Filomat 34:13 (2020), 4513–4522
https://doi.org/10.2298/FIL2013513U

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximating the Finite Mellin and Sumudu Transforms Utilizing Wavelet Transform

Fuat Usta[a], Hüseyin Budak[a], Mehmet Zeki Sarikaya[a]

[a]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*

Abstract. In this study, some approximates for the finite Wavelet transform of different classes of absolutely continues mappings are presented using Wavelet transform of unit function. Then, with the help of these approximates, some other approximates for the finite Mellin and Sumudu transforms are given.

1. Introduction

Integral transform method is one of the frequently used methods in solving ordinary and partial differential equations in mathematics. Indeed, increasing studies in applied mathematics, mathematical physics, and engineering have shown the first signal that integral transforms will be one of the important methods in operational mathematics. The increasing demand for mathematical methods that provide both theory and application in science and engineering has clearly demonstrated the usefulness and need for integral transforms. Leaving aside a number of the mathematical and physical applications of integral transform, the subject of integral transformation still constitutes one of the major areas of interest in scientific studies and research [3, 9].

The importance of integral transforms lies in providing powerful operational methods for solving initial value and initial-boundary value problems for linear differential and integral equations. Especially if the solution in the domain of the partial differential equation is quite time consuming and challenging, the integral transform equation provides the opportunity to move and solve it in a form where mathematical operations are much easier. In this way, after finding the solution in transformed form, using the inverse integral transform method, the solution of the original partial differential equation is obtained. So, the solution process in a partial differential equation solved using integral transform can be summarized as transform-solve-inverse transform [16].

In addition to these, in [4–6] and [13–15], some explicit bounds for the finite Hilbert transform are given utilizing the fundamental integral identity for absolutely continuous mappings. At the same studies, some numerical experiments for the obtained approximation are also presented.

In general, an integral transform of function $f(t)$ can be defined as following,

$$\mathcal{F}_K f(s) = \int_a^b K(s,t) f(t) dt,$$

2010 *Mathematics Subject Classification*. Primary 44A15; Secondary 26D15, 41A55, 41A80
Keywords. Finite Wavelet transform, Mellin transform, Sumudu transform, analytic inequalities
Received: 19 January 2020; Revised: 16 August 2020; Accepted: 23 August 2020
Communicated by Ljubiša D.R. Kočinac
Email addresses: fuatusta@duzce.edu.tr (Fuat Usta), hsyn.budak@gmail.com (Hüseyin Budak), sarikayamz@gmail.com (Mehmet Zeki Sarikaya)

where K is a kernel depend on s and t. To be advantageous to use such a transformation in any problem, $\mathcal{F}_K f(s)$ should be easier to identify and manipulate than $f(t)$.

Recently, the finite Wavelet transforms have been implemented successfully in the areas of sound processing, signal analysis, data compression (see, for details, [1, 11] and the references cited therein). Using the notation of inner product, the wavelet transform of a function $f(t)$ can be expressed as

$$W_\varphi f(a,b;s,u) = \langle f,\varphi\rangle = \frac{1}{\sqrt{s}}\int_a^b f(t)\varphi^*\left(\frac{t-u}{s}\right)dt,$$

where $u \in \mathbb{R}$ is a translation parameter and the symbol $s > 0$ represents the scaling or dilating parameter, which determines the time and frequency resolutions of the scaled base wavelet $\varphi\left(\frac{t-u}{s}\right)$. The specific values of s are inversely proportional to the frequency. The symbol u is the shifting parameter, which translates the scaled wavelet along the time axis. The symbol $\varphi^*(\cdot)$ denotes the complex conjugation of the base wavelet $\varphi(\cdot)$. As an example, if the Morlet wavelet $\varphi(t) = e^{i2\pi f_0 t}e^{-(\alpha t^2)/\beta^2}$ is chosen as the base wavelet, its scaled version will be expressed as

$$\varphi\left(\frac{t-u}{s}\right) = e^{i2\pi f_0 \frac{t-u}{s}} e^{-\alpha(t-u)^2/s^2\beta^2},$$

with the parameters f_0, α and β all being constants.

