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On Topological Equivalence of $\mathbb{R}$-Uniformly Continuous Fuzzy Metric Spaces

Changqing Li[a], Yanlan Zhang[b], Jing Zhang[a]

[a]*School of Mathematics and Statistics,
Minnan Normal University, Zhangzhou, Fujian 363000, China*
[b]*College of Computer,
Minnan Normal University, Zhangzhou, Fujian 363000, China*

Abstract. The notion of uniform continuity in fuzzy metric spaces was first introduced by George and Veeramani in 1995. Later, Gregori et al. gave some contributions to the theory. As a consequence of the study, we introduce the notion of *RUC* fuzzy metric space. Also, necessary and sufficient conditions for a fuzzy metric space to be an *RUC* fuzzy metric space are studied. In addition, several examples are given.

1. Introduction

Fuzzy metric spaces have been constructed by several authors from different points of view [3, 4, 18, 20]. In particular, Kramosil and Michalek introduced a concept of fuzzy metric with the help of continuous t-norms in [20]. Later, George and Veeramani [4] modified in a slight but appealing way the concept given by Kramosil and Michalek and proved that the topology generated by such a fuzzy metric is Hausdorff. Whereafter, Gregori and Romaguera [14] showed that the topological space generated by such a fuzzy metric is metrizable. The new version of fuzzy metric is more restrictive, but it determines the class of spaces that are tightly connected with the class of metrizable topological spaces. Thus it is interesting to explore it. Some classical theorems in metric spaces have been adapted to realm of fuzzy metric spaces [14], but Gregori and Romaguera [15] proved that there are non-completable fuzzy metric spaces, which is very different from the classical metric completion. So fuzzy metric spaces seem to be a structure that leads to a theory which seems to be a richer one than that of metric spaces. In the past more than 10 years, Gregori et al. [10–13, 16] gave much progress to the study of completable fuzzy metric spaces. In addition, Romaguera and Sanchis [26] introduced a notion of fuzzy metric group and investigated properties of the quotient subgroups of a fuzzy metric group. An arclength notion of continuous curves in fuzzy metric spaces was proposed, and some arclength properties, including invariance, additive, continuity and boundedness were explored by Chen et al. in [1]. Savchenko and Zarichnyi [28] studied fuzzy ultrametrics on the set

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Email addresses: helen_smile0320@163.com (Changqing Li), zyl_1983_2004@163.com (Yanlan Zhang), zhangjing86@126.com (Jing Zhang)

of probability measures. Kočinac [19] gave some selection properties of fuzzy metric spaces. Other more contributions to the study of fuzzy metric spaces can be found in [6–9, 21–25, 27, 29].

For the purpose of generalizing uniform continuity in metric spaces to realm of fuzzy metric spaces, George and Veeramani [5] introduced the notion of uniform continuity of mappings in fuzzy metric spaces. Later, Gregori et al. [17] gave appropriate fuzzy notions of equinormality and Lebesgue property, and studied several characterizations of uniform continuity in fuzzy metric spaces, in the sense of George and Veeramani. It is natural to investigate other more equivalent conditions of uniform continuity in fuzzy metric spaces. In this paper we do it. Here, we introduce the concept of $\mathbb{R}$-uniformly continuous fuzzy metric spaces, in the sense of George and Veeramani. Also, we explore several necessary and sufficient conditions for a fuzzy metric space to be an $\mathbb{R}$-uniformly continuous fuzzy metric space. Furthermore, two examples are given.

2. Preliminaries

Throughout the paper the letter $\mathbb{N}$ and $\mathbb{R}$ shall denote the set of all positive integers and the set of all real numbers, respectively. Our basic reference for general topology is [2].

Definition 2.1. ([4]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Obviously, $a * b = a \cdot b$ and $a * b = \min\{a, b\}$ are two common examples of continuous t-norms.

Definition 2.2. ([4]) A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0,\infty)$:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) the function $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, $(M, *)$ will be called a *fuzzy metric on X*.

Definition 2.3. ([4]) Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0,1), t > 0$ and $x \in X$. The set

$$B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$$

is called the *open ball with center x and radius r with respect to t*.

It is obvious that $\{B_M(x, r, t) | x \in X, t > 0, r \in (0,1)\}$ forms a base of a topology τ_M in X. $\{B_M(x, \frac{1}{n}, \frac{1}{n}) | n \in \mathbb{N}\}$ is a neighborhood base at x for the topology τ_M for all $x \in X$ (see [4]).

Proposition 2.4. ([4]) *Let $(X, M, *)$ be a fuzzy metric space. A sequence (x_n) in X converges to $x_0 \in X$ if and only if $\lim_{n\to\infty} M(x_n, x_0, t) = 1$ for all $t > 0$.*

The following is straightforward.

