\in \ \delta\partial f(\tilde{x}(1-2\delta), 1-2\delta),$$

$$x^*(1-\delta) - \Big[\delta P_{s-1}^* + sQ^*\Big]\lambda(1-s\delta) \ \in \ \delta\partial f(\tilde{x}(1-\delta), 1-\delta),$$

$$x^*(1) \ \in \ \delta\partial f(\tilde{x}(1), 1). \tag{22}$$

Now, remembering that under the conditions (22) $x^*(t+s\delta) = Q^*\lambda(t)$, we deduce

$$x^*(1-(s-1)\delta) - \sum_{j=0}^{s-2}(-1)^{s-j}\binom{s}{1+j}x^*(1-j\delta) - \sum_{j=0}^{s-2}\Big[\sum_{k=1+j}^{s-1}(-1)^{k+j+1}\binom{k}{1+j}\delta^{s-k}P_k^*\Big]\lambda(1-(s+j)\delta)$$

$$\in \ \delta\partial f(\tilde{x}(1-(s-1)\delta), 1-(s-1)\delta),$$

$$x^*(1-(s-2)\delta) + \sum_{j=0}^{s-3}(-1)^{s-j}\binom{s}{2+j}x^*(1-j\delta) - \sum_{j=0}^{s-3}\Big[\sum_{k=2+j}^{s-1}(-1)^{k+j}\binom{k}{2+j}\delta^{s-k}P_k^*\Big]\lambda(1-(s+j)\delta)$$

$$\in \ \delta\partial f(\tilde{x}(1-(s-2)\delta), 1-(s-2)\delta),$$

$$\dots\dots\dots\dots\dots$$

$$x^*(1-2\delta) - \sum_{j=0}^{1}(-1)^{1-j}\binom{s}{s-2+j}x^*(1-j\delta) - \sum_{j=0}^{1}\Big[\sum_{k=s-2+j}^{s-1}(-1)^{k+j+s}\binom{k}{s-2+j}\delta^{s-k}P_k^*\Big]$$

$$\lambda(1-(s+j)\delta) \in \ \delta\partial f(\tilde{x}(1-2\delta), 1-2\delta),$$

$$x^*(1-\delta) - \delta P_{s-1}^*\lambda(1-s\delta) - sx^*(1) \ \in \ \delta\partial f(\tilde{x}(1-\delta), 1-\delta),$$

$$x^*(1) \ \in \ \delta\partial f(\tilde{x}(1), 1). \tag{23}$$

Note that the function φ_0 in the problem (19) should be defined as follows

$$\varphi_0\Big(\tilde{x}(1-(s-1)\delta),\ \Delta\tilde{x}(1-(s-1)\delta),\dots,\Delta^{s-1}\tilde{x}(1-(s-1)\delta)\Big)$$

$$\equiv \phi\Big(\tilde{x}(1-(s-1)\delta)\ ,\ \tilde{x}(1-(s-2)\delta)\ ,\dots,\tilde{x}(1)\Big)$$

$$= \delta\Big(f\Big(\tilde{x}(1-(s-1)\delta),1-(s-1)\delta\Big)+f\Big(\tilde{x}(1-(s-2)\delta),1-(s-2)\delta\Big)+\cdots+f\Big(\tilde{x}(1),1\Big)\Big)$$

and then the transversality conditions (23) have the following forms

$$\Big(x^*(1-(s-1)\delta)-\sum_{j=0}^{s-2}(-1)^{s-j}\binom{s}{1+j}x^*(1-j\delta)-\sum_{j=0}^{s-2}\Big[\sum_{k=1+j}^{s-1}(-1)^{k+j+1}\binom{k}{1+j}\delta^{s-k}P_k^*\Big]\lambda(1-(s+j)\delta),$$

$$x^*(1-(s-2)\delta)+\sum_{j=0}^{s-3}(-1)^{s-j}\binom{s}{2+j}x^*(1-j\delta)-\sum_{j=0}^{s-3}\Big[\sum_{k=2+j}^{s-1}(-1)^{k+j}\binom{k}{2+j}\delta^{s-k}P_k^*\Big]\lambda(1-(s+j)\delta)$$

$$,\ \dots\ ,\ x^*(1-\delta)-\delta P_{s-1}^*\lambda(1-s\delta)-sx^*(1)\ ,\ x^*(1)\Big)$$

$$\in \partial\phi\Big(\tilde{x}(1-(s-1)\delta)\ ,\ \tilde{x}(1-(s-2)\delta)\ ,\dots,\tilde{x}(1)\Big).$$

Dividing both sides of this inclusion again by δ^{s-1}, we have

$$\Big(-\frac{1}{\delta^{s-1}}\sum_{j=0}^{s-1}(-1)^{s-j}\binom{s}{1+j}x^*(1-j\delta)-\sum_{j=0}^{s-2}\Big[\sum_{k=1+j}^{s-1}(-1)^{k+j+1}\binom{k}{1+j}\delta^{1-k}P_k^*\Big]\lambda(1-(s+j)\delta)\ ,$$

$$\frac{1}{\delta^{s-1}}\sum_{j=0}^{s-2}(-1)^{s-j}\binom{s}{2+j}x^*(1-j\delta)-\sum_{j=0}^{s-3}\Big[\sum_{k=2+j}^{s-1}(-1)^{k+j}\binom{k}{2+j}\delta^{1-k}P_k^*\Big]\lambda(1-(s+j)\delta)\ ,$$

$$,\ \cdots\ ,\ \frac{x^*(1-\delta)-sx^*(1)}{\delta^{s-1}}-\frac{P_{s-1}^*\lambda(1-s\delta)}{\delta^{s-2}}\ ,\ \frac{x^*(1)}{\delta^{s-1}}\Big)$$

$$\in \partial\phi\Big(\tilde{x}(1-(s-1)\delta)\ ,\ \tilde{x}(1-(s-2)\delta)\ ,\dots,\ \tilde{x}(1)\Big).$$

By applying the Theorem 3.1 and using the combinatorial identity, we can express the last inclusion in the subdifferential term φ_0:

$$\Big((-1)^{1+s}\Delta^{s-1}x^*(1-(s-1)\delta)+\sum_{j=1}^{s-1}(-1)^jP_j^*\Delta^{j-1}\lambda(1-(j+s-1)\delta)\ ,$$

$$(-1)^s\Delta^{s-2}x^*(1-(s-2)\delta)+\sum_{j=1}^{s-2}(-1)^jP_j^*\Delta^{j-1}\lambda(1-(j+s-2)\delta),$$

$$,\ \dots\ ,\ -\Delta x^*(1-\delta)-P_2^*\lambda(1-3\delta)\ ,\ x^*(1)\Big)$$

$$\in \partial\varphi_0\Big(\tilde{x}(1-(s-1)\delta),\ \Delta\tilde{x}(1-(s-1)\delta)\ ,\ \dots\ ,\ \Delta^{s-1}\tilde{x}(1-(s-1)\delta)\Big).$$

The transversality condition of the problem (19) can be rewritten as follows

$$\Big(\xi_1,\xi_2,\dots,\xi_s\Big)\in\partial\varphi_0\Big(\tilde{x}(1-(s-1)\delta),\ \Delta\tilde{x}(1-(s-1)\delta),\dots,\ \Delta^{s-1}\tilde{x}(1-(s-1)\delta)\Big),$$

$$\xi_k = (-1)^{k+s}\Delta^{s-k}x^*(1-(s-k)\delta) + \sum_{j=1}^{s-k}(-1)^j P_j^*\Delta^{j-1}\lambda(1-(j+s-k)\delta)\ ,\ k=1,2,\dots,s.$$

The proof of theorem has been concluded. □

4. Sufficient Conditions of Optimality For Higher Order PDFIs

In this section, the sufficient conditions of optimality for the original problem (*PC*) are formulated. Here we use the result of Theorem 3.2 dedicated to the optimization of the s-th order *PDSIs*. Namely, by passing to the limit under the terms of Theorem 3.2, we derive the sufficient conditions of optimality for *PDFIs*.

