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P-Hirano Inverses in Rings

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Abstract. We introduce and study a new class of generalized inverses in rings. An element a in a ring R has p-Hirano inverse if there exists $b \in R$ such that $bab = b, b \in comm^2(a), (a^2 - ab)^k \in J(R)$ for some $k \in \mathbb{N}$. We prove that $a \in R$ has p-Hirano inverse if and only if there exists $p = p^2 \in comm^2(a)$ such that $(a^2 - p)^k \in J(R)$ for some $k \in \mathbb{N}$. Multiplicative and additive properties for such generalized inverses are thereby obtained. We then completely determine when a 2×2 matrix over local rings has p-Hirano inverse.

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. Following Wang and Chen [8], an element a in R has p-Drazin inverse (that is, pseudo Drazin inverse) if there exists $b \in R$ such that

$$b = bab, b \in comm^2(a), a^k - a^{k+1}b \in J(R)$$

for some $k \in \mathbb{N}$. Here, $J(R)$ denotes the Jacobson radical of the ring R. The preceding b is unique, if such element exists, and called the p-Drazin inverse of a and denote b by a^{pD}. Pseudo Drazin inverses in a ring are extensively studied in both matrix theory and Banach algebra (see [2, 3, 8, 11] and [12]). Recently, Mosic [6], has introdeced and studied new classes of generalized Drazin inverses and pseudo n-strong Drazin inverses.

We shall see that $a \in R$ has p-Drazin inverse if and only if there exists $b \in R$ such that

$$b = bab, b \in comm^2(a), (a - a^2b)^k \in J(R)$$

for some $k \in \mathbb{N}$ (see Lemma 2.1). The motivation of this paper is to discuss the dual of pseudo Drazin inverses in a ring. We introduce and study a new class of generalized inverses in a ring. An element $a \in R$ has pseudo Hirano inverse (p-Hirano inverse as an abbreviation) if there exists $b \in R$ such that

$$b = bab, b \in comm^2(a), (a^2 - ab)^k \in J(R)$$

for some $k \in \mathbb{N}$. We shall prove that the preceding b is unique, if such element exists, and call b the p-Hirano inverse of a, and denote b by a^{ph}.

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In Section 2, the relations of p-Hirano and p-Drazin inverses are obtained, and we prove that $a \in R$ has p-Hirano inverse if and only if there exists $p = p^2 \in comm^2(a)$ such that $(a^2 - p)^k \in J(R)$ for some $k \in \mathbb{N}$.

Let $a, b \in R$. Then ab has p-Drazin inverse if and only if ba has p-Drazin inverse and $(ba)^{pd} = b((ab)^{pd})^2a$. This was known as Cline's formula for p-Drazin inverses (see [5, 6, 8]). In Section 3, we establish Cline's formula for p-Hirano inverses. Further, we explore multiplicative property of p-Hirano inverses for elements in a Banach algebra.

In [7, Theorem 2.6], the authors proved that for any elements $a, b \in R$, $1 - ab$ has p-Drazin inverse if and only if $1 - ba$ has p-Drazin inverse. This is the extension of Jacobson's lemma for Drazin and generalized Drazin inverses in a ring (see [6, 7, 10]). In Section 4, we investigate Jacobson's lemma for p-Hirano inverses and prove that $1 - ab \in R$ has p-Hirano inverse if and only if $1 - ba \in$ has p-Hirano inverse.

A ring R is local if R has only one maximal right ideal . Finally, in the last section, we completely determine when a 2×2 matrix over local rings has p-Hirano inverse. This provides many 2×2 matrices over a local ring additively generated by tripotnt and nilpotent matrices.

Throughout the paper, all rings are associative with an identity and all Banach algebras are complex. $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. We use $N(R)$ to denote the set of all nilpotent elements in R. $\mathbb{N}$ stands for the set of all natural numbers and $GL_2(R)$ is the group of 2×2 invertible matrices over R.

2. Pseudo Drazin inverses

The goal of this section is to investigate elementary properties of p-Hirano inverses and explore the relations between p-Drazin and p-Hirano inverses. We begin with

Lemma 2.1. *Let R be a ring and $a \in R$. Then a has p-Drazin inverse if and only if there exists $b \in R$ such that*

$$b = bab, b \in comm^2(a), (a - a^2b)^k \in J(R)$$

for some $k \in \mathbb{N}$.

