Filomat 34:13 (2020), 4445–4451
https://doi.org/10.2298/FIL2013445Z

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Results on Rectifiable Spaces

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Abstract. In this paper, it is mainly proved that (1) if G is a rectifiable space, then $In_l(G) \leq e(G)$; (2) if G is a rectifiable space and A is a discrete rectifiable subspace of G, then $|A| \leq \ell(G)$; (3) every locally compact NSS rectifiable space G is first-countable. The above results improve the corresponding results in topological groups.

1. Introduction

A topological space G is said to be a rectifiable space provided that there exists a homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a rectification on G. M.M. Choban [6] proved the next theorem.

Theorem 1.1. *A topological space G is a rectifiable space if and only if there exist $e \in G$ and two continuous maps $p : G \times G \to G$, $q : G \times G \to G$ such that for any $x \in G$, $y \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y \text{ and } q(x, x) = e.$$

In fact, we can assume that $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$ in Theorem 1.1. Fixing a point $x \in G$, we get that the maps $f_x, g_x : G \to G$ defined by $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$ for each $y \in G$, are homeomorphisms.

The above map $p : G \times G \to G$ will be called multiplication on G. Let G be a rectifiable space, and let p be the multiplication on G. Therefore, $q(x, y)$ is an element such that $p(x, q(x, y)) = y$. Since $p(x, e) = p(x, q(x, x)) = x$ and $p(x, q(x, e)) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x.

Recall that a topological group G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A paratopological group G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous. It is well known that rectifiable spaces are good generalization of topological groups. In fact, for a topological group with the neutral element e, as it is easy to see, the map $\varphi(x, y) = (x, x^{-1}y)$ is a rectification

2010 *Mathematics Subject Classification*. Primary 54H11; Secondary 22A05, 54A25, 54D20
Keywords. Cardinal invariants, rectifiable spaces, locally compact spaces, topological groups
Received: 11 January 2020; Revised: 09 April 2020; Accepted: 11 April 2020
Communicated by Ljubiša D.R. Kočinac
Research supported by the NSFC (No.11801254), the SDNSF (No. ZR2018MA013), the Institute of Meteorological Big Data-Digital Fujian and Fujian Key Laboratory of Date Science and Statistics
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on G. The 7-dimensional sphere S_7 is a rectifiable space but not a topological group [20]. Further, it is easy to see that paratopological groups and rectifiable spaces are all homogeneous. W. Atiponrat [3] introduced the concept of topological gyrogroups as a generalization of topological groups. In [5], the authors proved that each topological gyrogroup is a rectifiable space.

Cardinal functions are an interesting topic in general topology (see [9, 10]). Many topologists have investigated cardinal invariants in topological groups and paratopological groups extensively ([2, 16, 17]). In 1996, A.S. Gul'ko [8] proves that if G is a rectifiable space, then (1) $\pi\chi(G) = \chi(G)$; (2) $\omega(G) \leq k(G)\chi(G)$; (3) $\omega(G) = \pi\omega(G) = d(G)\chi(G)$. F. Lin, C. Liu and R. Shen in [11–13] also study cardinal functions in rectifiable spaces.

The notations ω, $w(G)$, $\pi w(G)$, $\chi(G)$ and $\ell(G)$ denote the first infinite ordinal, the weight of a space G, the π-weight of a space G, the character of a space G and the Lindelöf number of a space G, respectively. The symbol κ denotes an infinite cardinal. The letter e denotes the neutral element of a group and the right neutral element of a rectifiable space, respectively. The readers may consult [2, 7] for notation and terminology not explicitly given here. All spaces are assumed to be T_2.

2. The Index of Narrowness of Rectifiable Spaces

The next two problems about the index of narrowness was posed by A.V. Arhangel'skii, A. Bella [1] and F. Lin [11], respectively.

Problem 2.1. *Let G be a paratopological (semitopological) (Hausdorff, regular) group of countable extent. Must G be ω-narrow?*

Problem 2.2. *Is every Souslin rectifiable space G left ω-narrow?*

In [19], we prove that if G is a Hausdorff quasitopological group of countable extent, then G is ω-narrow.