Theorem 4.1. *Let $x(t)$ be a feasible trajectory lying interior to domF. Therefore, for optimality of $\tilde{x}(t), t\in[0,1]$ in problem (PC) with s-th order PDFIs, it is sufficient that there exists an absolutely continuous function $x^*(t)$ satisfying the following Euler-Lagrange form PDFI almost everywhere*

$$(a)\ (-1)^s\frac{d^sx^*(t)}{dt^s}\ \in\ \sum_{k=0}^{s-1}(-1)^kP_k^*\frac{d^k\lambda(t)}{dt^k}+\partial f(\tilde{x}(t),t)\ ,\ a.e.\ \ t\in[0,1],$$

$$x^*(t)=Q^*\lambda(t),$$

$$(b)\ \left\langle\sum_{k=0}^{s-1}P_k\frac{d^k\tilde{x}(t)}{dt^k}-Q\frac{d^s\tilde{x}(t)}{dt^s}-d\ ,\ \lambda(t)\right\rangle=0\ ,\ a.e.\ t\in[0,1],$$

and the transversality conditions

$$(c)\ \Big(\psi_1,\psi_2,\dots,\psi_s\Big)\in\partial\varphi_0\Big(\tilde{x}(1),\ \tilde{x}'(1),\dots,\ \tilde{x}^{(s-1)}(1)\Big),\ \ where\ \psi_s=x^*(1),$$

$$\psi_j=(-1)^{s-j}\ \frac{d^{s-j}x^*(1)}{dt^{s-j}}+\sum_{i=j}^{s-1}(-1)^{i+j-1}P_i^*\frac{d^{i-j}\lambda(1)}{dt^{i-j}}\ ,\ j=1,2,\dots,s-1.$$

Proof. By definition of subdifferential for all feasible solutions, we rewrite the inclusion (a) and the transversality condition (c) in the form, respectively

$$f(x(t),t)-f(\tilde{x}(t),t)\geq\left\langle(-1)^s\frac{d^sx^*(t)}{dt^s}-\sum_{k=0}^{s-1}(-1)^kP_k^*\frac{d^k\lambda(t)}{dt^k}\ ,x(t)-\tilde{x}(t)\right\rangle, \tag{24}$$

$$\varphi_0\Big(x(1),x'(1),\dots,x^{(s-1)}(1)\Big)-\varphi_0\Big(\tilde{x}(1),\tilde{x}'(1),\dots,\tilde{x}^{(s-1)}(1)\Big)\geq\left\langle\psi_1\ ,x(1)-\tilde{x}(1)\right\rangle+\left\langle\psi_2,x'(1)-\tilde{x}'(1)\right\rangle$$

$$+\dots+\left\langle\psi_{s-1},\ x^{(s-2)}(1)-\tilde{x}^{(s-2)}(1)\right\rangle+\left\langle x^*(1)\ ,\ x^{(s-1)}(1)-\tilde{x}^{(s-1)}(1)\right\rangle. \tag{25}$$

Let's transform the right hand side of the inequality (24) as follows

$$\left\langle(-1)^s\frac{d^sx^*(t)}{dt^s}-\sum_{k=0}^{s-1}(-1)^kP_k^*\frac{d^k\lambda(t)}{dt^k}\ ,\ x(t)-\tilde{x}(t)\right\rangle=\left\langle(-1)^s\frac{d^sx^*(t)}{dt^s}\ ,\ x(t)-\tilde{x}(t)\right\rangle$$

$$+\left\langle\sum_{k=0}^{s-1}(-1)^{k+1}\frac{d^k\lambda(t)}{dt^k}\ ,\ P_k\Big(x(t)-\tilde{x}(t)\Big)\right\rangle. \tag{26}$$

Obviously, for a feasible solution $x(\cdot)$ and $\lambda(t)\geq 0, t\in[0,1]$ it can be written

$$\left\langle\sum_{k=0}^{s-1}P_k\frac{d^kx(t)}{dt^k}\ ,\ \lambda(t)\right\rangle\leq\left\langle Q\frac{d^sx(t)}{dt^s}+d\ ,\ \lambda(t)\right\rangle.$$

By using the theorem's second condition (b), we have

$$\Big\langle \sum_{k=0}^{s-1} P_k \frac{d^k \widetilde{x}(t)}{dt^k} \ , \ \lambda(t) \Big\rangle = \Big\langle Q \frac{d^s \tilde{x}(t)}{dt^s} + d \ , \ \lambda(t) \Big\rangle.$$

Then taking into account that $x^*(t) = Q^* \lambda(t)$ from the last two relations, we derive that

$$\Big\langle \sum_{k=0}^{s-1} P_k \frac{d^k \Big(x(t) - \tilde{x}(t)\Big)}{dt^k} \ , \ \lambda(t) \Big\rangle \leq \Big\langle \frac{d^s \Big(x(t) - \tilde{x}(t)\Big)}{dt^s} \ , \ x^*(t) \Big\rangle. \tag{27}$$

Then, the relations (26) and (27) imply that

$$\Big\langle (-1)^s \frac{d^s x^*(t)}{dt^s} - \sum_{k=0}^{s-1} (-1)^k P_k^* \frac{d^k \lambda(t)}{dt^k} \ , \ x(t) - \tilde{x}(t) \Big\rangle \geq \Big\langle (-1)^s \frac{d^s x^*(t)}{dt^s} \ , \ x(t) - \tilde{x}(t) \Big\rangle$$

$$- \Big\langle \frac{d^s \Big(x(t) - \tilde{x}(t)\Big)}{dt^s} \ , \ x^*(t) \Big\rangle + \sum_{k=1}^{s-1} \Big\langle (-1)^{k+1} P_k^* \frac{d^k \lambda(t)}{dt^k} \ , \ x(t) - \tilde{x}(t) \Big\rangle + \sum_{k=1}^{s-1} \Big\langle P_k^* \lambda(t) \ , \ \frac{d^k \Big(x(t) - \tilde{x}(t)\Big)}{dt^k} \Big\rangle. \tag{28}$$