Proof. $\Longrightarrow$ Since a has p-Drazin inverse, we can find $b \in comm^2(a)$ such that $b = bab$ and $a^k - a^{k+1}b \in J(R)$ for some $k \in \mathbb{N}$. Hence, $a^{k-1}(a - a^2b) \in J(R)$. This implies that $(a - a^2b)^k = a^{k-1}(a - a^2b)(1 - ab) \in J(R)$, as desired.

$\Longleftarrow$ By hypothesis, there exists $b \in R$ such that $b = b^2a, b \in comm^2(a), (a - a^2b)^k \in J(R)$ for some $k \in \mathbb{N}$. Then we have $c \in R$ such that $(a^k - a^{k+1}b) - (a - a^2b)^k = a^{k-1}(a - a^2b) - (a - a^2b)^k = (a - a^2b)(a^{k-1} - (a - a^2b)^{k-1}) = (a - a^2b)a^2bc = (ab - a^2b^2)a^2c = 0$. Thus $a^k - a^{k+1}b = (a - a^2b)^k \in J(R)$, as required. □

Lemma 2.2. *Let R be a ring and $a \in R$. If a has p-Hirano inverse, then it has p-Drazin inverse.*

Proof. Let $a \in R$. Then there exists some $b \in comm^2(a)$ such that $bab = b$ and $(a^2 - ab)^n \in J(R)$. Hence $(a^2 - a^2b^2)^n = (a^2 - a(bab))^n = (a^2 - ab)^n \in J(R)$. So we have

$$(a^2(1 - a^2b^2)^n) = (a^2 - a^2b^2)^n(1 - a^2b^2)^n \in J(R).$$

It follows that

$$(a(a - a^2b))^n = (a^2(1 - a^2b^2)^n) \in J(R).$$

We conclude that

$$(a - a^2b)^{2n} = (a(a - a^2b)(1 - ab))^n \in J(R).$$

Since $b \in comm^2(a)$ and $bab = b$ we deduce that a has p-Drazin inverse by Lemma 2.1. □

Let R be a ring and $a \in R$. Since the p-Drazin inverse of an element is unique, we see that a has at most one p-Hirano inverse in R, and if the p-Hirano inverse of a exists, it is exactly its p-Drazin inverse and we denote it by a^{ph}. We now give the relations of p-Hirano and p-Drazin inverses.

Theorem 2.3. *Let R be a ring and $a \in R$. Then a has p-Hirano inverse if and only if*

(1) *a and* $-a$ *have p-Drazin inverses;*
(2) $(a-a^3)^k \in J(R)$ *for some* $k \in \mathbb{N}$.

Proof. $\Longrightarrow$ Clearly, a has p-Hirano inverse if and only if so does $-a$. Thus, a and $-a$ have p-Drazin inverses by Lemma 2.2.

$\Longleftarrow$ By [8, Theorem 1.4], there exist idempotents $e, f \in comm^2(a)$ such that $a-e, a+f \in U(R)$. Clearly, a, e, f commute with each other. Since $(ae)^k, (af)^k \in J(R)$, we have

$$(ae-af)^{2k+1} = \sum_{i=0}^{2k+1} \binom{2k+1}{i} (ae)^{2k+1-i}(-af)^i \in J(R).$$

As $a+e, a+f \in U(R)$, we see that,

$$[(a+f)^{-1}(a-e)^{-1}(ae-af)]^{2k+1} \in J(R);$$

hence,

$$1-[(a+f)^{-1}(a-e)^{-1}(ae-af)]^{2k+1} \in U(R).$$

This implies that

$$1-(a+f)^{-1}(a-e)^{-1}(ae-af) \in U(R).$$

Hence,

$$\begin{array}{rcl} u & = & a^2-ef \\ & = & (a-e)(a+f)-af+ae \\ & = & (a-e)(a+f)[1-(a+f)^{-1}(a-e)^{-1}(ae-af)] \\ & \in & U(R). \end{array}.$$