Remark 2.5. A sequence (x_n) in a fuzzy metric space $(X, M, *)$ is convergent to $x_0 \in X$ if and only if for each $r \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, t) > 1 - r$ for all $n > n_0$.

Definition 2.6. ([4]) Let (X, d) be a metric space. Define $a * b = a \cdot b$ for all $a, b \in [0, 1]$, and let M_d be the real value mapping on $X \times X \times (0, \infty)$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and $(M_d, \cdot)$ is called *the standard fuzzy metric induced by d*.

Definition 2.7. ([4]) Let $(X, M, *)$ be a fuzzy metric space. A sequence (x_n) in X is called *Cauchy* if for each $r \in (0, 1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq n_0$. $(X, M, *)$ (or simply X) is called *complete* if every Cauchy sequence in X is convergent with respect to τ_M.

Definition 2.8. ([14]) A fuzzy metric space $(X, M, *)$ is called *compact* if (X, τ_M) is a compact topological space.

3. Equivalent Conditions of Uniform Continuity

We start this section by recalling the concept of uniformly continuous on two fuzzy metric spaces.

Definition 3.1. ([5]) A mapping f from a fuzzy metric space $(X_1, M_1, *_1)$ to a fuzzy metric space $(X_2, M_2, *_2)$ is called *uniformly continuous* if for each $r_2 \in (0, 1)$ and each $t_2 > 0$, there exist $r_1 \in (0, 1)$ and $t_1 > 0$ such that $M_2(f(x), f(y), t_2) > 1 - r_2$ whenever $x, y \in X_1$ and $M_1(x, y, t_1) > 1 - r_1$.

Definition 3.2. Two sequences (x_n) and (y_n) in a fuzzy metric space $(X, M, *)$ are called *asymptotic* if for each $r \in (0, 1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, y_n, t) > 1 - r$ for all $n > n_0$.

Clearly, for each subsequence $((x_{n_i}), (y_{n_i}))$ of $((x_n), (y_n))$ in $X \times X$, if (x_n) and (y_n) are asymptotic, then so are (x_{n_i}) and (y_{n_i}). Let $M(A, B, t) := \sup\{M(a, b, t) | a \in A, b \in B\}$ for all $A, B \subset X$ and $t > 0$ (see Definition 2.4 of [29]). It is immediate to see that $M(A, B, t) = 1$ for every $t > 0$ if (x_n) and (y_n) are asymptotic, where $A = \{x_n | n \in \mathbb{N}\}$ and $B = \{y_n | n \in \mathbb{N}\}$.

Lemma 3.3. *Let $(X, M, *)$ be a fuzzy metric space, $\varepsilon \in (0, 1)$ and $t > 0$. Assume that (x_n) and (y_n) are two sequences in X such that for each $n \in \mathbb{N}$, $M(x_n, y_n, t) \leq 1 - \varepsilon$. Then there exists a subsequence $((x_{n_i}), (y_{n_i}))$ of $((x_n), (y_n))$ and $\varepsilon_0 \in (0, \varepsilon)$ such that $M(x_{n_k}, y_{n_l}, \frac{t}{4}) < 1 - \varepsilon_0$ for all $k, l \in \mathbb{N}$, where $n_i < n_j$ if $i < j$.*

Proof. Let $t > 0$ and $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon < 1$ with $(1 - \varepsilon_0) * (1 - \varepsilon_0) > 1 - \varepsilon_1$ and $(1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon$. For each $n \in \mathbb{N}$, put

$$X_n = \{m \in \mathbb{N} | M(x_n, y_m, \frac{t}{4}) \geq 1 - \varepsilon_0\},$$

$$Y_n = \{m \in \mathbb{N} | M(y_n, x_m, \frac{t}{4}) \geq 1 - \varepsilon_0\}.$$

Suppose that $p \in X_n$ and $q \in X_n$ with $p \neq q$. Then

$$M(y_p, y_q, \frac{t}{2}) \geq M(y_p, x_n, \frac{t}{4}) * M(x_n, y_q, \frac{t}{4}) \geq (1 - \varepsilon_0) * (1 - \varepsilon_0) > 1 - \varepsilon_1.$$

We claim that

$$M(x_p, y_q, \frac{t}{2}) < 1 - \varepsilon_1,$$

which follows that

$$M(x_p, y_q, \frac{t}{4}) \leq M(x_p, y_q, \frac{t}{2}) < 1 - \varepsilon_1 < 1 - \varepsilon_0.$$