Denoting $\Omega_i(t) = \sum\limits_{k=i}^{s-1} (-1)^{k+i} P_k^* \dfrac{d^{k-i} \lambda(t)}{dt^{k-i}}$, $i = 1, 2, \ldots, s-1$, in view of the following equation

$$\sum_{k=1}^{s-1} \Big\langle (-1)^{k+1} P_k^* \frac{d^k \lambda(t)}{dt^k}, x(t) - \tilde{x}(t) \Big\rangle + \sum_{k=1}^{s-1} \Big\langle P_k^* \lambda(t), \frac{d^k \Big(x(t) - \tilde{x}(t)\Big)}{dt^k} \Big\rangle$$

$$= \sum_{i=1}^{s-1} \frac{d}{dt} \Big[\Big\langle \sum_{k=i}^{s-1} (-1)^{k+i} P_k^* \frac{d^{k-i} \lambda(t)}{dt^{k-i}} \ , \ \frac{d^{i-1} \Big(x(t) - \widetilde{x}(t)\Big)}{dt^{i-1}} \Big\rangle \Big]$$

the relation (28) can be rewritten as follows

$$\Big\langle (-1)^s \frac{d^s x^*(t)}{dt^s} - \sum_{k=0}^{s-1} (-1)^k P_k^* \frac{d^k \lambda(t)}{dt^k} \ , \ x(t) - \tilde{x}(t) \Big\rangle \geq \Big\langle (-1)^s \frac{d^s x^*(t)}{dt^s} \ , \ x(t) - \tilde{x}(t) \Big\rangle$$

$$- \Big\langle \frac{d^s \Big(x(t) - \tilde{x}(t)\Big)}{dt^s} \ , \ x^*(t) \Big\rangle + \sum_{i=1}^{s-1} \frac{d}{dt} \Big[\Big\langle \Omega_i(t) \ , \ \frac{d^{i-1} \Big(x(t) - \widetilde{x}(t)\Big)}{dt^{i-1}} \Big\rangle \Big]. \tag{29}$$

Finally, from the inequalities (24) and (29), we conclude that

$$f(x(t), t) - f(\tilde{x}(t), t) \geq \Big\langle (-1)^s \frac{d^s x^*(t)}{dt^s}, x(t) - \tilde{x}(t) \Big\rangle - \Big\langle \frac{d^s \Big(x(t) - \tilde{x}(t)\Big)}{dt^s}, x^*(t) \Big\rangle$$

$$+ \sum_{i=1}^{s-1} \frac{d}{dt} \Big[\Big\langle \Omega_i(t), \frac{d^{i-1} \Big(x(t) - \widetilde{x}(t)\Big)}{dt^{i-1}} \Big\rangle \Big]. \tag{30}$$

Recall that $x(\cdot), \tilde{x}(\cdot)$ are feasible solutions $\Big(x^{(k)}(0) = \widetilde{x}^{(k)}(0) = \theta_k, \ k = 0, 1, \ldots, s-1\Big)$, then integrating the inequality (30), we have

$$\int_0^1 \Big(f(x(t), t) - f(\tilde{x}(t), t) \Big) dt \geq \int_0^1 \Big[\Big\langle (-1)^s \frac{d^s x^*(t)}{dt^s}, x(t) - \tilde{x}(t) \Big\rangle - \Big\langle \frac{d^s \Big(x(t) - \tilde{x}(t)\Big)}{dt^s}, x^*(t) \Big\rangle \Big] dt$$

$$+\sum_{i=1}^{s-1}\Big\langle \Omega_i(1)\ ,\ \frac{d^{i-1}\Big(x(1)-\widetilde{x}(1)\Big)}{dt^{i-1}}\Big\rangle. \tag{31}$$

Let us denote the expression in the square brackets on the right hand side of (31) by Ψ

$$\Psi=\Big\langle(-1)^s\frac{d^sx^*(t)}{dt^s}\ ,\ x(t)-\tilde{x}(t)\Big\rangle-\Big\langle\frac{d^s\Big(x(t)-\tilde{x}(t)\Big)}{dt^s}\ ,\ x^*(t)\Big\rangle$$

and transform it. The first term of Ψ can be converted as follows

$$\Big\langle(-1)^s\frac{d^sx^*(t)}{dt^s},x(t)-\tilde{x}(t)\Big\rangle=\frac{d}{dt}\Big[\Big\langle(-1)^s\frac{d^{s-1}x^*(t)}{dt^{s-1}},x(t)-\tilde{x}(t)\Big\rangle\Big]+\Big\langle(-1)^{s+1}\frac{d^{s-1}x^*(t)}{dt^{s-1}}\ ,\ x'(t)-\tilde{x}'(t)\Big\rangle \tag{32}$$

and the second term of Ψ can be rewritten in the following form

$$\Big\langle\frac{d^s\Big(x(t)-\tilde{x}(t)\Big)}{dt^s}\ ,\ x^*(t)\Big\rangle=\frac{d}{dt}\Big[\Big\langle\frac{d^{s-1}\Big(x(t)-\tilde{x}(t)\Big)}{dt^{s-1}}\ ,\ x^*(t)\Big\rangle\Big]-\frac{d}{dt}\Big[\Big\langle\frac{d^{s-2}\Big(x(t)-\tilde{x}(t)\Big)}{dt^{s-2}}\ ,\ \frac{dx^*(t)}{dt}\Big\rangle\Big]$$

$$+\frac{d}{dt}\Big[\Big\langle\frac{d^{s-3}\Big(x(t)-\tilde{x}(t)\Big)}{dt^{s-3}}\ ,\ \frac{d^2x^*(t)}{dt^2}\Big\rangle\Big]-\cdots+(-1)^s\frac{d}{dt}\Big[\Big\langle\frac{d\Big(x(t)-\tilde{x}(t)\Big)}{dt}\ ,\ \frac{d^{s-2}x^*(t)}{dt^{s-2}}\Big\rangle\Big]$$

$$+(-1)^{s-1}\frac{d}{dt}\Big[\Big\langle x(t)-\tilde{x}(t)\ ,\ \frac{d^{s-1}x^*(t)}{dt^{s-1}}\Big\rangle\Big]+(-1)^s\Big\langle x(t)-\tilde{x}(t)\ ,\ \frac{d^sx^*(t)}{dt^s}\Big\rangle. \tag{33}$$

Thus by subtracting (32) and (33) we derive

$$\Psi=\frac{d}{dt}\Big\langle(-1)^s\frac{d^{s-1}x^*(t)}{dt^{s-1}}\ ,\ x(t)-\widetilde{x}(t)\Big\rangle-\frac{d}{dt}\Big\langle\frac{d^{s-1}\Big(x(t)-\tilde{x}(t)\Big)}{dt^{s-1}}\ ,\ x^*(t)\Big\rangle+\frac{d}{dt}\Big\langle\frac{d^{s-2}\Big(x(t)-\tilde{x}(t)\Big)}{dt^{s-2}}\ ,\ \frac{dx^*(t)}{dt}\Big\rangle$$

$$-\frac{d}{dt}\Big\langle\frac{d^{s-3}\Big(x(t)-\tilde{x}(t)\Big)}{dt^{s-3}}\ ,\ \frac{d^2x^*(t)}{dt^2}\Big\rangle+\cdots+(-1)^{s+1}\frac{d}{dt}\Big[\Big\langle\frac{d\Big(x(t)-\tilde{x}(t)\Big)}{dt}\ ,\ \frac{d^{s-2}x^*(t)}{dt^{s-2}}\Big\rangle\Big].$$