In addition to this, Mellin transform occurs in a number of areas of engineering and applied mathematics. The Mellin integral transform of a sufficiently well-behaved function $f(t)$ is defined as

$$\mathcal{M}\{f(t),s\} = \int_0^{+\infty} f(t)t^{s-1}dt.$$

For more information regarding the Mellin integral transform including its properties and particular cases we refer the interested reader to e.g. [2, 7, 8].

Recently, Watugala introduced a new transform and called as Sumudu transform which is defined by the following formula

$$\mathcal{S}\{f(t),s\} = \frac{1}{s}\int_0^{+\infty} f(t)e^{-t/s}dt, \qquad s \in (-\tau_1,\tau_2),$$

and applied this new transform to the solution of ordinary differential equations and control engineering problems, see [10, 12].

In this study, motivated by the some approximates for the finite Hilbert transform [4–6], we will show some approximates for the finite Wavelet transform of different classes of absolutely continues mappings. Then with the help of these approximates, some other approximates for the finite Mellin and Sumudu transforms will be given.

2. Some Convergence Analysis

The following theorem provides error bounds for finite Wavelet transform.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be such that $f' : (a,b) \to \mathbb{R}$ is absolutely continuous on (a,b) and $\varphi(\cdot)$ is a basis function for Wavelet transform. Then we have the following bounds:*

$$\left|W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) - \frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt - \frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt\right|$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{4\sqrt{s}} \int_a^b |t-u|^2 \left|\varphi\left(\dfrac{t-u}{s}\right)\right| dt, & \text{if } f'' \in L_\infty[a,b], \\ \dfrac{\|f''\|_p}{2\sqrt{s}(q+1)^{1/q}} \int_a^b |t-u|^{(q+1)/q} \left|\varphi\left(\dfrac{t-u}{s}\right)\right| dt, & \text{if } f'' \in L_p[a,b],\ p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \dfrac{\|f''\|_1}{2\sqrt{s}} \int_a^b |u-t| \left|\varphi\left(\dfrac{t-u}{s}\right)\right| dt, & \text{if } f'' \in L_1[a,b], \end{cases}$$

for all $t \in (a,b)$, where $\|\cdot\|_p$ are the usual Lebesque norms in $L_p[a,b]$ $(1 \leq p \leq \infty)$.

Proof. Let start with Wavelet transform of unit function. In other words, for $f(t) = 1$, we have

$$W_\varphi 1(a,b;s,u) = \frac{1}{\sqrt{s}} \int_a^b \varphi\left(\frac{t-u}{s}\right) dt =: \Phi_\varphi(a,b;s,u).$$

Then, obviously

$$\begin{aligned} W_\varphi f(a,b;s,u) &= \frac{1}{\sqrt{s}} \int_a^b [f(t) - f(u) + f(u)] \varphi\left(\frac{t-u}{s}\right) dt, \\ &= \frac{1}{\sqrt{s}} \int_a^b [f(t) - f(u)] \varphi\left(\frac{t-u}{s}\right) dt + f(u)\Phi_\varphi(a,b;s,u). \end{aligned}$$

Using the following elementary identity, which can be proved using the integration by parts formula, which is

$$\int_\alpha^\beta u(v)dv = \frac{u(\alpha)+u(\beta)}{2}(\beta-\alpha) + \int_\alpha^\beta \left(\frac{\alpha+\beta}{2} - v\right) u'(v)dv,$$

we obtain

$$\begin{aligned} W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) &= \frac{1}{\sqrt{s}} \int_a^b \left[\int_u^t f'(x)dx\right] \varphi\left(\frac{t-u}{s}\right) dt, \\ &= \frac{1}{\sqrt{s}} \int_a^b \left\{ \frac{f'(t)+f'(u)}{2}(t-u) + \int_u^t \left(\frac{t+u}{2} - x\right) f''(x)dx \right\} \varphi\left(\frac{t-u}{s}\right) dt, \\ &= \frac{1}{2\sqrt{s}} \int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right) dt + \frac{f'(u)}{2\sqrt{s}} \int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right) dt \\ &+ \frac{1}{\sqrt{s}} \int_a^b \int_u^t \left(\frac{t+u}{2} - x\right) f''(x) \varphi\left(\frac{t-u}{s}\right) dxdt. \end{aligned}$$