By hypothesis, $\overline{a-a^3} \in N(R/J(R))$, and so $\overline{a^2-a^4} \in N(R/J(R))$. In view of [9, Lemma 3.5], we can find $g \in \mathbb{Z}[a]$ such that $(a^2-g)^m, g-g^2 \in J(R)$ for some $m \in \mathbb{N}$. Hence, $(u+ef-g)^m \in J(R)$. This shows that $ef-g \in U(R)$. Clearly, $(ef-g)^3-(ef-g) \in J(R)$, we see that $(ef-g)((ef-g)^2-1) \in J(R)$. Hence $g-(1-2ef)(1-ef) \in J(R)$. This implies that $(a^2-(1-ef))^m \in J(R)$. We complete the proof by Lemma 2.2. □

A ring R is strongly 2-nil-clean if every element in R is the sum of a tripotent and a nilpotent that commute (see [1]). It follows by Theorem 2.3 that a ring R is strongly 2-nil-clean if and only if every element in R has p-Hirano inverse and $J(R)$ is nil.

Lemma 2.4. *Let R be a ring and* $a \in R$. *Then the following are equivalent:*

(1) *a has p-Hirano inverse.*
(2) *There exists* $b \in R$ *such that*

$$b = ba^2b, b \in comm^2(a), (a^2-a^2b)^k \in J(R)$$

for some $k \in \mathbb{N}$.

Proof. It follows from [6, Theorem 3.2]. □

We are now ready to prove the following.

Theorem 2.5. *Let R be a ring and* $a \in R$. *Then the following are equivalent:*

(1) *a has p-Hirano inverse.*
(2) *There exists* $p^2 = p \in comm^2(a)$ *such that* $(a^2-p)^k \in J(R)$ *for some* $k \in \mathbb{N}$.

Proof. It is obvious by [6, Theorem 3.1]. □

Corollary 2.6. *Let R be a ring and* $a \in R$. *Then the following are equivalent:*

(1) *a has p-Hirano inverse.*
(2) *There exists* $b \in R$ *such that*

$$ab = (ab)^2, b \in comm^2(a), (a^2-ab)^k \in J(R)$$

for some $k \in \mathbb{N}$.

3. Multiplicative Property

The aim of this section is to generalize Cline's formula from p-Drazin inverses to p-Hirano inverses. We record [8, Theorem 2.1].

Lemma 3.1. *Let R be a ring, and let $a, b \in R$. Then ab has p-Drazin inverse if and only if ba has p-Drazin inverse.*

Theorem 3.2. *Let R be a ring, and let $a, b \in R$. Then ab has p-Hirano inverse if and only if ba has p-Hirano inverse, and*

$$(ba)^{ph} = b((ab)^{ph})^2 a.$$

Proof. It is a consequence of [6, Corollary 3.2]. □

Corollary 3.3. *Let R be a ring, and let $a, b \in R$. If $(ab)^k$ has p-Hirano inverse, then so is $(ba)^k$.*

In a Banach algebra, it shall be suffice to require that the p-Hirano inverse of an element a merely commutes with a. That is, we have

Lemma 3.4. *Let A be a Banach algebra and $a \in A$. Then the following are equivalent:*

(1) *a has p-Hirano inverse.*
(2) *There exists $p^2 = p \in comm(a)$ such that $(a^2 - p)^k \in J(A)$ for some $k \in \mathbb{N}$.*
(3) *There exists $b \in comm(a)$ such that*

$$b = bab, (a^2 - ab)^k \in J(A)$$

for some $k \in \mathbb{N}$.

Proof. (1) ⇒ (3) This is trivial.

(3) ⇒ (2) Set $p = ab$. Then $p^2 = p \in comm(a)$ with $(a^2 - p)^k \in J(A)$, as required.

(2) ⇒ (1) Set $w = a^2 - p$. Then $w^k \in J(A)$. Let $\lambda \in \mathbb{C}$. Then $(\lambda w)^k = \lambda^k w^k \in J(A)$. Hence, $1 - (\lambda w)^k \in U(A)$. That is,

$$(1 - \lambda w)[1 + \lambda w + (\lambda w)^2 + \cdots + (\lambda w)^{k-1}] \in U(A).$$

This implies that $1 - \lambda w \in U(A)$. By the definition of quasinilpotent element, we get $w \in A^{qnil}$.