Then we need to calculate the following integral,

$$\int_0^1\Psi dt=(-1)^s\Big\langle\frac{d^{s-1}x^*(1)}{dt^{s-1}},x(1)-\widetilde{x}(1)\Big\rangle-\Big\langle\frac{d^{s-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-1}},x^*(1)\Big\rangle+\Big\langle\frac{d^{s-2}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-2}},\frac{dx^*(1)}{dt}\Big\rangle$$

$$-\Big\langle\frac{d^{s-3}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-3}},\frac{d^2x^*(1)}{dt^2}\Big\rangle+\cdots+(-1)^{s+1}\Big\langle\frac{d\Big(x(1)-\tilde{x}(1)\Big)}{dt},\ \frac{d^{s-2}x^*(1)}{dt^{s-2}}\Big\rangle. \tag{34}$$

By using (34) from the inequality (31) we deduce

$$\int_0^1\Big(f(x(t),t)-f(\tilde{x}(t),t)\Big)dt\geq(-1)^s\Big\langle\frac{d^{s-1}x^*(1)}{dt^{s-1}},x(1)-\widetilde{x}(1)\Big\rangle-\Big\langle\frac{d^{s-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-1}},x^*(1)\Big\rangle$$

$$+\Big\langle\frac{d^{s-2}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-2}},\frac{dx^*(1)}{dt}\Big\rangle+\cdots+(-1)^{s+1}\Big\langle\frac{d\Big(x(1)-\tilde{x}(1)\Big)}{dt},\ \frac{d^{s-2}x^*(1)}{dt^{s-2}}\Big\rangle+\Big\langle\Omega_1(1),x(1)-\widetilde{x}(1)\Big\rangle$$

$$+\Big\langle\Omega_2(1),\frac{d\Big(x(1)-\widetilde{x}(1)\Big)}{dt}\Big\rangle+\Big\langle\Omega_3(1),\frac{d^2\Big(x(1)-\widetilde{x}(1)\Big)}{dt^2}\Big\rangle+\cdots+\Big\langle\Omega_{s-1}(1),\frac{d^{s-2}\Big(x(1)-\widetilde{x}(1)\Big)}{dt^{s-2}}\Big\rangle.$$

As a result of rewriting this inequality that we have

$$\int_0^1 \Big(f(x(t),t)-f(\tilde{x}(t),t)\Big)dt \geq \Big\langle (-1)^s \frac{d^{s-1}x^*(1)}{dt^{s-1}} + \Omega_1(1),\ x(1)-\widetilde{x}(1)\Big\rangle$$

$$+\Big\langle (-1)^{s+1}\frac{d^{s-2}x^*(1)}{dt^{s-2}} + \Omega_2(1),\ x'(1)-\widetilde{x}'(1)\Big\rangle + \Big\langle (-1)^{s+2}\frac{d^{s-3}x^*(1)}{dt^{s-3}} + \Omega_3(1),\ x''(1)-\widetilde{x}''(1)\Big\rangle$$

$$+\cdots+\Big\langle \frac{dx^*(1)}{dt} + \Omega_{s-1}(1),\ x^{(s-2)}(1)-\widetilde{x}^{(s-2)}(1)\Big\rangle - \Big\langle x^*(1),\ x^{(s-1)}(1)-\widetilde{x}^{(s-1)}(1)\Big\rangle.$$

Therefore, taking into account the expression $\Omega_i(1)$, $i=1,2,\ldots,s-1$ we can write

$$\int_0^1 \Big(f(x(t),t)-f(\tilde{x}(t),t)\Big)dt \geq \Big\langle (-1)^s \frac{d^{s-1}x^*(1)}{dt^{s-1}} + \sum_{k=1}^{s-1}(-1)^{k+1}P_k^*\frac{d^{k-1}\lambda(1)}{dt^{k-1}},\ x(1)-\widetilde{x}(1)\Big\rangle$$

$$+\Big\langle (-1)^{s+1}\frac{d^{s-2}x^*(1)}{dt^{s-2}} + \sum_{k=2}^{s-1}(-1)^{k+2}P_k^*\frac{d^{k-2}\lambda(1)}{dt^{k-2}},\ x'(1)-\widetilde{x}'(1)\Big\rangle$$

$$+\Big\langle (-1)^{s+2}\frac{d^{s-3}x^*(1)}{dt^{s-3}} + \sum_{k=3}^{s-1}(-1)^{k+3}P_k^*\frac{d^{k-3}\lambda(1)}{dt^{k-3}},\ x''(1)-\widetilde{x}''(1)\Big\rangle$$

$$+\cdots+\Big\langle \frac{dx^*(1)}{dt} + P_{s-1}^*\lambda(1),\ x^{(s-2)}(1)-\widetilde{x}^{(s-2)}(1)\Big\rangle - \Big\langle x^*(1),\ x^{(s-1)}(1)-\widetilde{x}^{(s-1)}(1)\Big\rangle. \tag{35}$$

Finally by summing up the inequalities (25) and (35), we derive that

$$\int_0^1 f(x(t),t)dt + \varphi_0(x(1),x'(1),\ldots,x^{(s-1)}(1)) \geq \int_0^1 f(\tilde{x}(t),t)dt + \varphi_0(\tilde{x}(1),\tilde{x}'(1),\ldots,\widetilde{x}^{(s-1)}(1)).$$

As a result of all feasible solutions $x(t)$, $J(x(\cdot))-J(\tilde{x}(\cdot))\geq 0$ and therefore $\tilde{x}(t)$ is the optimal trajectory. □

Corollary 4.2. *Consider the Bolza problem with cost functional (4) and differential inclusion (5) with the following boundary value conditions*

$$x^{(k)}(0)\in G_k\ ,\quad x^{(k)}(1)\in H_k\ ,\ k=0,1,\ldots,s-1,$$

where G_k and H_k are polyhedral sets. Then for the optimality of the trajectory $\widetilde{x}(t)$ in the Bolza problem the transversality conditions at points 0 and 1 should be as follows:

$$(d)\qquad (-1)^{s-j}\ \frac{d^{s-j}x^*(0)}{dt^{s-j}} + \sum_{i=j}^{s-1}(-1)^{i+j-1}P_i^*\frac{d^{i-j}\lambda(0)}{dt^{i-j}} \in K^*_{G_{j-1}}\Big(\tilde{x}^{(j-1)}(0)\Big)\ ,\ j=1,2,\ldots,s-1,$$

$$x^*(0)\in K^*_{G_{s-1}}\Big(\tilde{x}^{(s-1)}(0)\Big),$$

$$(e)\qquad \sum_{i=j}^{s-1}(-1)^{i+j}P_i^*\frac{d^{i-j}\lambda(1)}{dt^{i-j}} \in K^*_{H_{j-1}}\Big(\tilde{x}^{(j-1)}(1)\Big)\ ,\ j=1,2,\ldots,s-1,$$

$$-x^*(1)\in K^*_{H_{s-1}}\Big(\tilde{x}^{(s-1)}(1)\Big),$$

$$\left((-1)^{s-1}\ \frac{d^{s-1}x^*(1)}{dt^{s-1}}\ ,\ (-1)^s\ \frac{d^{s-2}x^*(1)}{dt^{s-2}}\ ,\ (-1)^{s-1}\ \frac{d^{s-3}x^*(1)}{dt^{s-3}}\ ,\ \ldots\ ,\ -\frac{dx^*(1)}{dt}\ ,\ x^*(1)\right)$$

$$\in \partial\varphi_0\Big(\tilde{x}(1),\ \tilde{x}'(1),\ldots,\ \tilde{x}^{(s-1)}(1)\Big).$$

