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The Absolute Center of p-Groups of Maximal Class

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Abstract. The purpose of this paper is to determine $L(G)$, the absolute center of the group G, when G is a p-group of maximal class. Particularly we find $L(G)$ for metabelian p-groups of maximal class, all p-groups of maximal class of order less than p^6 and p-groups of maximal class for $p = 2, 3$.

1. Introduction

In 1994, Hegarty [6] introduced $L(G)$, the absolute center of a group G as follows: $L(G) = \{g \in G \mid g^\alpha = g$ for all $\alpha \in \mathrm{Aut}(G)\}$. As we see there is an analogue between $L(G)$ and $Z(G)$, on the other hand we may define $Z(G) = \{g \in G \mid g^\alpha = g$ for all $\alpha \in \mathrm{Inn}(G)\}$. Obviously $L(G) \leq Z(G)$. Hegarty [6] proved an analogue of Schur's theorem for the absolute center, that is, if G is a group such that $G/L(G)$ is finite, then $\langle g^{-1}g^\alpha \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle$ is also finite. Moreover Meng and Guo [12] explore the relationship between $L(G)$ and the Frattini subgroup $\Phi(G)$ for a finite group G, they also determine the structure of the absolute center of all finite minimal non-abelian p-groups.

In this paper we study $L(G)$ for p-groups of maximal class. As the definition of $L(G)$ shows, studying $L(G)$ directly depends on the structure of $\mathrm{Aut}(G)$. Therefore we use a structure of the Sylow p-subgroup of $\mathrm{Aut}(G)$ for metabelian p-groups of maximal class from our paper [5] and also the structure of p'-automorphism of p-groups of maximal class from [13] to prove our main theorem. Moreover we need the concept of the degree of commutativity of p-groups of maximal class. Specially we prove that $|L(G)| = 2$ for all 2-groups of maximal class, $L(G) = 1$ for all 3-groups of maximal class and also $L(G) = 1$ for p-groups of maximal class of order p^4. Moreover we show that there is only one group of maximal class of order p^5 with $|L(G)| = p$ and all other groups of maximal class of order p^5 have trivial absolute center(See Theorem 2.12). Furthermore we determine the absolute center for all metabelian p-groups of maximal class(See corollaries 2.5, 2.6 and Theorem 2.8).

Throughout this paper the following notation is used. The terms of the lower and the upper central series of G are denoted by $\gamma_i(G)$ and $Z_i(G)$, respectively. The centre of G is denoted by $Z = Z(G)$. If α is an automorphism of G and x is an element of G, we write x^α for the image of x under α. For a normal subgroup N of G, we let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms of G centralizing G/N. Let $H \leq G$ and $A \leq \mathrm{Aut}(G)$, we note that $C_A(H) = \{\alpha \in A \mid h^\alpha = h, \forall h \in H\}$ and $C_H(A) = \{h \in H \mid h^\alpha = h, \forall \alpha \in A\}$. The Frattini subgroup of G is denoted by $\Phi = \Phi(G)$ and $\mathrm{Aut}_p(G)$ for the Sylow p-subgroup of $\mathrm{Aut}(G)$. Also we use the notation $x \equiv y \pmod H$ to indicate that $Hx = Hy$, where H is a subgroup of a group G and $x, y \in G$. Let $(a, p) = 1$, we note that $ord_p(a)$ is the smallest positive integer t such that $a^t \equiv 1 \pmod p$. All unexplained notation is standard and follows that of [9].

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2. Main results

Let G be a p-group of maximal class of order p^n ($n \geq 3$), where p is a prime. We note that if $n = 3$, then $L(G) = 1$ for $p > 2$ and $L(G) = Z(G)$ for $p = 2$. Therefore in the rest of the paper we assume that $n \geq 4$. Following [9], we define the 2-step centralizer K_i in G to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of G is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+l}$ for all $i, j \geq 1$ if P_1 is not abelian and $l = n-2$ if P_1 is abelian.

Take $s \in G - \bigcup_{i=2}^{n-2} K_i$, $s_1 \in P_1 - P_2$ and $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n-1$. It is easily seen that $\{s, s_1\}$ is a generating set for G and $P_i(G) = \langle s_i, \ldots, s_{n-1}\rangle$ for $1 \leq i \leq n-1$ and so $Z(G) = P_{n-1}(G) = \langle s_{n-1}\rangle$. For the rest of the paper we fix the above notation.

By [9, Corollary 3.2.7] and [2, Corollary p.59] we have the following result.

Lemma 2.1. *Let G be a p-group of maximal class of order p^n.*

(i) The degree of commutativity of G is positive if and only if the 2-step centralizers of G are all equal.

(ii) If G is metabelian then G has positive degree of commutativity.