If not, then

$$M(x_p, y_p, t) \geq M(x_p, y_q, \frac{t}{2}) * M(y_q, y_p, \frac{t}{2}) \geq (1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon,$$

which contradicts $M(x_n, y_n, t) \le 1-\varepsilon$ for every $n \in \mathbb{N}$. It is easy to see that the sequence $((x_{n_i}),(y_{n_i}))$ is as required if X_n is infinite for some $n \in \mathbb{N}$, where $n_i \in X_n$ and $n_i < n_j$ if $i<j$. Similarly, this can be proved to Y_n. To complete the proof, it suffices to consider the case that both X_n and Y_n are finite for every $n \in \mathbb{N}$. Put $n_1 = 1$. Let $m_1' = \max\{m | m \in X_{n_1} \cup Y_{n_1}\}$ and $n_2 = \max\{n_1, m_1'\}+1$. Then $M(x_{n_1}, y_{n_2}, \frac{t}{4}) < 1-\varepsilon_0$, $M(x_{n_2}, y_{n_1}, \frac{t}{4}) < 1-\varepsilon_0$, $M(x_{n_1}, y_{n_1}, \frac{t}{4}) \le M(x_{n_1}, y_{n_1}, t) \le 1-\varepsilon < 1-\varepsilon_0$ and $M(x_{n_2}, y_{n_2}, \frac{t}{4}) \le M(x_{n_2}, y_{n_2}, t) \le 1-\varepsilon < 1-\varepsilon_0$. Let $m_2' = \max\{m | m \in \bigcup_{j=1}^{2}(X_{n_j} \cup Y_{n_j})\}$ and $n_3 = \max\{n_2, m_2'\}+1$. Then $M(x_{n_k}, y_{n_l}, \frac{t}{4}) < 1-\varepsilon_0$, where $k,l \in \{1,2,3\}$. So, by induction on i, we can obtain a strictly increasing sequence (n_i) of positive integers such that $M(x_{n_k}, y_{n_l}, \frac{t}{4}) < 1-\varepsilon_0$ for all $k,l \in \mathbb{N}$. We are done. □

Theorem 3.4. *Let $f : X \to Y$ be a mapping between two fuzzy metric spaces $(X, M, *)$ and $(Y, N, \star)$. Then the following are equivalent.*

(i) *f is uniformly continuous.*

(ii) *If (x_n) and (y_n) are asymptotic sequences in X, then $(f(x_n))$ and $(f(y_n))$ are asymptotic sequences in Y.*

(iii) *Let A and B be a pair of nonempty subsets of X. Then $N(f(A), f(B), t) = 1$ for every $t > 0$ if $M(A, B, t) = 1$ for every $t > 0$.*

Proof. (i) ⇒ (ii) Assume that f is uniformly continuous. Then, for each $\varepsilon \in (0,1)$ and $t>0$, there exist $r \in (0,1)$ and $s>0$ such that $N(f(x), f(y), t) > 1-\varepsilon$ whenever $x, y \in X$ and $M(x,y,s) > 1-r$. Since (x_n) and (y_n) are asymptotic sequences in X, we can find $N_0 \in \mathbb{N}$ such that $M(x_n, y_n, s) > 1-r$ for all $n > N_0$. So $N(f(x_n), f(y_n), t) > 1-\varepsilon$ for all $n > N_0$, which means that $(f(x_n))$ and $(f(y_n))$ are asymptotic sequences in Y.

(ii) ⇒ (iii) Let A and B be a pair of nonempty subsets of X. Assume that $M(A,B,t) = 1$ for every $t>0$. Then, for each $n \ge 2$, we can pick $x_n \in A$ and $y_n \in B$ such that $M(x_n, y_n, \frac{1}{n}) > 1-\frac{1}{n}$. So (x_n) and (y_n) are asymptotic sequences in X. Due to (ii), $(f(x_n))$ and $(f(y_n))$ are asymptotic sequences in Y, which implies that $N(f(A), f(B), t) = 1$ for every $t>0$.