Proof. Integrating inequality (30) we have

$$\int_0^1 \Big(f(x(t),t)-f(\tilde{x}(t),t)\Big)dt \geq \int_0^1 \Big[\Big\langle (-1)^s \frac{d^s x^*(t)}{dt^s}, x(t)-\tilde{x}(t)\Big\rangle - \Big\langle \frac{d^s\Big(x(t)-\tilde{x}(t)\Big)}{dt^s}, x^*(t)\Big\rangle\Big]dt$$

$$+\sum_{i=1}^{s-1}\Big\langle \Omega_i(1)\ ,\ \frac{d^{i-1}\Big(x(1)-\widetilde{x}(1)\Big)}{dt^{i-1}}\Big\rangle - \sum_{i=1}^{s-1}\Big\langle \Omega_i(0)\ ,\ \frac{d^{i-1}\Big(x(0)-\widetilde{x}(0)\Big)}{dt^{i-1}}\Big\rangle. \tag{36}$$

Similarly, by denoting the expression in square brackets on the right hand side of (36) by Ψ, we calculate the following integral

$$\int_0^1 \Psi dt = (-1)^s\Big\langle \frac{d^{s-1}x^*(1)}{dt^{s-1}}, x(1)-\widetilde{x}(1)\Big\rangle - \Big\langle \frac{d^{s-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-1}}, x^*(1)\Big\rangle + \Big\langle \frac{d^{s-2}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-2}}, \frac{dx^*(1)}{dt}\Big\rangle$$

$$-\Big\langle \frac{d^{s-3}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-3}}, \frac{d^2x^*(1)}{dt^2}\Big\rangle + \cdots + (-1)^{s+1}\Big\langle \frac{d\Big(x(1)-\tilde{x}(1)\Big)}{dt}, \frac{d^{s-2}x^*(1)}{dt^{s-2}}\Big\rangle - (-1)^s\Big\langle \frac{d^{s-1}x^*(0)}{dt^{s-1}}, x(0)-\widetilde{x}(0)\Big\rangle$$

$$+\Big\langle \frac{d^{s-1}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{s-1}}, x^*(0)\Big\rangle - \Big\langle \frac{d^{s-2}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{s-2}}, \frac{dx^*(0)}{dt}\Big\rangle + \Big\langle \frac{d^{s-3}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{s-3}}, \frac{d^2x^*(0)}{dt^2}\Big\rangle$$

$$+\cdots + (-1)^s\Big\langle \frac{d\Big(x(0)-\tilde{x}(0)\Big)}{dt}, \frac{d^{s-2}x^*(0)}{dt^{s-2}}\Big\rangle. \tag{37}$$

By using the formula (37) and by rewriting the inequality (36), we deduce

$$\int_0^1 \Big(f(x(t),t)-f(\tilde{x}(t),t)\Big)dt \geq (-1)^s\Big\langle \frac{d^{s-1}x^*(1)}{dt^{s-1}}, x(1)-\widetilde{x}(1)\Big\rangle - \Big\langle \frac{d^{s-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-1}}, x^*(1)\Big\rangle$$

$$+\Big\langle \frac{d^{s-2}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{s-2}}, \frac{dx^*(1)}{dt}\Big\rangle + \cdots + (-1)^{s+1}\Big\langle \frac{d\Big(x(1)-\tilde{x}(1)\Big)}{dt}, \frac{d^{s-2}x^*(1)}{dt^{s-2}}\Big\rangle$$

$$-(-1)^s\Big\langle \frac{d^{s-1}x^*(0)}{dt^{s-1}}, x(0)-\widetilde{x}(0)\Big\rangle + \Big\langle \frac{d^{s-1}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{s-1}}, x^*(0)\Big\rangle - \Big\langle \frac{d^{s-2}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{s-2}}, \frac{dx^*(0)}{dt}\Big\rangle$$

$$-\cdots + (-1)^s\Big\langle \frac{d\Big(x(0)-\tilde{x}(0)\Big)}{dt}, \frac{d^{s-2}x^*(0)}{dt^{s-2}}\Big\rangle + \Big\langle \Omega_1(1), x(1)-\widetilde{x}(1)\Big\rangle + \Big\langle \Omega_2(1), \frac{d\Big(x(1)-\widetilde{x}(1)\Big)}{dt}\Big\rangle$$

$$+\cdots + \Big\langle \Omega_{s-1}(1), \frac{d^{s-2}\Big(x(1)-\widetilde{x}(1)\Big)}{dt^{s-2}}\Big\rangle - \Big\langle \Omega_1(0), x(0)-\widetilde{x}(0)\Big\rangle - \Big\langle \Omega_2(0), \frac{d\Big(x(0)-\widetilde{x}(0)\Big)}{dt}\Big\rangle$$

$$-\cdots - \Big\langle \Omega_{s-1}(0), \frac{d^{s-2}\Big(x(0)-\widetilde{x}(0)\Big)}{dt^{s-2}}\Big\rangle,$$

and by writing the definition of $\Omega_i(t)$, $i = 1, 2, \ldots, s-1$, in this inequality, we have

$$\int_0^1 \Big(f(x(t),t)-f(\tilde{x}(t),t)\Big)dt \geq \Big\langle (-1)^s\frac{d^{s-1}x^*(1)}{dt^{s-1}} + \sum_{k=1}^{s-1}(-1)^{k+1}P_k^*\frac{d^{k-1}\lambda(1)}{dt^{k-1}},\ x(1)-\widetilde{x}(1)\Big\rangle$$

$$+\Big\langle (-1)^{s+1}\frac{d^{s-2}x^*(1)}{dt^{s-2}} + \sum_{k=2}^{s-1}(-1)^{k+2}P_k^*\frac{d^{k-2}\lambda(1)}{dt^{k-2}},\ x'(1)-\widetilde{x}'(1)\Big\rangle$$

$$+\cdots+\Big\langle\frac{dx^*(1)}{dt}+P^*_{s-1}\lambda(1),\ x^{(s-2)}(1)-\widetilde{x}^{(s-2)}(1)\Big\rangle-\Big\langle x^*(1),\ x^{(s-1)}(1)-\widetilde{x}^{(s-1)}(1)\Big\rangle$$

$$-\Big\langle(-1)^s\frac{d^{s-1}x^*(0)}{dt^{s-1}}+\sum_{k=1}^{s-1}(-1)^{k+1}P^*_k\frac{d^{k-1}\lambda(0)}{dt^{k-1}},\ x(0)-\widetilde{x}(0)\Big\rangle$$

$$-\Big\langle(-1)^{s+1}\frac{d^{s-2}x^*(0)}{dt^{s-2}}+\sum_{k=2}^{s-1}(-1)^{k+2}P^*_k\frac{d^{k-2}\lambda(0)}{dt^{k-2}},\ x'(0)-\widetilde{x}'(0)\Big\rangle$$

$$-\cdots-\Big\langle\frac{dx^*(0)}{dt}+P^*_{s-1}\lambda(0),\ x^{(s-2)}(0)-\widetilde{x}^{(s-2)}(0)\Big\rangle+\Big\langle x^*(0),\ x^{(s-1)}(0)-\widetilde{x}^{(s-1)}(0)\Big\rangle. \tag{38}$$