Lemma 2.2. [7, Hilfssatz III. 14.13] *If G is a p-group of maximal class of order p^n and $s \notin K_i$ for $2 \leq i \leq n-2$, then $C_G(s) = \langle s\rangle P_{n-1}(G)$ and $s^p \in P_{n-1}(G)$.*

Theorem 2.3. [3, Theorem 3.2] *Let $G = \langle a, b\rangle$ be a two-generated metabelian group. Then the following are equivalent:*

(i) For all $u, v \in G'$, there is an automorphism of G that maps a to au and b to bv;

(ii) G is nilpotent.

By the above theorem we see that if G is a metabelian p-group of maximal class of order p^n, then for any elements $x, y \in G' = \Phi(G)$ there is an automorphism that maps s to sx and s_1 to s_1y hence $|\mathrm{Aut}^{\Phi}(G)| = p^{2n-4}$. Moreover $\frac{\mathrm{Aut}(G)}{\mathrm{Aut}^{\Phi}(G)} \hookrightarrow \mathrm{Aut}(\frac{G}{\Phi(G)})$ and so $|\mathrm{Aut}_p(G) : \mathrm{Aut}^{\Phi}(G)|$ divides p, since $\frac{G}{\Phi(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Lemma 2.4. *If G is a p-group of maximal class of order p^n, then $\mathrm{Aut}_p(G)$ fix $Z(G)$ elementwise.*

Proof. Consider the action of $\mathrm{Aut}_p(G)$ on $Z(G)$. It is obvious that $C_{Z(G)}(\mathrm{Aut}_p(G)) \neq 1$ since $\mathrm{Aut}_p(G)$ and $Z(G)$ are p-groups. As $|Z(G)| = p$, we have $C_{Z(G)}(\mathrm{Aut}_p(G)) = Z(G)$, which completes the proof. □

Corollary 2.5. *If G is a p-group of maximal class of order p^n and $\mathrm{Aut}(G)$ is also a p-group, then $L(G) = Z(G)$.*

Proof. This is obvious by the fact that $L(G) \leq Z(G) \cong \mathbb{Z}_p$ and Lemma 2.4. □

Corollary 2.6. *Let G be a 2-group of maximal class of order 2^n, then $L(G) = Z(G)$.*

Proof. By [5, Theorem 5.9], we see that $\mathrm{Aut}(G)$ is also a 2-group which completes the proof by using Corollary 2.5. □

Lemma 2.7. *Let G be a p-group of maximal class of order p^n. If $\delta \in \mathrm{Aut}(G)$ with $s^\delta = s^a x$ and $s_1^\delta = s_1^c y$, where $x, y \in \Phi(G)$ and $0 < a, c < p$. Then $s_{n-1}^\delta = s_{n-1}^{a^{n-2}c}$.*

Proof. By induction on m we have $[s_i^m, s] \equiv s_{i+1}^m \pmod{\gamma_{i+2}(G)}$ and so $[s_i^m, s^\ell] \equiv s_{i+1}^{m\ell} \pmod{\gamma_{i+2}(G)}$ for $\ell, i \geq 1$. Therefore by induction on i we see that $s_i^\delta \equiv s_i^{a^{i-1}c} \pmod{\gamma_{i+1}(G)}$, as required. □

Now for the rest of paper by using corollaries 2.5 and 2.6 we may assume that G is a metabelian p-group of maximal class of order $p^n (p > 2)$ and Aut(G) is not p-group. It is straightforward to see that when p is odd, Aut(G) is supersolvable and is a split extension of $\mathrm{Aut}_p(G)$ by a subgroup of the direct product of two cyclic groups of order $p-1$. On the other hand, if H be a p'-subgroup of Aut(G), then we have $\mathrm{Aut}(G) = \mathrm{Aut}_p(G) \rtimes H$ and H is embeded in $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$ (see [1] Section 1). Since $P_1(G)$ and $\Phi(G)$ are characteristic subgroups of G , $G/\Phi(G)$ and $P_1/\Phi(G)$ are invariant under H. So by Maschke's Theorem there exists $s \in G - P_1$ such that $G/\Phi(G) = P_1/\Phi(G) \times \langle \Phi(G), s\rangle/\Phi(G)$ and $\langle \Phi(G), s\rangle/\Phi(G)$ is invariant under H. In the rest of the paper s will be as above. Therefore if $\delta \in H$ then $s^\delta = s^a x$ and $s_1^\delta = s_1^c y$, where $x, y \in \Phi(G)$ and $0 < a, c < p$. We recall that if G is metabelian p-group of maximal class, then G has positive degree of commutativity and $|s|$ divides p^2 by Lemma 2.2. In the next theorem we find the absolute center for finite metabelian p-group of maximal class when $H \neq 1$.