We see that

$$a^2 + (1 - p) = 1 + w \in U(R), a^2(1 - p) = w(1 - p) \in A^{qnil}, (1 - p)^2 = 1 - p \in comm(a^2).$$

That is, $1 - p$ is the spectral idempotent of a^2. In light of [4, Theorem 7.5.3], we see that $1 - p \in comm^2(a^2) \subseteq comm^2(a)$. This implies that $p \in comm^2(a)$, as desired. □

Theorem 3.5. *Let A be a Banach algebra, and let $a, b \in A$. If a, b have p-Hirano inverses and $ab = ba$, then ab has p-Hirano inverse.*

Proof. Since a, b have p-Hirano inverses, we can find $x \in comm^2(a)$, $y \in comm^2(b)$ such that

$$x = x^2 a, y = y^2 b, (a^2 - ax)^k, (b^2 - by)^k \in J(A)$$

for some $k \in \mathbb{N}$. Obviously, x, y, a, b commute with each other. We easily check that

$$(ab)^2 - (ab)(yx) = -(b^2 - by)(a^2 - ax) + a^2(b^2 - by) + (a^2 - ax)b^2.$$

Then we have some $m \in \mathbb{N}$ such that $((ab)^2 - (ab)(yx))^m \in J(A)$. Moreover, we see that $yx = (yx)^2(ab)$ and $yx \in comm(ab)$. Therefore ab has p-Hirano inverse, by Lemma 3.4. □

Corollary 3.6. *Let A be a Banach algebra. If $a \in A$ has p-Hirano inverse, then a^n has p-Hirano inverse for all $n \in \mathbb{N}$.*

4. Jacobson's Lemma for p-Hirano inverses

Jacobon's Lemma states that for any $a, b \in R$, $1-ab \in R$ is invertible if and only if $1-ba \in R$ is invertible. An element $a \in R$ has generalized Drazin inverse in case there exists $b \in R$ such that

$$b = bab, b \in comm^2(a), a - a^2b \in R^{qnil}.$$

In [10, Theorem 2.1], the authors generalized Jacobson's Lemma to generalized Drazin invertibility. Motivated by this known theorem, we consider Jacobson's Lemma for p-Hirano inverses. The main theorem in this section is the following.

Theorem 4.1. *Let R be a ring, and let $a, b \in R$. Then $1-ab$ has p-Hirano inverse if and only if $1-ba$ has p-Hirano inverse, and*

$$(1-ba)^{ph} = 1 + b((1-ab)^{ph} - (1-ab)^{\pi}(1-(1-ab)^{\pi}(1-ab))^{-1})a,$$

where $(1-ab)^{\pi} = 1-(1-ab)^{ph}(1-ab)$.

Proof. $\Longrightarrow$ Let $\alpha = (1-ab)^2$. Then $\alpha = 1-cd$, where $c = a$ and $d = (2-ba)b$. In view of Lemma 2.4, there exists $x \in R$ such that

$$x = x\alpha x, x \in comm^2(1-ab), (\alpha - \alpha x)^k \in J(R)$$

for some $k \in \mathbb{N}$. One easily checks that $\alpha(1-\alpha x) = (\alpha-\alpha x)(1-\alpha x)$. Hence, $(\alpha-\alpha^2 x)^k = (\alpha-\alpha x)^k(1-\alpha x) \in J(R)$. Let $\beta = (1-ba)^2$. Then $\beta = 1-dc$. Denote $p = 1-\alpha x$. Since $(\alpha-\alpha^2x)^k \in J(R)$, we see that $1-p\alpha = 1-(\alpha-\alpha^2 x) \in U(R)$. Set $q = dp(1-p\alpha)^{-1}c$. Then

$$\begin{aligned} q^2 &= dp(1-p\alpha)^{-1}(cd)p(1-p\alpha)^{-1}c \\ &= d(1-p\alpha)^{-1}p(1-p\alpha)(1-p\alpha)^{-1}pc \\ &= q. \end{aligned}$$

Clearly, $\beta d = (1-dc)d = d(1-cd) = d\alpha$ and $a\beta = a(1-dc) = a(1-da) = (1-ad)a = (1-cd)c = \alpha c$. Further, we check that $\beta q = \beta dp(1-p\alpha)^{-1}c = q\beta$. We claim that $q \in comm^2(\beta)$. Let $y \in R$ be such that $y\beta = \beta y$. Then $y(1-dc) = (1-dc)y$, and so $ydc = dcy$. This implies that $(cyd)cd = cd(cyd)$. We infer that $(cyd)(1-\alpha) = (1-\alpha)(cyd)$, and then $(cyd)\alpha = \alpha(cyd)$. As $p \in comm^2(\alpha)$, we get $(cyd)p = p(cyd)$.