For all feasible solutions, the transversality condition (e) can be written in the form

$$\varphi_0\Big(x(1),x'(1),\dots,x^{(s-1)}(1)\Big)-\varphi_0\Big(\tilde{x}(1),\tilde{x}'(1),\dots,\tilde{x}^{(s-1)}(1)\Big)$$

$$\geq\Big\langle(-1)^{s-1}\ \frac{d^{s-1}x^*(1)}{dt^{s-1}}\ ,\ x(1)-\tilde{x}(1)\Big\rangle+\Big\langle(-1)^s\ \frac{d^{s-2}x^*(1)}{dt^{s-2}}\ ,\ x'(1)-\tilde{x}'(1)\Big\rangle$$

$$+\cdots+\Big\langle-\frac{dx^*(1)}{dt}\ ,\ x^{(s-2)}(1)-\tilde{x}^{(s-2)}(1)\Big\rangle+\Big\langle x^*(1)\ ,\ x^{(s-1)}(1)-\tilde{x}^{(s-1)}(1)\Big\rangle. \tag{39}$$

Then, by adding the inequalities (38) and (39) we derive

$$\int_0^1\Big(f(x(t),t)-f(\tilde{x}(t),t)\Big)dt+\varphi_0\Big(x(1),x'(1),\dots,x^{(s-1)}(1)\Big)-\varphi_0\Big(\tilde{x}(1),\tilde{x}'(1),\dots,\tilde{x}^{(s-1)}(1)\Big)$$

$$\geq\Big\langle\sum_{k=1}^{s-1}(-1)^{k+1}P^*_k\frac{d^{k-1}\lambda(1)}{dt^{k-1}},\ x(1)-\widetilde{x}(1)\Big\rangle+\Big\langle\sum_{k=2}^{s-1}(-1)^{k+2}P^*_k\frac{d^{k-2}\lambda(1)}{dt^{k-2}},\ x'(1)-\widetilde{x}'(1)\Big\rangle$$

$$+\cdots+\Big\langle P^*_{s-1}\lambda(1),\ x^{(s-2)}(1)-\widetilde{x}^{(s-2)}(1)\Big\rangle-\Big\langle(-1)^s\frac{d^{s-1}x^*(0)}{dt^{s-1}}+\sum_{k=1}^{s-1}(-1)^{k+1}P^*_k\frac{d^{k-1}\lambda(0)}{dt^{k-1}},\ x(0)-\widetilde{x}(0)\Big\rangle$$

$$-\Big\langle(-1)^{s+1}\frac{d^{s-2}x^*(0)}{dt^{s-2}}+\sum_{k=2}^{s-1}(-1)^{k+2}P^*_k\frac{d^{k-2}\lambda(0)}{dt^{k-2}},\ x'(0)-\widetilde{x}'(0)\Big\rangle$$

$$-\cdots-\Big\langle\frac{dx^*(0)}{dt}+P^*_{s-1}\lambda(0)\ ,\ x^{(s-2)}(0)-\widetilde{x}^{(s-2)}(0)\Big\rangle+\Big\langle x^*(0)\ ,\ x^{(s-1)}(0)-\widetilde{x}^{(s-1)}(0)\Big\rangle. \tag{40}$$

Taking into account the other transversality conditions we have

$$\int_0^1 f(x(t),t)dt+\varphi_0(x(1),x'(1),\dots,x^{(s-1)}(1))\geq\int_0^1 f(\tilde{x}(t),t)dt+\varphi_0(\tilde{x}(1),\tilde{x}'(1),\dots,\widetilde{x}^{(s-1)}(1)),$$

this means that for all feasible solutions $x(t)$, $J(x(\cdot))-J(\tilde{x}(\cdot))\geq 0$ and therefore $\tilde{x}(t)$ is the optimal trajectory. □

By using the results obtained in this section, we are able to formulate sufficient conditions of optimality for the continuous problem (4)-(6) given by second order ($s=2$) polyhedral differential inclusions.

Corollary 4.3. *[22] For trajectory $\tilde{x}(t)$, $t \in [0,1]$, lying interior to domF to be optimal in Bolza problem with second order ($s = 2$) polyhedral differential inclusion (PC), it is sufficient that there is an absolutely continuous function $x^*(t)$ satisfying the following Euler-Lagrange differential inclusion almost everywhere*

$$(i)\ \frac{d^2x^*(t)}{dt^2} \in P_0^*\lambda(t) - P_1^*\frac{d\lambda(t)}{dt} + \partial f(\tilde{x}(t),t)\ ,\ a.e.\ \ t \in [0,1],$$

$$x^*(t) = Q^*\lambda(t),\ \lambda(t) \geq 0,$$

$$(ii)\ \Big\langle P_0\tilde{x}(t) + P_1\frac{d\tilde{x}(t)}{dt} - Q\frac{d^2\tilde{x}(t)}{dt^2} - d\ ,\ \lambda(t)\Big\rangle = 0\ ,\ a.e.\ t \in [0,1],$$

and transversality condition

$$(iii)\ \Big(-\frac{dx^*(1)}{dt} - P_1^*\lambda(1)\ ,\ x^*(1)\Big) \in \partial\varphi_0\Big(\tilde{x}(1),\tilde{x}'(1)\Big).$$

5. Numerical Application

This section describes some of the interesting applications of the Theorem 2.2. It should be noted that according to Theorem 2.2, the numerical solution of the second order polyhedral discrete problem can be calculated. Let $T = 15$, $f(x_T,T) = x_T$, $f(x_t,t) = 0$, $t = 2,3,\ldots,14$ be given for this purpose. Let's look at the following example:

Example 5.1.

$$\begin{gathered} minimize \sum_{t=2}^{15} f(x_t,t) \\ x_{t+2} \in F(x_t,x_{t+1})\ ,\quad t = 0,1,\ldots,13\ , \\ x_0 = 0,\ x_1 = 1, \end{gathered} \tag{41}$$

where $F(x,v_1) = \Big\{v_2 : \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} x + \begin{pmatrix} -0.1 \\ -0.2 \\ -0.3 \end{pmatrix} v_1 - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} v_2 \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\Big\}$ *is a polyhedral multi-valued mapping. In fact, the graphF of F is a cone.*

We assume that the objective function has a form $\sum_{t=2}^{T} f(x_T,T)$ in the problem (41) with second-order polyhedral discrete inclusions where $f(x_T,T)$ is not identically zero, i.e., $f(x_T,T) \not\equiv 0$. This means that $x_T^* \neq 0$ and then the transversality condition for T consists of the following inclusion $x_T^* \in \partial f(\tilde{x}_T,T)$.

For optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^{15}$ of polyhedral discrete inclusions in the second order discrete polyhedral optimization problem (41), it is necessary and sufficient that there are vectors x_t^*, $t = 0,\ldots,15$ not all equal to zero satisfying the Euler-Lagrange discrete inclusion

$$\begin{gathered} x_t^* = P_0^*\lambda_t + P_1^*\lambda_{t-1} + u_t^*\ ,\quad u_t^* \in \partial f(\tilde{x}_t,t), \\ x_{t+2}^* = Q^*\lambda_t,\ \ \lambda_t \geq 0\ ,\ t = 0,\ldots,13,\ \partial f(\tilde{x}_0,0) = \partial f(\tilde{x}_1,1) = \{0\},\ \lambda_{-1} \equiv 0, \\ \Big\langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\tilde{x}_t + \begin{pmatrix} -0.1 \\ -0.2 \\ -0.3 \end{pmatrix}\tilde{x}_{t+1} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\tilde{x}_{t+2}\ ,\ \lambda_t\Big\rangle = 0, \end{gathered} \tag{42}$$

and transversality conditions

$$\begin{gathered} x_{14}^* - P_1^*\lambda_{13} \in \partial f(\tilde{x}_{14},14), \\ x_{15}^* \in \partial f(\tilde{x}_{15},15). \end{gathered}$$

Here $P_0^* = (1 \ -2 \ 1)$, $P_1^* = (-0.1 \ -0.2 \ -0.3)$ and $Q^* = (1 \ -1 \ 1)$ are transposed matrices. Since f is continuously differentiable function, we have

$$x_t^* = P_0^* \lambda_t + P_1^* \lambda_{t-1}, \quad t = 0, 1, \ldots, 13,$$

and transversality condition

$$x_{14}^* - P_1^* \lambda_{13} = 0,$$
$$x_{15}^* = 1.$$

By sequentially resolving these equations, it is not difficult to calculate λ_t $(t = 0, \ldots, 13)$ and x_t^* $(t = 0, \ldots, 15)$. We have

$$\lambda_{13} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \lambda_{12} = \begin{pmatrix} 0 \\ 0.1 \\ 0 \end{pmatrix}, \ \lambda_{11} = \begin{pmatrix} 0.98 \\ 0 \\ 0 \end{pmatrix}, \ \lambda_{10} = \begin{pmatrix} 0 \\ 0.298 \\ 0 \end{pmatrix}, \ \lambda_9 = \begin{pmatrix} 0.9204 \\ 0 \\ 0 \end{pmatrix},$$

$$\lambda_8 = \begin{pmatrix} 0 \\ 0.688 \\ 0 \end{pmatrix}, \ \lambda_7 = \begin{pmatrix} 0.7828 \\ 0 \\ 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 \\ 1.4543 \\ 0 \end{pmatrix}, \ \lambda_5 = \begin{pmatrix} 0.4919 \\ 0 \\ 0 \end{pmatrix}, \ \lambda_4 = \begin{pmatrix} 0 \\ 2.9579 \\ 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 0 \\ 0.0996 \\ 0 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 0 \\ 5.9357 \\ 0 \end{pmatrix}, \ \lambda_1 = \begin{pmatrix} 0 \\ 1.3864 \\ 0 \end{pmatrix}, \ \lambda_0 = \begin{pmatrix} 0 \\ 12.1488 \\ 0 \end{pmatrix}.$$

The values of the adjoint variables $\{x_t^*\}_{t=0}^{15}$ which are calculated and the optimal trajectory $\{\tilde{x}_t\}_{t=0}^{15}$ obtained by using these values are given in Table 1. In addition, the graphical representation of the optimal trajectory $\{\tilde{x}_t\}_{t=0}^{15}$ is given in detail in Figure 1.

t	x_t^*
15	1
14	-0,1
13	0,98
12	-0,298
11	0.9204
10	-0,68804
9	0.78279
8	-1,45435
7	0.49192
6	-2,95791
5	-0,09966
4	-5,93575
3	-1,38647
2	-12,1488
1	-5,2027
0	-24,2976

t	$\tilde{x}_t$
0	0
1	1
2	0,2
3	2,04
4	0,808
5	4,2416
6	2,46432
7	3,99516
8	5,72767
9	3,4224
10	12,13982
11	2,20841
12	24,72133
13	-0,26371
14	49,3899
15	-5,2027

Table 1: The values of adjoint variables $\{x_t^*\}_{t=0}^{15}$ and the values of optimal trajectory $\{\tilde{x}_t\}_{t=0}^{15}$

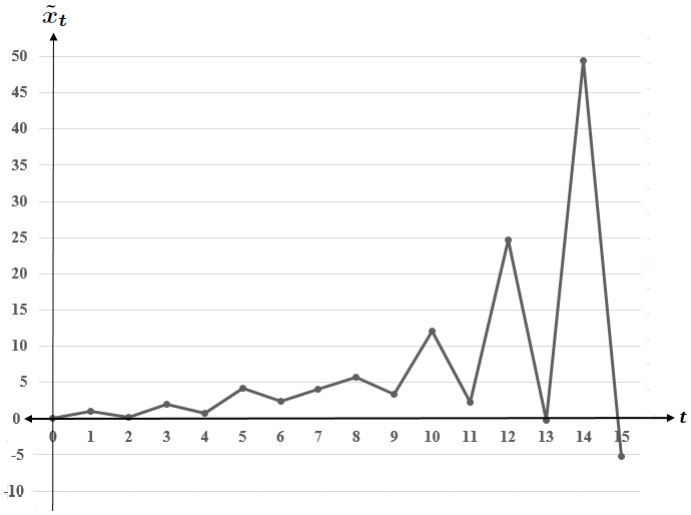

Figure 1: The exact graph of $\{\tilde{x}_t\}_{t=0}^{15}$ for $t = 0, \ldots, 15$.

6. Conclusions

This paper presents a new method of discretization to solve the optimization problem described by multi-valued polyhedral mappings for discrete and differential inclusions of higher order that are often used to describe different processes in science and engineering. The problem of higher-order discrete-approximation inclusions is investigated according to the proposed discretization approach. This approach plays a much larger role in the derivation of discrete and differential inclusions of higher-order adjoints. Equivalence theorems for subdifferential inclusions are basic tools for the analysis of conditions of optimality for discrete and discrete-approximation problems. Therefore, necessary and sufficient conditions of optimality are deduced for such problems. Finally, the numerical approach is presented with a second order polyhedral discrete inclusion to solve the optimal control problem. Besides, it is clear that the investigation of the conditions of optimality for problems with polyhedral discrete and differential inclusions will make a significant contribution to modern development of the optimal control theory with polyhedral differential inclusions. Thus, we can conclude that the proposed method is effective in solving various optimization problems with higher order discrete and differential inclusions.