Theorem 2.8. *Let G be a metabelian p-group of maximal class of order $p^n(p > 2)$ and $H \neq 1$. If H is not cyclic, then $L(G) = 1$. Let H be cyclic such that $H = \langle\delta\rangle$ with $s^\delta = s^a x$, $s_1^\delta = s_1^c y$, where $1 \le a, c < p$ and $x, y \in \Phi(G)$.*

(i) If $|s| = p^2$, then $L(G) = 1$.

(ii) If all elements out of P_1 have order p, then

(a) if $ord_p(c) \nmid ord_p(a)$, then $L(G) = 1$.

(b) if $ord_p(c) \mid ord_p(a)$, then there exists $0 \le r < ord_p(a)$ such that $c \equiv a^r \pmod p$. On setting $ord_p(a) = t$ we have $L(G) = Z(G)$ when $t \mid n-2+r$ and $L(G) = 1$ when $t \nmid n-2+r$.

Proof. By [13, Theorem A], we have $C_H(Z(G))$ is cyclic. Hence there exists $\alpha \in H$ such that $C_{Z(G)}(\alpha) \neq Z(G)$. As $|Z(G)| = p$ we deduce that $C_{Z(G)}(\alpha) = 1$, which completes the proof, since $L(G) \le C_{Z(G)}(\alpha)$.
(i) Since $\delta \notin \mathrm{Aut}^\Phi(G)$, we have $(a, c) \neq (1, 1)$. By Lemma 2.7, if $a = 1$ then $s_{n-1}^\delta = s_{n-1}^c \neq s_{n-1}$, as desired. If $a > 1$, then by Theorem 2.3, the map β defined by $s^\beta = su^{-1}$ and $s_1^\beta = s_1 w^{-1}$, where $u^\delta = x$ and $w^\delta = y$, is an automorphism of G lying in $\mathrm{Aut}^\Phi(G)$. On setting $\alpha = \beta\delta$, we see that $s^\alpha = s^a$ and $s_1^\alpha = s_1^c$ and so $(s^p)^\alpha = s^{ap} \neq s^p$. Moreover by Lemma 2.2, $Z(G) = \langle s^p\rangle$, which completes the proof.
(ii)(a) The map β defined by $s^\beta = su^{-1}$ and $s_1^\beta = s_1 w^{-1}$, where $u^\delta = x$ and $w^\delta = y$, is an automorphism of G lying in $\mathrm{Aut}^\Phi(G)$. On setting $\alpha = \beta\delta$ and $ord_p(a) = t$, we have $s^{\alpha^t} = s$ and $s_1^{\alpha^t} = s_1^{c^t}$ and so by Lemma 2.7, $s_{n-1}^{\alpha^t} = s_{n-1}^{c^t} \neq s_{n-1}$ since $ord_p(c) \nmid t$.
(ii)(b) First we see that $1, a, \ldots, a^{t-1}$ are all distinct roots of the equation $x^t \equiv 1 \pmod p$. Therefore there exists $0 \le r < t$ such that $c \equiv a^r \pmod p$. Now by Lemma 2.7, $s_{n-1}^\delta = s_{n-1}^{a^{n-2+r}}$, which completes the proof. □

In what follows first we find the absolute center for all finite 3-groups of maximal class and finally we obtain the absolute center for all p-groups of maximal class of order p^n, where $4 \le n \le 5$.

Lemma 2.9. *Let G be a p-group of maximal class of order $p^n(p > 2)$ and $H \neq 1$. If P_1 is abelian, then $L(G) = 1$.*

Proof. First we may assume that $|s| = p$ by Theorem 2.8. Now we see that any element of G is uniquely determined by $s^t u$, where $0 \le t < p$ and $u \in P_1$. Assume that $1 < b < p$, we define $\beta : G \to G$ by $(s^t u)^\beta = s^t u^b$, and we show that β is an automorphism. Let $g_1 = s^t u$ and $g_2 = s^{t'} u'$, where $0 \le t, t' < p$ and $u, u' \in P_1$. We may write $g_1 g_2 = s^{t+t'}[s^{t'}, u^{-1}]uu'$. If $t + t' \equiv r(\bmod p)$, then $s^{t+t'} = s^r$ since $|s| = p$ and so $(g_1g_2)^\beta = s^r([s^{t'}, u^{-1}]uu')^b$. Moreover $g_1^\beta g_2^\beta = s^{t+t'}[s^{t'}, u^{-b}]u^b u'^b$. We have $[s^{t'}, u^{-b}] = [s^{t'}, u^{-1}]^b$ since P_1 is abelian and so β is a homomorphism. Also β is onto since $G = \langle s, s_1^b\rangle$. Thus β is an automorphism. Furthermore $s_{n-1}^\beta = s_{n-1}^b \neq s_{n-1}$, which completes the proof since $L(G) \le Z(G) = \langle s_{n-1}\rangle$. □