Recall that a rectifiable space G is said to have the property $*$ if for each open neighborhood U of e in G, there exists an open neighborhood V of e in G such that $q(p(x, V), x) \subset U$ for every $x \in G$. In [18], we show that if G is a rectifiable space with the property *, then $In_l(G) \leq e(G)$ (The definition of $In_l(G)$ see bellow before Theorem 2.11.). In this section, further, we show that if G is a rectifiable space, then $In_l \leq e(G)$, which improves the related results in [18]. First of all, we give some concepts and technical lemmas.

Recall that if U is an open neighborhood of the right neutral element e of a rectifiable space G, a subset A of G is called U-discrete if $b \notin p(a, U)$, for any distinct $a, b \in A$ [18].

The next Lemma 2.3 and Lemma 2.4 can be found in [18]. For completeness of our proof, the specific proof of the next two lemmas are given again.

Lemma 2.3. *Let G be a rectifiable space, A be a subset of G and U be an open neighborhood of e in G . Then A is U-discrete if and only if $q(A, A) \cap U = \{e\}$.*

Proof. Firstly, we shall verify that if A is U-discrete, then we have $q(A, A) \cap U = \{e\}$. Suppose to the contrary that, there are two distinct elements $a, b \in A$ such that $q(a, b) \in U$. Then $b \in p(a, U)$, thus contradicting the assumption that the set A is U-discrete.

If $q(A, A) \cap U = \{e\}$, the following we shall verify that A is U-discrete. If there exist $a, b \in G$ and $a \neq b$ such that $b \in p(a, U)$, then $e \neq q(a, b) \in U \cap q(A, A)$, which is a contradiction. □

Lemma 2.4. *Let G be a rectifiable space and $A \subset G$. Then for each open neighborhood U of the neutral element e in G, A is U-discrete if and only if $\overline{A}$ is U-discrete.*

Proof. It is only need to verify that if A is U-discrete, then $\overline{A}$ is U-discrete. It follows from Lemma 2.3 that $q(A, A) \cap U = \{e\}$. We only need to prove that $q(\overline{A}, \overline{A}) \cap U = \{e\}$. The following we shall prove that for arbitrary distinct elements $a, b \in \overline{A}$, we can conclude $q(a, b) \notin U$.

Case 1: If $a, b \in A \subset \overline{A}$, since A is U-discrete, it is obvious that $q(a, b) \notin U$.

Case 2: If $a \in \overline{A} \setminus A$, $b \in A$ and $q(a,b) = u \in U$, then there exists an open neighborhood V of e in G such that $q(p(a,V),b) \subset U$. Since $a \in \overline{A}$, there is $a' \in p(a,V) \cap A$ where $a' \neq b$. We have $e \neq q(a',b) \in q(p(a,V),b) \subset U$, contradicting the assumption that A is U-discrete.

Case 3: If $a \in A$, $b \in \overline{A} \setminus A$ and $q(a,b) \in U$, then there is an open neighborhood V of e in G such that $q(a,p(b,V)) \subset U$. Since $b \in \overline{A}$, there is $b' \in p(b,V) \cap A$ where $b' \neq a$. It is easy to see that $e \neq q(a,b') \in q(a,p(b,V) \subset U$, contradicting the assumption that A is U-discrete.

Case 4: If $a,b \in \overline{A} \setminus A$ and $q(a,b) = u \in U$, then there is an open neighborhood W of e in G such that $q(p(a,W),p(b,W)) \subset U$. Since $a,b \in \overline{A}$, there are $a' \in p(a,W) \cap A$ and $b' \in p(b,W) \cap A$ where $a' \neq b'$. Thus $e \neq q(a',b') \in U$, contradicting the assumption that A is U-discrete. □

Recall that if A is a subspace of a rectifiable space G, then A is called a rectifiable subspace of G [12] if we have $p(A,A) \subset A$ and $q(A,A) \subset A$.

Lemma 2.5. ([12]) *Let G be a rectifiable space and A be a rectifiable subspace of G. Then $\overline{A}$ is also a rectifiable subspace of G.*

It is quite easy to verify the next theorem.