References

[1] B. Ahmad, S.K. Ntouyas, H. Al-Sulami, Existence of Solutions or Nonlinear nth-order Differential Equations and Inclusions with Nonlocal and Integral Boundary Conditions via Fixed Point Theory, Filomat, 28:10 (2014), 2149-2162.
[2] J.-P. Aubin, A. Cellina, Differential Inclusions, Springer-Verlag, Grundlehren der Math., Wiss, 1984.
[3] D. Azzam-Laouir, S. Melit, Existence of Solutions for a Second Order Boundary Value Problem with the Clarke Subdifferential, Filomat, 31:9 (2017), 2763-2771.
[4] G.M. Bahaa, Fractional Optimal Control Problem for Differential System with Control Constraints, Filomat, 30:8 (2016), 2177-2189.
[5] A. Cernea, On the existence of viable solutions for a class of second-order differential inclusions, Discussiones Mathematicae, Differential Inclusions, Control and Optimization, 22:1 (2002), 67-78.
[6] F.H. Clarke, Optimization and Nonsmooth Analysis, John Wiley and Sons, New York, 1983.
[7] B. Damjanovic, B. Samet, C. Vetro, Common fixed point theorems for multi-valued maps, Acta Mathematica Scientia, 32:2 (2012), 818-824.
[8] W. Gautschi, G. Mastroianni, T. M. Rassias, Approximation and Computation: In Honor of Gradimir V. Milovanović, Springer-Verlag New York Inc., 2010.
[9] I.M. Gelfand, S.V. Fomin, Calculus of Variations, Dover Publications; Dover Books on Mathematics edition, 2000.
[10] R. Gupta, M. Srivastava, Constraint Qualifications in Nonsmooth Multiobjective Optimization Problem, Filomat, 31:3 (2017), 781-797.
[11] T. Haddad, M. Yarou, Existence of solutions for nonconvex second order differential inclusions in the infinite dimensional space, Electronic J. of Differential Equations, 33 (2006), 1-8.

[12] T. Kamran, C. Vetro , M.U. Ali, M. Waheed, A Fixed Point Theorem for G-Monotone Multivalued Mapping with Application to Nonlinear Integral Equations, Filomat, 31:7 (2017), 2045-2052.
[13] A.B. Kurzhanski, V.M. Veliov, Set-valued Analysis and Differential Inclusions, Birkhaeuser Boston Inc, 1993.
[14] J. Li, Z. Yang, The Von Neumann analysis and modified equation approach for finite difference schemes, Applied Mathematics and Computation, 225 (2013), 610-621.
[15] E.N. Mahmudov, Approximation and Optimization of Discrete and Differential Inclusions, Elsevier, Boston, 2011.
[16] E.N. Mahmudov, The optimality principle for discrete and the first order differential inclusions, J. of Mathematical Analysis and Applications, 308 (2005), 605-619.
[17] E.N. Mahmudov, Locally adjoint mappings and optimization of the first boundary value problem for hyperbolic type discrete and differential inclusions, Nonlinear Analysis: Theory Methods & Appl., 67:10 (2007), 2966-2981.
[18] E.N. Mahmudov, Optimal Control of Cauchy Problem for first order Discrete and Partial Differential Inclusions, J. of Dynamical and Control Systems, 15:4 (2009), 587-610.
[19] E.N. Mahmudov, Approximation and Optimization of Darboux Type Differential Inclusions with Set-valued Boundary Conditions, Optimization Letters, 7 (2013), 871-891.
[20] E.N. Mahmudov, Approximation and Optimization of Higher Order Discrete and Differential Inclusions, Nonlinear Differential Equations and Appl., 21 (2014), 1-26.
[21] E.N. Mahmudov, Optimization of Second Order Discrete Approximation Inclusions, Numerical Functional Analysis and Optimization, 36:5 (2015), 624-643.
[22] E.N. Mahmudov, Mathematical programming and polyhedral optimization of second order discrete and differential inclusions, Pacific J. of Optimization, 11:3, (2015), 511-525.
[23] E.N. Mahmudov, Convex optimization of second order discrete and differential inclusions with inequality constraints, J. of Convex Analysis, 25:1 (2018), 293-318.
[24] E.N. Mahmudov, Optimization of Lagrange Problem with Higher Order Differential Inclusions and Endpoint Constraints, Filomat, 32:7 (2018), 2367-2382.
[25] E.N. Mahmudov, Optimization of Mayer Problem with Sturm-Liouville-Type Differential Inclusions, J. of Optimization Theory and Appl., 177:2 (2018), 345-375.
[26] E.N. Mahmudov, Optimal control of higher order differential inclusions with functional constraints, ESAIM: Control, Optimisation and Calculus of Variations, (2019), DOI:https://doi.org/10.1051/cocv/2019018.
[27] E.N. Mahmudov, On Duality in Second-Order Discrete and Differential Inclusions with Delay, J. of Dynamical and Control Systems, (2020), DOI:https://doi.org/10.1007/s10883-019-09471-4.
[28] C. Min, Z. Liu, L. Zhang , N. Huang, On a System of Fuzzy Differential Inclusions, Filomat, 29:6 (2015), 1231-1244.
[29] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. 330 and 331, Springer, Berlin, 2006.
[30] R.V. Namm, G. Woo, Sensitivity Functionals in Convex Optimization Problem, Filomat, 30:14 (2016), 3681-3687.
[31] D. O'Regan, R.P. Agarwal, Set Valued Mappings with Applications to Nonlinear Analysis, Taylor & Francis, London, 2002.
[32] M. Oveisiha, M. Aghabagloo, Scalarized Solutions of Set-Valued Optimization Problems and Generalized Variational-like Inequalities, Filomat, 31:12 (2017), 3953-3963.
[33] N.S. Papageorgiou, C. Vetro, F. Vetro, Extremal solutions and strong relaxation for nonlinear multivalued systems with maximal monotone terms, Journal of Mathematical Analysis and Applications, 461:1 (2018), 401-421.
[34] N.S. Papageorgiou, C. Vetro, F. Vetro, Nonlinear vector Duffing inclusions with no growth restriction on the orientor field, Topological Methods in Nonlinear Analysis, 54:1 (2019), 257-274.
[35] N.S. Papageorgiou, C. Vetro and F. Vetro, Relaxation for a class of control systems with unilateral constraints, Acta Applicandae Mathematicae (2019), doi:10.1007/s10440-019-00270-4.
[36] N.S. Papageorgiou, C. Vetro, F. Vetro, Nonlinear multivalued Duffing systems, Journal of Mathematical Analysis and Applications, 468:1 (2018), 376-390.
[37] R.T. Rockefellar, Convex Analysis, Princeton University Press, New Jersey, 1972.
[38] M. Usman Ali, F. Tchier, C. Vetro, On the Existence of Bounded Solutions to a Class of Nonlinear Initial Value Problems with Delay, Filomat, 31:11 (2017), 3125-3135.