Then by rearranging the above equations, we can obtain the following identity,

$$W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) - \frac{1}{2\sqrt{s}} \int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right) dt - \frac{f'(u)}{2\sqrt{s}} \int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right) dt$$

$$= \frac{1}{\sqrt{s}} \int_a^b \int_u^t \left(\frac{t+u}{2} - x\right) f''(x) \varphi\left(\frac{t-u}{s}\right) dxdt.$$

Using the properties of modulus, we obtain

$$\begin{aligned}&\left|W_{\varphi}f(a,b;s,u)-f(u)\Phi_{\varphi}(a,b;s,u)-\frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt-\frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt\right|\\ =\;&\frac{1}{\sqrt{s}}\left|\int_a^b\left(\int_u^t\left(\frac{t+u}{2}-x\right)f''(x)dx\right)\varphi\left(\frac{t-u}{s}\right)dt\right|,\\ \leq\;&\frac{1}{\sqrt{s}}\int_a^b\left|\int_u^t\left(\frac{t+u}{2}-x\right)f''(x)dx\right|\left|\varphi\left(\frac{t-u}{s}\right)\right|dt.\end{aligned}$$

So if we choose $f''\in L_\infty[a,b]$, we have

$$\begin{aligned}&\left|W_{\varphi}f(a,b;s,u)-f(u)\Phi_{\varphi}(a,b;s,u)-\frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt-\frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt\right|\\ \leq\;&\frac{\|f''\|_\infty}{\sqrt{s}}\int_a^b\int_u^t\left|\left(\frac{t+u}{2}-x\right)\right|dx\left|\varphi\left(\frac{t-u}{s}\right)\right|dt,\\ \leq\;&\frac{\|f''\|_\infty}{4\sqrt{s}}\int_a^b|t-u|^2\left|\varphi\left(\frac{t-u}{s}\right)\right|dt.\end{aligned}$$

Now if we choose $f''\in L_p[a,b]$ and apply the well-known Hölder inequality, we can obtain the following inequality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}\left|\int_u^t\left(\frac{t+u}{2}-x\right)f''(x)dx\right|&\leq\left|\int_u^t|f''(x)|^pdx\right|^{1/p}\left|\int_u^t\left|\frac{t+u}{2}-x\right|^qdx\right|^{1/q},\\ &\leq\left(\int_a^b|f''(x)|^pdx\right)^{1/p}\left|\int_u^{(t+u)/2}\left(\frac{t+u}{2}-x\right)^qdx+\int_{(t+u)/2}^t\left(x-\frac{t+u}{2}\right)^qdx\right|^{1/q},\\ &=\|f''\|_p\frac{|u-t|^{(q+1)/q}}{2(q+1)^{1/q}},\end{aligned}$$

for all $u,t\in(a,b)$. Then,

$$\begin{aligned}&\left|W_{\varphi}f(a,b;s,u)-f(u)\Phi_{\varphi}(a,b;s,u)-\frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt-\frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt\right|\\ \leq\;&\frac{\|f''\|_p}{2\sqrt{s}(q+1)^{1/q}}\int_a^b|t-u|^{(q+1)/q}\left|\varphi\left(\frac{t-u}{s}\right)\right|dt.\end{aligned}$$

and the second bound of theorem is proved.

Finally, we observe that

$$\left|\int_u^t \left(\frac{t+u}{2}-x\right) f''(x)dx\right| \leq \sup_{x\in[u,t]}\left|\frac{t+u}{2}-x\right|\left|\int_u^t f''(x)dx\right|, \leq \frac{|u-t|}{2}\left|\int_a^b f''(x)dx\right|, \leq \frac{|u-t|}{2}\|f''\|_1.$$