Thus, we get $(cyd)p(1-p\alpha)^{-1} = p(cyd)(1-p\alpha)^{-1} = p(1-p\alpha)^{-1}(cyd)$. Hence,

$$\begin{aligned} (dcy)q &= dcy(dp(1-p\alpha)^{-1}c) \\ &= q(ydc), \end{aligned}$$

and so $(1-\beta)yq = qy(1-\beta)$. Therefore

$$\begin{aligned} yq(1-\beta q) &= yq(1-\beta) \\ &= y(1-\beta)q \\ &= qy(1-\beta). \end{aligned}$$

Multiplying the above by q on the right side yields $yq(1-\beta q) = qyq(1-\beta q)$. As

$$\begin{aligned} 1-c\beta dp(1-p\alpha)^{-1} &= 1-c(1-dc)dp(1-p\alpha)^{-1} \\ &= 1-p\alpha \\ &\in U(R), \end{aligned}$$

we see that $1-\beta q = 1-\beta dp(1-p\alpha)^{-1}c \in U(R)$. This implies that $yq = qyq$. As $(1-\beta)qy = (1-\beta)yq$, we deduce that $(1-\beta q)qy = (1-q\beta)qyq$, and so $qy = qyq$. Therefore $yq = qyq = qy$, and so $q \in comm^2(\beta)$.

Write $r = \big(p(1-p\alpha)^{-1}-1\big)c$. Then $rd = \big(p(1-p\alpha)^{-1}-1\big)cd = p-1+\alpha$, and so $(1-p\alpha)(1+rd) = (1-p\alpha)(\alpha+p)$. As $1-p\alpha \in U(R)$, we see that $1+rd = \alpha+p$, and so $\alpha-\alpha x = \alpha-1+p = rd$. This shows that $(rd)^k \in J(R)$, $(dr)^{k+1} \in J(R)$. On the other hand, $\beta+q = 1+dr$, and so $(\beta-(1-q))^{k+1} = (dr)^{k+1} \in J(R)$. Here $(1-q)^2 = 1-q \in comm^2(\beta)$, and so $1-q \in comm^2(1-ba)$. Therefore $1-ba \in R$ has p-Hrano inverse. The formula is then obtained by Lemma 2.2 and [7, Theorem 2.6].

$\Longleftarrow$ This is symmetric. $\square$

Corollary 4.2. *Let R be a ring, let $n \in \mathbb{N}$, and let $a, b \in R$. Then $(1-ab)^n$ has p-Hirano inverse if and only if $(1-ba)^n$ has p-Hirano inverse.*

Proof. $\Longrightarrow$ Since $b(1-ab) = (1-ba)b$, we have $b(1-ab)^2 = [b(1-ab)](1-ab) = [(1-ba)b](1-ab) = (1-ba)[b(1-ab)] = (1-ba)^2 b$. By induction, we have

$$b(1-ab)^m = (1-ba)^m b$$

for any $m \in \mathbb{N}$. Then we have

$$\begin{array}{rl} & 1-(1-ab)^n \\ = & [1-(1-ab)][1+(1-ab)+(1-ab)^2+\cdots+(1-ab)^{n-1}] \\ = & ab[1+(1-ab)+(1-ab)^2+\cdots+(1-ab)^{n-1}] \\ = & a[b+b(1-ab)+b(1-ab)^2+\cdots+b(1-ab)^{n-1}] \\ = & a[b+(1-ba)b+(1-ab)^2 b+\cdots+(1-ab)^{n-1}b] \\ = & a[1+(1-ba)+(1-ba)^2+\cdots+(1-ba)^{n-1}]b. \end{array}$$

Then we have

$$(1-ab)^n = 1-a[1+(1-ba)+(1-ba)^2+\cdots+(1-ba)^{n-1}]b.$$

Likewise, we have

$$(1-ba)^n = 1-b[1+(1-ab)+(1-ab)^2+\cdots+(1-ab)^{n-1}]a.$$

Therefore

$$\begin{array}{rl} & (1-ba)^n \\ = & 1-[b+b(1-ab)+b(1-ab)^2+\cdots+b(1-ab)^{n-1}]a \\ = & 1-[b+(1-ba)b+(1-ba)^2 b+\cdots+(1-ba)^{n-1}b]a \\ = & 1-[1+(1-ba)+(1-ba)^2+\cdots+(1-ba)^{n-1}]ba. \end{array}.$$