Theorem 2.6. *Let G be a rectifiable space and A be a rectifiable subspace of G. Then A is a discrete rectifiable subspace if and only if $\overline{A}$ is a discrete rectifiable subspace.*

Proof. If A is discrete, then there is an open neighborhood U of the right neutral element e in G such that $U \cap A = \{e\}$. Since A is a rectifiable subspace, $U \cap q(A,A) = \{e\}$. It follows from Lemma 2.4 and Lemma 2.5 that $\overline{A}$ is U-discrete rectifiable subspace.

It is easy to see that the following corollaries are true. □

Corollary 2.7. *Every discrete rectifiable subspace A of a countably compact rectifiable space G is finite.*

Corollary 2.8. *Every discrete subgroup A of a countably compact topological group G is finite.*

Corollary 2.9. *Let G be a rectifiable space and A be a discrete rectifiable subspace of G. Then $|A| \leq \ell(G)$.*

Proof. By Theorem 2.6, $\overline{A}$ is a discrete rectifiable subspace of H. Since $\ell(\overline{A}) \leq \ell(G)$ and $\overline{A}$ is discrete, $|A| \leq |\overline{A}| \leq \ell(\overline{A}) \leq \ell(G)$. □

Corollary 2.10. *Let G be a topological group and A be a discrete subgroup of G. Then $|A| \leq \ell(G)$.*

Let G be a rectifiable space and $\mathcal{N}(e)$ the family of open neighborhoods of the right neutral element e in G. The left index of narrowness $In_l(G)$ and the right index of narrowness $In_r(G)$ of G are defined, respectively, as follows:

$$In_l(G) = \min\{\kappa \geq \omega : (\forall U \in \mathcal{N}(e))(\exists F \subset G)(p(F,U) = G \wedge |F| \leq \alpha)\},$$

$$In_r(G) = \min\{\kappa \geq \omega : (\forall U \in \mathcal{N}(e))(\exists F \subset G)(p(U,F) = G \wedge |F| \leq \alpha)\}.$$

If G satisfies $In_l(G) \leq \kappa (In_r(G) \leq \kappa)$, then G is left (right) κ-narrow [11]. We also define the index of narrowness of G by

$$In(G) = In_l(G) \cdot In_r(G).$$

Given a space X, we denote by $e(X)$ the supremum of cardinality of closed discrete subsets of X.

It is proved that the inequality $In(H) \leq e(H)$ hold in each topological group H [2]. The next theorem generalizes the above result.

Theorem 2.11. *Let H be a rectifiable space. Then $In_l(H) \leq e(H)$.*

Proof. Let $\kappa = e(H)$. It suffices to show that $In_l(H) \le \kappa$. If $In_l(H) > \kappa$, then there is an open neighborhood U of e in H such that for an arbitrary subset $A \subset H$ with $|A| \le \kappa$, we have $G \setminus p(A, U) \ne \emptyset$. The family $\mathcal{E}$ of all U-discrete subset of H is (partially) ordered by inclusion, and the union of any chain of U-discrete sets is also a U-discrete set. Therefore, according to Zorn's Lemma, there exists a maximal element A of the family $\mathcal{E}$. It follows from Lemma 2.4 and $\kappa = e(H)$ that A is closed and $|A| \le \kappa$. Then there is $x_1 \in G \setminus p(A, U)$. Since $x_1 \notin A$, there is an open neighborhood V_1 of e in H such that $V_1 \subset U$ and $p(x_1, V) \cap A = \emptyset$. Put $A_1 = A \cup \{x_1\}$. It is clear that A_1 is V_1-discrete and $|A_1| \le \kappa$.

Assume that we have defined open neighborhood V_α of e and V_α-discrete subset A_α with $|A_\alpha| \le \kappa$ for some infinite cardinal α . Thus $G \setminus p(A_\alpha, U) \ne \emptyset$. By Lemma 2.4, it is clear that $\overline{A_\alpha}$ is V_α-discrete, so $|\overline{A_\alpha}| \le \kappa$. Then there are $x_{\alpha+1} \in G \setminus p(\overline{A_\alpha}, U)$ and an open neighborhood $V_{\alpha+1}$ of e such that $p(x_{\alpha+1}, V_{\alpha+1}) \cap A_\alpha = \emptyset$, where $V_{\alpha+1} \subset U$. Put $A_{\alpha+1} = A_\alpha \cup \{x_{\alpha+1}\}$, then $A_{\alpha+1}$ is $V_{\alpha+1}$-discrete. By induction, we shall have a V-discrete subset B such that $|B| > \kappa$ for some open neighborhood V of e. It follows from Lemma 2.4 that $\overline{B}$ is V-discrete. Since $|\overline{B}| \ge |B| > \kappa$, this contradicts the definition of κ. Hence $In_l(H) \le \kappa = e(H)$. □