Consequently,

$$\left|W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) - \frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt - \frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt\right|$$

$$\leq \frac{\|f''\|_1}{2\sqrt{s}}\int_a^b |u-t|\left|\varphi\left(\frac{t-u}{s}\right)\right|dt,$$

thus the theorem is proved. □

Corollary 2.2. *If we choose* $\varphi(t) = \frac{1}{\pi}PV\frac{1}{\sqrt{s}t}$ *with the same conditions in Theorem 2.1, we have the following result*

$$\left|W_\varphi f(a,b;s,u) - \frac{f(u)}{\pi}\ln\frac{b-u}{u-a} - \frac{1}{2\pi}[f(b)-f(a)+f'(u)(b-a)]\right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{4\pi}\left[\frac{(b-a)^2}{4}+\left(u-\frac{a+b}{2}\right)^2\right], & \text{if } f''\in L_\infty[a,b],\\ \frac{q\|f''\|_p}{2\pi(q+1)^{(q+1)/q}}[(u-a)^{(q+1)/q}+(b-u)^{(q+1)/q}], & \text{if } f''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{\|f''\|_1}{2\pi}(b-a), & \text{if } f''\in L_1[a,b], \end{cases}$$

given in [5].

Corollary 2.3. *If we take* $\varphi(t) = \sqrt{s}(ts+u)^{s-1}$ *with the same conditions in Theorem 2.1, we get the following error bound for the Mellin transform*

$$\left|\mathcal{M}\{f(t),s\} - f(u)\frac{b^s-a^s}{s} + \frac{1}{2}\int_a^b f'(t)(t-u)t^{s-1}dt - \frac{f'(u)}{2}\left(\frac{b^{s+1}}{s+1}-\frac{ub^s}{s}-\frac{s^{s+1}}{s+1}+\frac{ua^s}{s}\right)\right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{4}\left(\frac{b^{s+2}}{s+2}-2u\frac{b^{s+1}}{s+1}+u^2\frac{b^s}{s}-\frac{a^{s+2}}{s+2}+2u\frac{a^{s+1}}{s+1}-u^2\frac{a^s}{s}\right), & \text{if } f''\in L_\infty[a,b],\\ \frac{\|f''\|_p}{2(q+1)^{1/q}}\int_a^b |t-u|^{(q+1)/q}\left|t^{s-1}\right|dt, & \text{if } f''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{\|f''\|_1}{2}\left(u\frac{b^s}{s}-\frac{b^{s+1}}{s+1}-u\frac{a^s}{s}+\frac{a^{s+1}}{s+1}\right), & \text{if } f''\in L_1[a,b]. \end{cases}$$

Proof. Let start with the substituting $\varphi(t) = \sqrt{s}(ts+u)^{s-1}$ in the result of Theorem 2.1, we have

$$\left|\mathcal{M}\{f(t),s\} - f(u)\int_a^b t^{s-1}dt - \frac{1}{2}\int_a^b f'(t)(t-u)t^{s-1}dt - \frac{f'(u)}{2}\int_a^b (t-u)t^{s-1}dt\right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{4}\int_a^b |t-u|^2\,|t|^{s-1}\,dt, & \text{if } f'' \in L_\infty[a,b], \\ \frac{\|f''\|_p}{2(q+1)^{1/q}}\int_a^b |t-u|^{(q+1)/q}\,|t|^{s-1}\,dt, & \text{if } f'' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\|f''\|_1}{2}\int_a^b |u-t|\,|t|^{s-1}\,dt, & \text{if } f'' \in L_1[a,b], \end{cases}$$

for all $t \in (a,b)$, where $\|\cdot\|_p$ are the usual Lebesque norms in $L_p[a,b]$ $(1 \leq p \leq \infty)$. And then, by calculating the above integrals, we have obtained the desired results, thus the proof is completed. □

Corollary 2.4. *If we take $\varphi(t) = \frac{1}{\sqrt{s}}e^{-(t+u/s)}$ with the same conditions in Theorem 2.1, we get the following error bound for the Sumudu transform*

$$\left|\mathcal{S}\{f(t),s\} - f(u)[e^{-a/s} - e^{-b/s}] - \frac{1}{2s}\int_a^b f'(t)(t-u)e^{-t/s}dt - \frac{f'(u)}{2}[e^{-a/s}(a+s-u) - e^{-b/s}(b+s-u)]\right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{4}[e^{-a/s}(2s(a-u)+(a-u)^2+2s^2) - e^{-b/s}(2s(b-u)+(b-u)^2+2s^2)], & \text{if } f'' \in L_\infty[a,b], \\ \frac{\|f''\|_p}{2s(q+1)^{1/q}}\int_a^b |t-u|^{(q+1)/q}e^{-t/s}dt, & \text{if } f'' \in L_p[a,b], p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\|f''\|_1}{2}[e^{-b/s}(b+s-u) - e^{-a/s}(a+s-u)], & \text{if } f'' \in L_1[a,b]. \end{cases}$$