(iii) ⇒ (i) Assume that f is not uniformly continuous. Then there exist $\varepsilon_0 \in (0,1)$ and $t_0 > 0$ such that for each $r \in (0,1)$ and each $s>0$, there exist $x_0, y_0 \in X$ such that $M(x_0, y_0, s) > 1-r$ and $N(f(x_0), f(y_0), t_0) \le 1-\varepsilon_0$. Therefore, for each $n \ge 2$, we can choose $x_n, y_n \in X$ such that $M(x_n, y_n, \frac{1}{n}) > 1-\frac{1}{n}$ and $N(f(x_n), f(y_n), t_0) \le 1-\varepsilon_0$. Without loss of generality, we may suppose that $\{x_n | n \ge 2\} \cap \{y_n | n \ge 2\} = \emptyset$. So (x_n) and (y_n) are asymptotic sequences in X. Moreover, by Lemma 3.3, we can take a subsequence $((x_{n_i}),(y_{n_i}))$ of $((x_n),(y_n))$ and $\varepsilon' \in (0, \varepsilon_0)$ such that $N(f(x_{n_k}), f(y_{n_l}), \frac{t_0}{4}) < 1-\varepsilon'$ for all $k,l \in \mathbb{N}$, where $n_i < n_j$ if $i<j$. Fix $A = \{x_{n_i} | i \in \mathbb{N}\}$ and $B = \{y_{n_i} | i \in \mathbb{N}\}$. We deduce that $N(f(A), f(B), \frac{t_0}{4}) \le 1-\varepsilon' < 1$. Since (x_{n_i}) and (y_{n_i}) are asymptotic sequences in X, we get that $M(A,B,t) = 1$ for every $t>0$. It follows from (iii) that $N(f(A), f(B), \frac{t_0}{4}) = 1$, which is a contradiction. We finish the proof. □

4. On *RUC* Fuzzy Metric Spaces

In the following discussion, we always assume that the metric of $\mathbb{R}$ is the Euclidian metric $|\cdot|$, and its corresponding standard fuzzy metric is $M_{|\cdot|}(x,y,t) = \frac{t}{t+|x-y|}$.

Definition 4.1. ([17]) A fuzzy metric $(M,*)$ on a set X is called *equinormal* if for each pair of disjoint nonempty closed subsets A and B of (X, τ_M), there exists $s>0$ such that $\sup\{M(a,b,s) | a \in A, b \in B\} < 1$.

Definition 4.2. ([17]) A real value mapping f on a fuzzy metric space $(X, M, *)$ is said to be $\mathbb{R}$-*uniformly continuous* provided that for each $\varepsilon > 0$ there exist $r \in (0,1)$ and $s>0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $M(x,y,s) > 1-r$.

It is trivial to check that a mapping f from a fuzzy metric space $(X, M, *)$ to the fuzzy metric space $(\mathbb{R}, M_{|\cdot|}, \cdot)$ is uniformly continuous if and only if it is $\mathbb{R}$-uniformly continuous.

Lemma 4.3. ([17]) *Let $(X, M, *)$ be a fuzzy metric space. Then every real valued continuous mapping on (X, τ_M) is $\mathbb{R}$-uniformly continuous if and only if $(M,*)$ is an equinormal fuzzy metric on X.*

Definition 4.4. A fuzzy metric space $(X, M, *)$ is said to be $\mathbb{R}$-*uniformly continuous* (or simply *RUC*) if every real valued continuous mapping on (X, τ_M) is $\mathbb{R}$-uniformly continuous.

Immediately, we can see that each compact fuzzy metric space is *RUC* using the proof in [17]. However, the converse is false in general. We illustrate this fact with an example.

Example 4.5. Let $X = (0, 1)$. For any $a, b \in [0, 1]$, denote $a * b = a \cdot b$. For any $x, y \in X$ and $t > 0$, define M by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ xyt, & x \neq y,\ t \leq 1, \\ xy, & x \neq y,\ t > 1. \end{cases}$$

Then $(X, M, *)$ is a fuzzy metric space and τ_M is the discrete topology (see [9]). So $(X, M, *)$ is *RUC*. However, it is clear that $(X, M, *)$ fails to be compact.

Lemma 4.6. *Let (x_{n_i}) be a subsequence of a Cauchy sequence (x_n) in X . If (x_{n_i}) is convergent to $x_0 \in X$, then (x_n) is convergent to x_0.*

Proof. Let $\varepsilon \in (0, 1)$ and $t > 0$. Then we can pick $\varepsilon_1 \in (0, \varepsilon)$ such that

$$(1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon.$$

Since (x_{n_i}) is a subsequence, which converges to x_0, of the Cauchy sequence (x_n) in X , we can find $n_0 \in \mathbb{N}$ such that $M(x_k, x_l, \frac{t}{2}) > 1 - \varepsilon_1$ and $M(x_{n_k}, x_0, \frac{t}{2}) > 1 - \varepsilon_1$ whenever $k, l, n_k \geq n_0$. Hence

$$M(x_k, x_0, t) \geq M(x_k, x_{n_k}, \frac{t}{2}) * M(x_{n_k}, x_0, \frac{t}{2}) \geq (1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon,$$

which implies that (x_n) converges to x_0. □

Lemma 4.7. *Let (x_n) and (y_n) be two sequences in a fuzzy metric space $(X, M, *)$. Let (t_n) be a sequence in $(0, \infty)$ with $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} M(x_n, y_n, t_n) = 1$. If $\xi \in X$ is an accumulation point of (x_n), then ξ is an accumulation point of (y_n).*