$\Longleftarrow$ This is symmetric. □

For rectangular matrices over a ring, we derive

Proposition 4.3. *Let R be a ring, and let $A \in M_{m\times n}(R), B \in M_{n\times m}(R)$. Then $I_m + AB \in M_m(R)$ has p-Hirano inverse if and only if $I_n + BA \in M_n(R)$ has p-Hirano inverse.*

Proof. Let $k = m + n$. Set

$$C = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in M_{k\times k}(R).$$

Then we observe that

$$I_k + CD = \begin{pmatrix} I_n & 0 \\ 0 & I_m + AB \end{pmatrix}, I_k + DC = \begin{pmatrix} I_n + BA & 0 \\ 0 & I_m \end{pmatrix}.$$

Theorem 4.1 is applied to give the following result: $I_k + CD \in M_{k\times k}(R)$ has p-Hirano inverse if and only if so has $I_k + DC \in M_{k\times k}(R)$. Therefore we easily obtain the result. □

Corollary 4.4. *Let A be a Banach algebra, $a, b \in A$ and $ab = 0$. If a, b have p-Hirano inverse. Then $a + b$ has p-Hirano inverse.*

Proof. Let $C = \begin{pmatrix} 1 \\ a \end{pmatrix}$ and $B = (b, 1)$. Clearly, $CB = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$. By virtue of Theorem 2.3 and [8, Theorem 5.3], $CB \in M_2(A)$ has p-Hirano inverse. In light of Proposition 4.3, $BC = a + b \in A$ has p-Hirano inverse, as asserted. □

5. Matrices over local rings

The goal of this section is to completely determine when a 2×2 matrix over a local ring has p-Hirano inverse. Recal that a local ring R is called co-bleached if for any $j\in J(R)$ and $u\in U(R), l_u-r_j$ and l_j-r_u are injective, Where l_u and r_j will denote the abelian group endomorphisms of R given by left or right multiplication by u or j. The following lemma is crucial.

Lemma 5.1. *([3, Theorem 3.5]) Let R be a local ring and $A\in M_2(R)$. Then A has p-Drazin inverse if and only if*

(1) *$A\in GL_2(R)$; or*
(2) *$A^2\in M_2(J(R))$; or*
(3) *A is similar to $\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$, where $l_\alpha-r_\beta, l_\beta-r_\alpha$ are injective and $\alpha\in U(R), \beta\in J(R)$.*

We come now to characterize 2×2 matrices over a local ring by means of the similarity.

Theorem 5.2. *Let R be a local ring, and let $A\in M_2(R)$. Then A has p-Hirano inverse if and only if*

(1) *$A^2\in M_2(J(R))$, or $(I_2-A^2)^2\in M_2(J(R))$, or*
(2) *A is similar to $\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$, where $l_\alpha-r_\beta, l_\beta-r_\alpha$ are injective and $\alpha\in \pm 1+J(R), \beta\in J(R)$.*

Proof. $\Longrightarrow$ In light of Theorem 2.5, we may write $(A^2-E)^k\in M_2(J(R))$ for some $k\in\mathbb{N}$, where $E^2=E\in comm^2(A)$. By virtue of Theorem 2.3, A has p-Drazin inverse, i.e., it is pseudopolar. Then we have three cases.

Case 1. $A\in GL_2(R)$. Hence $E\in GL_2(R)$; and so $E=I_2$. This shows that $(I_2-A^2)^2\in M_2(J(R))$.

Case 2. $A^2\in M_2(J(R))$.

Case 3. A is similar to $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$, where $l_\lambda-r_\mu, l_\mu-r_\lambda$ are injective and $\lambda\in U(R), \mu\in J(R)$. In light of Theorem 2.3,

$$(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}^3)^k\in M_2(J(R))$$

for some $k\in\mathbb{N}$, and so $(\lambda-\lambda^3)^k\in J(R)$. This shows that $\lambda\in\pm 1+J(R)$.

$\Longleftarrow$ Case 1. $A^2\in M_2(J(R))$. Then A has p-Hirano inverse.

Case 2. $(I_2-A^2)^2\in M_2(J(R))$. In light of Theorem 2.5, A has p-Hirano inverse.