At the end of this section, we give two-element properties of U-discrete subsets preserved by homomorphism maps.

Let G, H be rectifiable spaces and $f : G \to H$ be a map from G to H. The map is called a homomorphism if for arbitrary $x, y \in G$ we have $f(p_G(x, y)) = p_H(f(x), f(y))$. Moreover, if f is a one-to-one homomorphism map from G onto H, then f is called an isomorphism [11].

Lemma 2.12. ([11]) *Let G, H be rectifiable spaces and $f : G \to H$ be a homomorphism from G to H. Then $f(e_G) = e_H$ and $f(q_G(x, y)) = q_H(f(x), f(y))$ for arbitrary $x, y \in G$.*

Making use of Lemma 2.3 and the definition of homomorphism, it is easy to deduce the propositions below.

Proposition 2.13. *Let G, H be rectifiable spaces and $f : G \to H$ be a isomorphism from G to H. If A a is U-discrete subset of H, then $f^{-1}(A)$ is a W-discrete subset of G for each open neighborhood W of e_G in G with $f(W) \subset U$.*

Proof. According to Lemma 2.3, it is only need to prove that the equation $W \cap q_G(f^{-1}(A), f^{-1}(A)) = \{e_G\}$ hold, where e_G denotes the right neutral element of G. If not, then there exist $a_1, a_2 \in f^{-1}(A)$, $b_1, b_2 \in A$ and $w \in W$ where $w \ne e_G$, $a_1 \ne a_2$ such that $f(a_1) = b_1, f(a_2) = b_2$ and $q_G(a_1, a_2) = w$. Thus $f(q_G(a_1, a_2)) = f(w)$. Since f is a homomorphism, $f(w) = f(q_G(a_1, a_2)) = q_H(f(a_1), f(a_2)) = q_H(b_1, b_2)$. Since f is a isomorphism, $e_H \ne f(w) \in f(W) \cap q_H(A, A) \subset U \cap q_H(A, A)$, which is a contradiction. □

Proposition 2.14. *Let G, H be rectifiable spaces and $f : G \to H$ be a homomorphism from G to H. If A a is U-discrete subset of G, then $f(A)$ is a V-discrete subset of H for each open neighborhood V of e_H with $f^{-1}(V) \subset U$.*

Proof. According to Lemma 2.3, it is only need to prove that the equation $V \cap q_H(f(A), f(A)) = \{e_H\}$ hold. If not, then there exist $a, b \in A$ and $v \in V$ where $v \ne e_H$, $f(a) \ne f(b)$ such that $q_H(f(a), f(b)) = v$. Since f is a homomorphism, $v = q_H(f(a), f(b)) = f(q_G(a, b))$. Then $q_G(a, b) \in f^{-1}(V) \cap q_G(A, A) \subset U \cap q_G(A, A)$, which is a contradiction. □

It is well known that every first-countable rectifiable space is metrizable [8]. It is easy to deduce that the following result is true.

Theorem 2.15. *Suppose that f is an open continuous homomorphism of a metrizable rectifiable space G onto a rectifiable space H, then H is also a metrizable rectifiable space.*

Proof. Since f is open and continuous, and the space G is first-countable, the space H is also first-countable. Thus H is a metrizable rectifiable space. □

3. Locally Compact NSS-Rectifiable Spaces

It is known that in every locally compact totally disconnected topological group G, there exists a local base $\mathcal{B}$ of G at e such that every element of $\mathcal{B}$ is an open compact subgroup of G.