Proof. The similar proof can be performed by following the similar step for the proof of Corollary 2.3. □

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be such that $f'' : (a,b) \to \mathbb{R}$ is absolutely continuous on (a,b) and $\varphi(\cdot)$ is a basis function for Wavelet transform. Then we have the following bounds:*

$$\left|W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) - \frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt - \frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt\right|$$

$$\leq \begin{cases} \frac{\|f'''\|_\infty}{12\sqrt{s}}\int_a^b |t-u|^2\left|\varphi\left(\frac{t-u}{s}\right)\right|dt, & \text{if } f''' \in L_\infty[a,b], \\ \frac{\|f'''\|_p[B(q+1,q+1)]^{1/q}}{4\sqrt{s}(q+1)^{1/q}}\int_a^b |t-u|^{2+1/q}\left|\varphi\left(\frac{t-u}{s}\right)\right|dt, & \text{if } f''' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\|f'''\|_1}{4\sqrt{s}}\int_a^b (u-t)^2\left|\varphi\left(\frac{t-u}{s}\right)\right|dt, & \text{if } f''' \in L_1[a,b], \end{cases}$$

for all $t \in (a,b)$, where $\|\cdot\|_p$ are the usual Lebesque norms in $L_p[a,b]$ $(1 \leq p \leq \infty)$ and $B(\cdot,\cdot)$ is beta function.

Proof. By following the similar procedure to that in previous theorem, we can easily obtain that

$$\begin{aligned}
&\left| W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) - \frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt - \frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt \right| \\
\leq\ & \frac{1}{2\sqrt{s}}\int_a^b \left| \int_u^t ((x-t)(u-x)) f'''(x)dx \right| \left| \varphi\left(\frac{t-u}{s}\right) \right| dt,
\end{aligned}$$

using the following integration by parts formula

$$\int_\alpha^\beta u(v)dv = \frac{u(\alpha)+u(\beta)}{2}(\beta-\alpha) - \frac{1}{2}\int_\alpha^\beta ((v-\alpha)(\beta-v))\, u''(v)dv.$$

So if we choose $f''' \in L_\infty[a,b]$, we have

$$\begin{aligned}
&\left| W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) - \frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt - \frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt \right| \\
\leq\ & \frac{\|f'''\|_\infty}{2\sqrt{s}}\int_a^b \int_u^t |((x-t)(u-x))|\, dx \left| \varphi\left(\frac{t-u}{s}\right) \right| dt, \\
\leq\ & \frac{\|f'''\|_\infty}{12\sqrt{s}}\int_a^b |t-u|^2 \left| \varphi\left(\frac{t-u}{s}\right) \right| dt.
\end{aligned}$$

Now if we choose $f''' \in L_p[a,b]$ and apply well-known Hölder inequality, we can state for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, that

$$\begin{aligned}
\left| \int_u^t ((x-t)(u-x)) f'''(x)dx \right| &\leq \left| \int_u^t |f'''(x)|^p dx \right|^{1/p} \left| \int_u^t |(x-t)|^q\, |(u-x)|^q\, dx \right|^{1/q}, \\
&\leq \|f'''\|_p |u-t|^{2+1/q} [B(q+1,q+1)]^{1/q},
\end{aligned}$$

for all $u, t \in (a,b)$. Then,

$$\begin{aligned}
&\left| W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) - \frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt - \frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt \right| \\
\leq\ & \frac{\|f'''\|_p [B(q+1,q+1)]^{1/q}}{2\sqrt{s}(q+1)^{1/q}}\int_a^b |t-u|^{2+1/q} \left| \varphi\left(\frac{t-u}{s}\right) \right| dt.
\end{aligned}$$

and the second bound of theorem is proved.