Case 3. A is similar to $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$, where $l_\lambda-r_\mu, l_\mu-r_\lambda$ are injective and $\lambda\in\pm 1+J(R), \mu\in J(R)$.

We see that

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}^2=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\lambda^2-1 & 0\\ 0 & \mu^2\end{pmatrix},$$

where $\begin{pmatrix}\lambda^2-1 & 0\\ 0 & \mu^2\end{pmatrix}\in M_2(J(R))$. Let $\begin{pmatrix}x & s\\ t & y\end{pmatrix}\in comm\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$. Then $\lambda s=s\mu$ and $\mu t=t\lambda$; hence, $s=t=0$. This implies that

$$\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix}.$$

Therefore $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\in comm^2\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$, as needed. □

We have at our disposal all the information necessary to characterize p-Hirano inverses of matrices over a cobleached local ring by means of the solvability of quadratic equations.

Theorem 5.3. *Let R be a cobleached local ring, and let $A\in M_2(R)$. Then A has p-Hirano inverse if and only if*

(1) *$A^2 \in M_2(J(R))$, or $(I_2 - A^2)^2 \in M_2(J(R))$, or*

(2) *A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - x\mu - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$.*

Proof. $\Longrightarrow$ As in the proof of Theorem 5.2, we may assume

$$U^{-1}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}U = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

for some $U \in GL_2(R)$, where $\alpha, \mu \in \pm 1 + J(R), \beta, \lambda \in J(R)$ Write $U^{-1} = \begin{pmatrix} z & y \\ s & t \end{pmatrix}$. It follows from

$$\begin{pmatrix} z & y \\ s & t \end{pmatrix}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} z & y \\ s & t \end{pmatrix}$$

that

$$\begin{gathered} y = \alpha z; \\ z\lambda + y\mu = \alpha y; \\ t = \beta s; \\ s\lambda + t\mu = \beta t. \end{gathered}$$

Clearly, $t = \beta s \in J(R)$. If y or s in $J(R)$, then U is not invertible, a contradiction. Since R is local, we see that $y, s \in U(R)$. If $z \in J(R)$, then $y = \alpha z \in J(R)$, a contradiction. This implies that $z \in U(R)$. Let $\delta = y^{-1}\alpha y$ and $\gamma = s^{-1}\beta s$ Then $\delta \in \pm 1 + J(R), \gamma \in J(R)$. We compute that

$$\begin{aligned} \delta^2 - \delta\mu &= y^{-1}\alpha^2 y - y^{-1}\alpha y\mu \\ &= (y^{-1}\alpha)(\alpha y - y\mu) \\ &= (y^{-1}\alpha)z\lambda \\ &= y^{-1}(\alpha z)\lambda \\ &= \lambda; \end{aligned}$$

hence, $\delta^2 - \delta\mu - \lambda = 0$. Moreover we check that

$$\begin{aligned} & \gamma^2 - \gamma\mu \\ =\ & (s^{-1}\beta)(\beta s - s\mu) \\ =\ & s^{-1}(\beta t - t\mu) \\ =\ & s^{-1}(s\lambda) \\ =\ & \lambda. \end{aligned}$$

Therefore the equation $x^2 - x\mu - \lambda = 0$ has a root $\delta \in \pm 1 + J(R)$ and a root $\gamma \in J(R)$, as desired.

$\Longleftarrow$ Suppose that the equation $x^2 - x\mu - \lambda = 0$ has a root $\alpha \in \pm 1 + J(R)$ and a root $\beta \in J(R)$. Then $\alpha^2 = \alpha\mu + \lambda; \beta^2 = \beta\mu + \lambda$. Hence,

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \beta - \alpha \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in GL_2(R).$$

Therefore $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in \pm 1 + J(R)$ and $\beta \in J(R)$. By virtue of Theorem 5.2, we complete the proof. □

Corollary 5.4. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A has p-Hirano inverse if and only if*

(1) $A^2 \in M_2(J(R))$, *or* $(I_2 - A^2)^2 \in M_2(J(R))$, *or*
(2) $x^2 - tr(A)x + det(A)$ *has a root* $\alpha \in \pm 1 + J(R)$ *and a root* $\beta \in J(R)$.

Proof. $\Longrightarrow$ By virtue of Theorem 5.3, we may assume that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$ and the equation $x^2 - \mu x - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$. Hence $\lambda = -det(A)$ and $\mu = tr(A)$, as desired.