Proof. Since $\xi \in X$ is an accumulation point of (x_n), we can choose a subsequence (x_{n_i}) of (x_n) such that (x_{n_i}) converges to ξ. Let $\varepsilon \in (0, 1)$ and $t > 0$. Then we can find $\varepsilon_1 \in (0, \varepsilon)$ such that

$$(1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon.$$

Now, we can choose $N_0 \in \mathbb{N}$ large enough such that $M(x_{n_i}, \xi, \frac{t}{2}) > 1 - \varepsilon_1$, $t_{n_i} < \frac{t}{2}$ and $M(x_{n_i}, y_{n_i}, \frac{t}{2}) \geq M(x_{n_i}, y_{n_i}, t_{n_i}) > 1 - \varepsilon_1$ for all $n_i > N_0$. Hence

$$M(y_{n_i}, \xi, t) \geq M(y_{n_i}, x_{n_i}, \frac{t}{2}) * M(x_{n_i}, \xi, \frac{t}{2}) \geq (1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon.$$

So $\lim_{n_i\to\infty} M(y_{n_i}, \xi, t) = 1$, which implies that ξ is an accumulation point of (y_n). □

Theorem 4.8. *Let $(X, M, *)$ be an RUC fuzzy metric space. Then $(X, M, *)$ is complete and the set $d(X)$ of all accumulation points in X is compact.*

Proof. Suppose that $(X, M, *)$ is not complete. Then there exists a Cauchy sequence (x_n) of distinct points in X such that it does not converge in X. Put $A = \{x_{2m} | m \in \mathbb{N}\}$ and $B = \{x_{2m-1} | m \in \mathbb{N}\}$. Due to Lemma 4.6, we get that A and B is a pair of disjoint nonempty closed subsets of X. According to Lemma 4.3, we can choose $s > 0$ such that $\sup\{M(a, b, s) | a \in A, b \in B\} < 1$. Set $\varepsilon_0 = 1 - \sup\{M(a, b, s) | a \in A, b \in B\}$. Then $0 < \varepsilon_0 < 1$. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$,

$$M(x_n, x_m, s) > 1 - \varepsilon_0 = \sup\{M(a, b, s) | a \in A, b \in B\}.$$

A contradiction occurs.

Next, we are going to prove that $d(X)$ is compact. Obviously, $d(X)$ is closed in X. Suppose that $d(X)$ is not compact. Then there exists a sequence (x_n) of distinct points in $d(X)$ such that any subsequence of (x_n) is not convergent. Let $X_n = \{x_m | m \neq n\}$. Then X_n is closed. By Lemma 4.3, we can find $s > 0$ such that $\sup\{M(x_n, b, s) | b \in X_n\} < 1$. Pick $k(n) \in \mathbb{N}$ such that $\frac{1}{k(n)} < s$. Put $s_n = \min\{\frac{1}{n}, \frac{1}{k(n)}\}$. Then

$$c_n = \sup\{M(x_n, b, s_n) | b \in X_n\} \leq \sup\{M(x_n, b, s) | b \in X_n\} < 1.$$

Note that each x_n is an accumulation point of X, we can choose $y_n \in X$ with $y_n \neq x_n$ such that

$$1 > M(x_n, y_n, s_n) > \max\{1 - \frac{1}{n+1}, c_n\} = b_n$$

for every $n \in \mathbb{N}$. Hence $y_n \in B_M(x_n, 1 - b_n, s_n)$. Notice that, for any $y \in B_M(x_n, 1 - b_n, s_n)$,

$$M(x_n, y, s_n) > b_n \geq c_n = \sup\{M(x_n, b, s_n) | b \in X_n\}.$$

We deduce that $B_M(x_n, 1 - b_n, s_n) \cap X_n = \emptyset$. Fix $X' = \{x_n | n \in \mathbb{N}\}$ and $Y' = \{y_n | n \in \mathbb{N}\}$. Then $X' \cap Y' = \emptyset$ and X' is closed. Note that $\lim_{n\to\infty} s_n = 0$ and $\lim_{n\to\infty} M(x_n, y_n, s_n) = 1$. According to Lemma 4.6 and Lemma 4.7, we obtain that Y' is closed. Due to Lemma 4.3, we can take $s' > 0$ such that $\sup\{M(a, b, s') | a \in X', b \in Y'\} < 1$. Let $\varepsilon \in (0, 1)$. Then there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \min\{s', \varepsilon\}$. Thus

$$M(x_m, y_m, s') \geq M(x_m, y_m, \frac{1}{m}) \geq M(x_m, y_m, s_m) > b_m \geq 1 - \frac{1}{m+1} > 1 - \varepsilon,$$

which is a contradiction. We are done. □

Now we will show that the converse of the preceding theorem is not true, in general.