Finally, we observe that

$$\begin{aligned}\left|\int_u^t ((x-t)(u-x)) f'''(x)dx\right| &\leq \max_{x\in[u,t]} |(x-t)(u-x)| \left|\int_u^t f''(x)dx\right|, \\ &\leq \frac{(u-t)^2}{2}\left|\int_a^b f'''(x)dx\right|, \\ &\leq \frac{(u-t)^2}{2}\|f'''\|_1.\end{aligned}$$

Consequently,

$$\left|W_\varphi f(a,b;s,u) - f(u)\Phi_\varphi(a,b;s,u) - \frac{1}{2\sqrt{s}}\int_a^b f'(t)(t-u)\varphi\left(\frac{t-u}{s}\right)dt - \frac{f'(u)}{2\sqrt{s}}\int_a^b (t-u)\varphi\left(\frac{t-u}{s}\right)dt\right|$$

$$\leq \frac{\|f'''\|_1}{4\sqrt{s}}\int_a^b (u-t)^2\left|\varphi\left(\frac{t-u}{s}\right)\right|dt.$$

and the theorem is proved. □

Corollary 2.6. *If we choose* $\varphi(t) = \frac{1}{\pi}PV\frac{1}{\sqrt{s}t}$ *with the same conditions in Theorem 2.5 we have the following result*

$$\left|W_\varphi f(a,b;s,u) - \frac{f(u)}{\pi}\ln\frac{b-u}{u-a} - \frac{1}{2\pi}[f(b)-f(a)+f'(u)(b-a)]\right|$$

$$\leq \begin{cases} \dfrac{\|f'''\|_\infty}{12\pi}\left[\dfrac{(b-a)^2}{12}+\left(u-\dfrac{a+b}{2}\right)^2\right], & \text{if } f''' \in L_\infty[a,b], \\ \dfrac{q\|f'''\|_p[B(q+1,q+1)]^{1/q}}{2\pi(2q+1)}[(u-a)^{2+1/q}+(b-u)^{2+1/q}], & \text{if } f''' \in L_p[a,b],\ p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \dfrac{\|f'''\|_1}{8\pi}\left[\dfrac{(b-a)^2}{4}+\left(u-\dfrac{a+b}{2}\right)^2\right], & \text{if } f''' \in L_1[a,b], \end{cases}$$

given in [5].

Corollary 2.7. *If we take* $\varphi(t) = \sqrt{s}(ts+u)^{s-1}$ *with the same conditions in Theorem 2.5 we get the following error bound for the Mellin transform*

$$\left|\mathcal{M}f\{f(t),s\} - f(u)\frac{b^s-a^s}{s} + \frac{1}{2}\int_a^b f'(t)(t-u)t^{s-1}dt - \frac{f'(u)}{2}\left(\frac{b^{s+1}}{s+1} - \frac{ub^s}{s} - \frac{s^{s+1}}{s+1} + \frac{ua^s}{s}\right)\right|$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{12}\left(\dfrac{b^{s+2}}{s+2} - 2u\dfrac{b^{s+1}}{s+1} + u^2\dfrac{b^s}{s} - \dfrac{a^{s+2}}{s+2} + 2u\dfrac{a^{s+1}}{s+1} - u^2\dfrac{a^s}{s}\right), & \text{if } f'' \in L_\infty[a,b], \\ \dfrac{\|f'''\|_p[B(q+1,q+1)]^{1/q}}{4(q+1)^{1/q}} \int_a^b |t-u|^{2+1/q}\left|t^{s-1}\right| dt, & \text{if } f'' \in L_p[a,b],\ p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \dfrac{\|f''\|_1}{4}\left(\dfrac{b^{s+2}}{s+2} - 2u\dfrac{b^{s+1}}{s+1} + u^2\dfrac{b^s}{s} - \dfrac{a^{s+2}}{s+2} + 2u\dfrac{a^{s+1}}{s+1} - u^2\dfrac{a^s}{s}\right), & \text{if } f'' \in L_1[a,b]. \end{cases}$$

Proof. The similar proof can be performed by following the similar step for the proof of Corollary 2.3. □