$\Longleftarrow$ Case 1. A has p-Hirano inverse.

Case 2. Since $det(A) = \alpha\beta \in J(R)$, we see that $A \notin GL_2(R)$. As $tr(A) = \alpha + \beta \in \pm 1 + J(R)$, we have $det(I_2 - A) = 1 - tr(A) + det(A) \in J(R)$; hence, $I_2 - A \notin GL_2(R)$. In view of [5, Lemma 2.4], A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. Thus $\lambda = -det(A)$ and $tr(A) = u$, and so the equation $x^2 - \mu x - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$. Therefore A has p-Hirano inverse by Theorem 5.3. $\square$

We are ready to prove:

Theorem 5.5. *Let R be a commutative local ring, and let $A \in M_2(R)$. If $J(R)$ is nil, then the following are equivalent:*

(1) *A has p-Hirano inverse.*
(2) *A is the sum of a tripotent and a nilpotent that commute.*
(3) *A or $I_2 - A^2$ is nilpotent, or $x^2 - tr(A)x + det(A)$ has a root $\alpha \in \pm 1 + N(R)$ and a root $\beta \in N(R)$.*

Proof. (1) $\Leftrightarrow$ (3) This is obvious by Corollary 5.4, as $J(R) = N(R)$.

(1) $\Rightarrow$ (2) In light of Theorem 2.3, $(A - A^3)^2 \in M_2(J(R))$, and so $A - A^3 \in N(M_2(R))$.

Case 1. $2 \in J(R)$. Then $A - A^2 \in N(M_2(R))$. In view of [9, Lemma 3.5], there exists $E^2 = E \in \mathbb{Z}[A]$ such that $A - E \in N(M_2(R))$, as desired.

Case 2. $2 \notin J(R)$. Then $2 \in R$ is invertible. Let $B = \frac{A^2+A}{2}, C = \frac{A^2-A}{2}$. Then $A = B - C$. We have

$$B - B^2 = \frac{A^2 + A}{2} - (\frac{A^2 + A}{2})^2 =$$
$$\frac{A^2 + 2A - A^4 - 2A^3}{4} = \frac{(A - A^3)(A + 2I_2)}{4},$$
$$C - C^2 = \frac{A^2 - A}{2} = (\frac{A^2 - A}{2})^2 =$$
$$\frac{-A^4 + 2A^3 + A^2 - 2A}{4} = \frac{(A - A^3)(A - 2I_2)}{4}.$$

Since $A - A^3 \in N(M_2(R))$, we see that $B - B^2, C - C^2 \in N(M_2(R))$. In light o [9, Lemma 3.5], there exists idempotents $E, F \in \mathbb{Z}[A]$ such that $B - E, C - F \in N(M_2(R))$. Therefore $A = E - F + (B - E) - (C - F)$, where $(E - F)^3 = E - F \in \mathbb{Z}[A] \subseteq comm(A), (B - E) - (C - F) \in N(M_2(R))$. Therefore $(A - (E - F))^k = 0 \subseteq M_2(J(R))$ for some $k \in \mathbb{F}$, as desired.

(2) $\Rightarrow$ (1) Write $A = E + W, E^3 = E \in comm(A), W \in N(M_2(R))$. Then $A^2 - A^4 = ((2E + W) - 2E^2(2E + W) - (2E + W)^2)W)W \in N(M_2(R))$. According to [9, Lemma 3.5], there exists $F^2 = F \in \mathbb{Z}[A]$ such that $A^2 - F \in N(M_2(R))$. As $F \in comm^2(A)$, it follows by Theorem 2.5 that A has p-Hirano inverse, as asserted. $\square$

Corollary 5.6. *Let F be a field, and let $A \in M_2(F)$. Then the following are equivalent:*

(1) *A has p-Hirano inverse.*
(2) *A is the sum of a tripotent and a nilpotent that commute.*
(3) $A^2 = 0$, *or* $(I_2 - A^2)^2 = 0$, *or* $A^2 = \pm A$.

Proof. (1) $\Leftrightarrow$ (2) This is obvious, by Theorem 5.4.

(1) $\Rightarrow$ (3) This is clear by Theorem 5.5 and [3, Lemma 3.2], as $J(F) = 0$.

(3) $\Rightarrow$ (1) This is obtained by Theorem 5.5. $\square$

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