Example 4.9. Let $X = [0, \frac{1}{3}] \cup \mathbb{N} \cup \{n + \frac{1}{n} | n \in \mathbb{N}\}$. Then $(X, M_{|\cdot|}, \cdot)$ is a fuzzy metric spaces. It is immediate to see that X is complete and $d(X) = [0, \frac{1}{3}]$ is compact in X. Let $f : X \to \mathbb{R}$ be a mapping defined by

$$f(x) = \begin{cases} -1, & x \in [0, \frac{1}{3}], \\ 0, & x \in \mathbb{N}, \\ 1, & x \in \{n + \frac{1}{n} | n \in \mathbb{N}\}. \end{cases}$$

It is straightforward to show that f is continuous. However, f is not $\mathbb{R}$-uniformly continuous. In fact, fix $\varepsilon_0 = \frac{1}{2}$. For any $r \in (0, 1)$ and $s > 0$, there exists $N_0 = [\frac{1-r}{sr}] + 1 \in \mathbb{N}$ such that

$$M_{|\cdot|}(N_0, N_0 + \frac{1}{N_0}, s) = \frac{s}{s + \frac{1}{N_0}} = 1 - \frac{\frac{1}{N_0}}{s + \frac{1}{N_0}} > 1 - r.$$

But

$$|f(N_0) - f(N_0 + \frac{1}{N_0})| = |0 - 1| = 1 > \frac{1}{2} = \varepsilon_0.$$

Consequently, $(X, M_{|\cdot|}, \cdot)$ is not *RUC*.

5. Equivalent Characterizations of *RUC* Fuzzy Metric Spaces

In the section, we will study several equivalent characterizations of $\mathbb{R}$-uniformly continuous fuzzy metric spaces.

Definition 5.1. Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0, 1), t > 0$ and $A \subseteq X$. The set

$$B_M(A, r, t) = \{y \in X | M(A, y, t) > 1 - r\}$$

is called the *open ball with center A and radius r with respect to t*.

Theorem 5.2. *Let $(X, M, *)$ be a fuzzy metric space. Then $(X, M, *)$ is RUC if and only if for any pair of disjoint nonempty closed subsets A and B of X, there exist $r_0 \in (0, 1)$ and $t_0 > 0$ such that $B_M(A, r_0, t_0) \cap B_M(B, r_0, t_0) = \emptyset$.*

Proof. Suppose that $(X, M, *)$ is *RUC*. Let A and B be a pair of disjoint nonempty closed subsets of X. Assume that $B_M(A, r, t) \cap B_M(B, r, t) \neq \emptyset$ for any $r \in (0, 1)$ and $t > 0$. Then for each $n \geq 2$ there exists $z_n \in B_M(A, \frac{1}{n}, \frac{1}{n}) \cap B_M(B, \frac{1}{n}, \frac{1}{n})$. Therefore we can choose $x_n \in A$ and $y_n \in B$ such that $z_n \in B_M(x_n, \frac{1}{n}, \frac{1}{n}) \cap B_M(y_n, \frac{1}{n}, \frac{1}{n})$ for every $n \geq 2$. Let $\varepsilon \in (0, 1)$ and $s > 0$. Then there exists sufficiently large $N_0 \in \mathbb{N}$ such that $\frac{2}{n} < s$ and $(1 - \frac{1}{n}) * (1 - \frac{1}{n}) > 1 - \varepsilon$ for all $n > N_0$. Hence

$$M(x_n, y_n, s) \geq M(x_n, y_n, \frac{2}{n}) \geq M(x_n, z_n, \frac{1}{n}) * M(z_n, y_n, \frac{1}{n}) \geq (1 - \frac{1}{n}) * (1 - \frac{1}{n}) > 1 - \varepsilon$$

for all $n > N_0$. It follows that $\sup\{M(a, b, s) | a \in A, b \in B\} = 1$ for any $s > 0$, which contradicts that $(X, M, *)$ is *RUC* by Lemma 4.3.

Conversely, Let A and B be a pair of disjoint nonempty closed subsets of X. Suppose that there exist $r_0 \in (0, 1)$ and $t_0 > 0$ such that $B_M(A, r_0, t_0) \cap B_M(B, r_0, t_0) = \emptyset$. Then $B_M(a, r_0, t_0) \cap B_M(b, r_0, t_0) = \emptyset$ for any $a \in A$ and $b \in B$. Hence $M(a, b, t_0) \leq 1 - r_0$ for any $a \in A$ and $b \in B$, which means that $\sup\{M(a, b, t_0) | a \in A, b \in B\} \leq 1 - r_0 < 1$. According to Lemma 4.3, we conclude that $(X, M, *)$ is *RUC*. The proof is finished. □

Definition 5.3. Let $(X, M, *)$ be a fuzzy metric space, $Y \subset X$, $r \in (0, 1)$ and $t > 0$. Y is said to be *fuzzy r discrete with respect to t* if $M(x, y, t) < 1 - r$ whenever $x, y \in Y$ and $x \neq y$.