Corollary 2.8. *If we take* $\varphi(t) = \dfrac{1}{\sqrt{s}}e^{-(t+u/s)}$ *with the same conditions in Theorem 2.5 we get the following error bound for the Sumudu transform*

$$\left|\mathcal{S}\{f(t),s\} - f(u)[e^{-a/s} - e^{-b/s}] - \frac{1}{2s}\int_a^b f'(t)(t-u)e^{-t/s}dt - \frac{f'(u)}{2}[e^{-a/s}(a+s-u) - e^{-b/s}(b+s-u)]\right|$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{12}[e^{-a/s}(2s(a-u)+(a-u)^2+2s^2) - e^{-b/s}(2s(b-u)+(b-u)^2+2s^2)], & \text{if } f'' \in L_\infty[a,b], \\ \dfrac{\|f'''\|_p[B(q+1,q+1)]^{1/q}}{4(q+1)^{1/q}} \int_a^b |t-u|^{2+1/q}\left|e^{-t/s}\right| dt, & \text{if } f'' \in L_p[a,b], \\ & p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1. \\ \dfrac{\|f''\|_1}{4}[s(-e^{-b/s})(2s^2+2s(b-u)+(b-u)^2) - s(-e^{-a/s})(2s^2+2s(a-u)+(a-u)^2)], & \text{if } f'' \in L_1[a,b]. \end{cases}$$

Proof. The similar proof can be performed by following the similar step for the proof of Corollary 2.3. □

3. Concluding Remarks

All in all, the finite Wavelet transform plays a significant role in scientific and engineering computing. In order to find some new approximations of the finite Wavelet transform, different classes of absolutely continues functions for finite Wavelet transform have been studied with the help of some fundamental identities. Then using these approximates, some other approximates for the finite Mellin and Sumudu transforms are presented.

References

[1] C.S. Burrus, R.A. Gopinath, H. Guo, Introduction to Wavelets and Wavelet Transforms: a Primer, Prentice-Hall, Upper Saddle River, NJ, 1998.

[2] G. Davies, Integral Transforms and Their Applications, 2nd ed., Springer-Verlag, New York, NY, 1984.

[3] B. Davis, Integral Transforms and their Applications, Applied Mathematical Sciences 26, Springer-Verlag, New York, 1978.

[4] S.S. Dragomir, Approximating the finite Hilbert transform via an Ostrowski type inequality for functions of bounded variation, J. Inequal. Pure Appl. Math. 3 (2002), Article 51, 19 pages.

[5] N.M. Dragomir, S.S. Dragomir, P. Farrell, Approximating the finite Hilbert transform via trapezoid type inequalities, Comput. Math. Appl. 43 (2002), 1359–1369.

[6] N.M. Dragomir, S.S. Dragomir, P.M. Farrell, G.W. Baxter, A quadrature rule for the finite Hilbert transform via trapezoid type inequalities, J. Appl. Math. Comput. 13 (2003), 67–84.

[7] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Tables of Integral Transfer, McGraw-Hill Book Co., New York, NY, 1954.

[8] B. Jacqueline, P. Bertrand, J.-P. Ovarlez, The Mellin transform, In: Transforms and Applications Handbook, Alexander Poularikas (ed,), CRC Press, Boca Raton, Florida, 1996.
[9] J.W. Miles, Integral Transforms in Applied Mathematics, Cambridge University Press, Cambridge, 1971.
[10] F. Oberhettinger , Tables of Mellin Transform, 2nd ed., Springer-Verlag, New York, NY, 1974.
[11] J.T. Olkkonen, Discrete Wavelet Transforms: Theory and Applications, Intech, 2011.
[12] I.N. Sneddon, The Use of Integral Transform, McGraw-Hill Book Co., New York, NY, 1972.
[13] F. Usta, Approximating the finite Hilbert transform for absolutely continuous mappings and applications in numerical integration, Adv. Appl. Clifford Algebr. 28, 78 (2018).
[14] F. Usta, A note on approximating finite Hilbert transform and quadrature formula, Mediterr. J. Math. (2019) 16:125
[15] F. Usta, On approximating the finite hilbert transform and applications in numerical integration, Math. Methods Appl. Sci. 42 (2019) 5174–5183.
[16] E.J. Watson, Laplace Transforms and Applications, Van NostrandReinhold, New York, 1981