Lemma 2.10. *Let G be a 3-group of maximal class of order 3^n $(n \ge 4)$, then $L(G) = 1$.*

Proof. First we see that for $n = 4$, G is metabelian; and for $n \geq 5$, G has degree of commutativity $n - 4$ by [2, Theorem 3.13] and so is metabelian. Moreover by [5, Theorem 5.8], , we have $H \neq 1$. Now if P_1 is abelian, then by Lemma 2.9, $L(G) = 1$. Furthermore if P_1 is not abelian, then by observing the proof of [5, Theorem 5.6 (i)], we have $H = \langle \beta_2 \rangle$ when n is odd and $H = \langle \beta_3 \rangle$ when n is even, where $s^{\beta_2} = s^{-1}$, $s_1^{\beta_2} = s_1$ and $s^{\beta_3} = s^{-1}$, $s_1^{\beta_3} = s_1^{-1}$. Note that $s^{-1} = s^2 s^{-3}$ and $s^{-3} \in \Phi(G)$. Therefore Lemma 2.7 completes the proof. □

Lemma 2.11. *Let G be a p-group of maximal class of order p^4 ($p > 2$). Then $L(G) = 1$.*

Proof. First we see that $H \neq 1$ by [11, Lemma 9], . Since $P_1 = C_G(\gamma_2(G))$, we have $\gamma_2(G) \leq Z(P_1) \leq P_1$ which implies that $P_1/Z(P_1)$ is cyclic and so P_1 is abelian, as desired. □

Now for $p > 3$, Curran [4, Corollary 5] shown that there is only one group of order p^5 whose automorphism group is also a p-group in which $(p - 1, 3) = 1$. The presentation of this group is as follows:

$$G_0 = \langle a_1, a \mid a^p = [a_1, a]^p = [a_1, a, a]^p = [a_1, a, a, a]^p = [a_1, a, a, a, a] = 1$$

$$a_1^p = [a_1, a, a, a] = [a_1, a, a_1]^{-1} \rangle.$$

We note that G_0 is of maximal class. By this observation we state the following theorem.

Theorem 2.12. *Let G be a p-group of maximal class of order p^5 with $p > 3$. If $G = G_0$ then $L(G) = Z(G)$, for otherwise $L(G) = 1$.*

Proof. First we claim that G is metabelian. To prove this we have $[\gamma_2(G), Z_2(G)] = 1$ and so $\gamma_3(G) = Z_2(G) \leq Z(\gamma_2(G)) \leq \gamma_2(G)$, which implies that $\gamma_2(G)$ is abelian. If $G = G_0$ then Corollary 2.5 completes the proof. Therefore for the rest of the proof we may assume that $H \neq 1$. Since $p \geq 5$, by using [9, Proposition 3.3.2] we have $\exp(G/Z(G)) = \exp(G') = p$ which yields that $\mho_1(G) \leq Z(G) \cong \mathbb{Z}_p$. Moreover by [9, Lemma 1.2.11] G is regular. Now if $\mho_1(G) = Z(G)$, then $|\Omega_1(G)| = p^4$. Hence $\Omega_1(G)$ is a maximal subgroup of G and $\Omega_1(G) = \{x \in G | x^p = 1\}$ since G is regular. On setting $s \in G - (P_1 \cup \Omega_1(G))$, we have $|s| = p^2$ and so $L(G) = 1$ by Theorem 2.8. If $\mho_1(G) = 1$, then $\exp(G) = p$. Now from Jame's list [8], there are only two families Φ_9 and Φ_{10} of groups of maximal class of order p^5. By observing the presentation of these groups, we see that only $\Phi_9(1^5)$ and $\Phi_{10}(1^5)$ are of exponent p. Now if $G = \Phi_9(1^5)$ with the following presentation :

$$\langle s, s_1, \ldots, s_4 \mid [s_i, s] = s_{i+1}, s^p = s_i^p = 1 \ (1 \leq i \leq 4) \rangle,$$

then obviously P_1 is abelian and so $L(G) = 1$ by Lemma 2.9. Furthermore if $G = \Phi_{10}(1^5)$ with the presentation

$$\langle s, s_1, \ldots, s_4 \mid [s_i, s] = s_{i+1}, [s_1, s_2] = s_4, s^p = s_i^p = 1 \ (1 \leq i \leq 4) \rangle,$$

then the map α defined by $s^\alpha = s^{-1}, s_1^\alpha = s_1$ is an automorphism of order 2 and it is easily seen that $s_4^\alpha = s_4^{-1}$, completing the proof. □

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