Definition 5.4. Let $(X, M, *)$ be a fuzzy metric space and $Y \subset X$. Y is called a *fuzzy uniformly discrete set* if it is fuzzy r discrete with respect to t for some $r \in (0, 1)$ and $t > 0$.

According to Zorn's lemma, it is straightforward to show that, by the inclusion relationship of the sets, X has a maximal subset which is fuzzy r discrete with respect to t for all $r \in (0, 1)$ and $t > 0$.

Theorem 5.5. *Let $(X, M, *)$ be a fuzzy metric space. Then the following are equivalent.*

(i) *$(X, M, *)$ is RUC.*
(ii) *The set $d(X)$ of all accumulation points in X is compact and the set $X \backslash B_M(d(X), r, t)$ is fuzzy uniformly discrete for any $r \in (0, 1)$ and $t > 0$.*
(iii) *Let (x_n) be a sequence of pairwise distinct points in X. If $\lim_{n\to\infty} M(x_n, \{x_m | m \neq n\}, s) = 1$ for any $s > 0$, then (x_n) has an accumulation point in X.*

Proof. (i) ⇒ (ii) Assume that $(X, M, *)$ is *RUC*. Then we obtain that $d(X)$ is compact by Theorem 4.8. Suppose that $X \backslash B_M(d(X), r_0, t_0)$ is not fuzzy uniformly discrete for some $r_0 \in (0, 1)$ and $t_0 > 0$. Then, for any $n \geq 2$, there exist $x_n, y_n \in X \backslash B_M(d(X), r_0, t_0)$ such that $M(x_n, y_n, \frac{1}{n}) > 1 - \frac{1}{n}$. Without loss of generality, we may assume that $\{x_2, y_2, x_3, y_3, \cdots, x_n, y_n, \cdots\}$ is a set of pairwise distinct points. Notice that $X \backslash B_M(d(X), r_0, t_0)$ is an isolated set. It follows that $A = \{x_n | n \geq 2\}$ and $B = \{y_n | n \geq 2\}$ is a pair of disjoint nonempty closed subsets of X. By Theorem 5.2, there exist $r_0 \in (0, 1)$ and $t_0 > 0$ such that $B_M(A, r_0, t_0) \cap B_M(B, r_0, t_0) = \emptyset$. Consequently, for any $n \geq 2$, $M(x_n, y_n, t_0) \leq 1 - r_0$. On the other hand, we can take $N_0 \in \mathbb{N}$ such that $\frac{1}{N_0} < r_0$ and $\frac{1}{N_0} < t_0$. Then

$$M(x_n, y_n, t_0) \geq M(x_n, y_n, \frac{1}{n}) > 1 - \frac{1}{n} > 1 - r_0$$

for all $n \geq N_0$, which is a contradiction.

(ii) ⇒ (iii) Assume that the set $d(X)$ of all accumulation points in X is compact and the set $X \backslash B_M(d(X), r, t)$ is fuzzy uniformly discrete for any $r \in (0, 1)$ and $t > 0$. Let (x_n) be a sequence of pairwise distinct points in X. Suppose that $\lim_{n\to\infty} M(x_n, \{x_m | m \neq n\}, s) = 1$ for any $s > 0$. We claim that $\{x_n | n \in \mathbb{N}\} \cap (X \backslash B_M(d(X), r, t))$ is a finite set for any $r \in (0, 1)$ and $t > 0$. If not, we can find $r_0 \in (0, 1)$ and $t_0 > 0$ such that $\{x_{n_i} | i \in \mathbb{N}\}$ is infinite, where $\{x_{n_i} | i \in \mathbb{N}\} = \{x_n | n \in \mathbb{N}\} \cap (X \backslash B_M(d(X), r_0, t_0))$. Since $X \backslash B_M(d(X), r_0, t_0)$ is fuzzy uniformly discrete, we can pick $r_1 \in (0, r_0)$ and $t_1 \in (0, t_0)$ such that $M(x, y, t_1) < 1 - r_1$ whenever $x, y \in \{x_{n_i} | i \in \mathbb{N}\}$ and $x \neq y$. Observe that, for any $x \in \{x_{n_i} | i \in \mathbb{N}\}$ and $z \in \{x_n | n \in \mathbb{N}\} \cap B_M(d(X), r_0, t_0)$,

$$M(x, z, t_1) \leq M(x, z, t_0) \leq 1 - r_0 < 1 - r_1.$$

Thus, for each $x_{n_j} \in \{x_{n_i} | i \in \mathbb{N}\}$, we get that $M(x_{n_j}, \{x_m | m \neq n_j\}, t_1) < 1 - r_1$. It follows that

$$\sup\{M(x_{n_j}, \{x_m | m \neq n_j\}, t_1) | j \in \mathbb{N}\} \leq 1 - r_1 < 1,$$

which contradicts $\lim_{n\to\infty} M(x_n, \{x_m | m \neq n\}, t_1) = 1$. Since $\{x_n | n \in \mathbb{N}\} \cap B_M(d(X), r, t)$ is an infinite set for any $r \in (0, 1)$ and $t > 0$, we can take $z_n \in \{x_n | n \in \mathbb{N}\} \cap B_M(d(X), \frac{1}{n}, \frac{1}{n})$ for every $n \geq 2$. Then, for any $n \geq 2$, there exists $y_n \in d(X)$ such that $M(z_n, y_n, \frac{1}{n}) > 1 - \frac{1}{n}$. Hence $\lim_{n\to\infty} M(z_n, y_n, \frac{1}{n}) = 1$. Since $d(X)$ is compact, we conclude that (y_n) has an accumulation point $\xi \in X$. According to Lemma 4.7, ξ is an accumulation point of (z_n). Observe that (z_n) is a subsequence of (x_n). We get that (x_n) has an accumulation point ξ in X.

(iii) ⇒ (i) Assume that $(X, M, *)$ is not *RUC*. Due to Lemma 4.3, we can find a pair of disjoint nonempty closed subsets A and B of X such that $\sup\{M(a, b, t) | a \in A, b \in B\} = 1$ for all $t > 0$. Therefore, for any $n \geq 2$, there exist $y_n \in A$ and $z_n \in B$ such that $M(y_n, z_n, \frac{1}{n}) > 1 - \frac{1}{n}$. Hence we can obtain a sequence (x_n) in X, where

$$x_n = \begin{cases} y_m, & n = 2m - 3 \text{ with } m \geq 2, \\ z_m, & n = 2m - 2 \text{ with } m \geq 2. \end{cases}$$

Without loss of generality, we may assume that (x_n) is a sequence of pairwise distinct points in X. Let $s > 0$. Obviously, for any $n \in \mathbb{N}$, $n = 2l - 3$ or $2l - 2$, where $l \geq 2$. Now, let n be sufficiently large. If $n = 2l - 3$ with $\frac{1}{l} < s$, then $x_n = y_l$, which implies that $z_l \in \{x_m | m \neq n\}$. Hence

$$M(x_n, \{x_m | m \neq n\}, s) = M(y_l, \{x_m | m \neq n\}, s) \geq M(y_l, z_l, s) \geq M(y_l, z_l, \frac{1}{l}) \geq 1 - \frac{1}{l}.$$

If $n = 2l - 2$ with $\frac{1}{l} < s$, then $x_n = z_l$, which implies that $y_l \in \{x_m | m \neq n\}$. So

$$M(x_n, \{x_m | m \neq n\}, s) = M(z_l, \{x_m | m \neq n\}, s) \geq M(z_l, y_l, s) \geq M(z_l, y_l, \frac{1}{l}) \geq 1 - \frac{1}{l}.$$

Since $l \to \infty$ when $n \to \infty$, we obtain that

$$\lim_{n\to\infty} M(x_n, \{x_m | m \neq n\}, s) \geq \lim_{l\to\infty} (1 - \frac{1}{l}) = 1,$$

which means that $\lim_{n\to\infty} M(x_n, \{x_m | m \neq n\}, s) = 1$. According to (iii), we conclude that (x_n) has an accumulation point $\xi \in X$. Since $A \cup B$ is closed, we get that $\xi \in A \cup B$. Without loss of generality, we may suppose that $\xi \in A$. Thus ξ is an accumulation point of the sequence (y_n). Note that $\lim_{n\to\infty} M(y_n, z_n, \frac{1}{n}) = 1$. Due to Lemma 4.7, we deduce that ξ is an accumulation point of the sequence (z_n). Hence $\xi \in B$, which implies that $\xi \in A \cap B$. A contradiction occurs. This completes the proof. □

6. Conclusion

In this work, we have introduced the concept of *RUC* fuzzy metric space and given several equivalent characterizations for a fuzzy metric space to be an *RUC* fuzzy metric space. Can we give other more equivalent conditions for a fuzzy metric space to be an *RUC* fuzzy metric space?

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