In [14], the authors proved that every locally σ-compact rectifiable space with a bc-base is locally compact or zero-dimensional and posed the next question:

Problem 3.1. *Does each totally disconnected locally compact rectifiable space have an open compact rectifiable subspace?*

In this section, we prove that each totally disconnected locally compact rectifiable space have a closed compact rectifiable subspace. We also show that every locally compact NSS rectifiable space G is first-countable.

First of all, we give a simple lemma which will be used in our proof.

Lemma 3.2. ([15]) *Let C be a compact subset and F be a closed subset of a rectifiable space G such that $C \cap F = \emptyset$. Then there exists an open neighborhood V of e in G such that $p(C, V) \cap F = \emptyset$.*

We can conclude the next propositions easily by Lemma 3.2.

Proposition 3.3. *Let G be a rectifiable space and C be a $\sigma-$compact subset of G and F be a closed subset of G such that $C \cap F = \emptyset$. Then there exists a G_δ-subset V of e such that $p(C, V) \cap F = \emptyset$.*

Proof. Assume that $C = \bigcup_{n\in N} C_n$, where each C_n is compact, there exists an open neighborhood V_n of e in G such that $p(C_n, V_n) \cap F = \emptyset$ according to Lemma 3.2. Put $V = \bigcap_{n\in N} V_n$, then V is a G_δ-subset of G and $p(C, V) \cap F = \emptyset$, which completes our proof. □

Proposition 3.4. *Let G be a rectifiable space, and let C be a non-empty compact G_κ-set in G. Then there exists a G_κ-set V in G such that $e \in V$ and $p(C, V) \subset C$.*

Proof. Let $C = \bigcap \gamma$, where γ is a family of open subsets of G and $|\gamma| \leq \kappa$. Take any $U \in \gamma$. By Lemma 3.2, there exists an open neighborhood V_U of e such that $p(C, V_U) \subset U$. Put $V = \bigcap\{V_U : U \in \gamma\}$. Then V is a G_κ-subset of G such that $e \in V$ and $p(C, V) \subset C$. □

The following theorem implies that each totally disconnected locally compact rectifiable space have a closed compact rectifiable subspace.

Theorem 3.5. *Suppose that G is a rectifiable space and C is a compact neighborhood of e in G. Then there exists a closed compact rectifiable subspace H of G such that $H \subset C$.*

Proof. Since C is a compact neighborhood of e in G , we can choose an open neighborhood V_1 of e in G such that $V_1 \subset \overline{V_1} \subset C$. Assume that open neighborhoods V_i of e in G are defined for each $i = 1, 2, \cdots, n$. Then there is an open neighborhood V_{n+1} of e in G such that $p(V_{n+1}, V_{n+1}) \subset V_n$, $q(V_{n+1}, V_{n+1}) \subset V_n$ and $\overline{V_{n+1}} \subset V_n$. Put $H = \bigcap_{n\in N} V_n$. Then$H = \bigcap_{n\in N} \overline{V_n}$. It is easy to check that H is a closed rectifiable subspace and $H \subset C$. Indeed, we only need to verify that H is a rectifiable subspace. For arbitrary $x, y \in H$ and for each $n \in N$, we have $x, y \in V_{n+1}$. Therefore $p(x, y) \in p(V_{n+1}, V_{n+1}) \subset V_n$ and $q(x, y) \in q(V_{n+1}, V_{n+1}) \subset V_n$, which implies that $p(H, H) \subset H$ and $q(H, H) \subset H$. Since C is compact and H is closed contained in C, H is a compact rectifiable subspace contained in C. □

It is clear that the next corollary gives a partial answer to Problem 3.1. We call a space X a P-space if every G_δ-set in X is open.

Corollary 3.6. *Suppose that G is a totaly disconnected locally compact rectifiable space and G is a P-space. Then there exists a local base $\mathcal{B}$ of G at the right neutral element e such that each element of $\mathcal{B}$ is an open compact rectifiable subspace.*

Proof. Since G is a totaly disconnected locally compact space, there exists a base $\mathcal{P}$ of G at e consisting of open compact subsets of G. It follows from Theorem 3.5 that for every $V \in \mathcal{P}$, there is a compact rectifiable subspace H_V which is also a G_δ-subset in G such that $H_V \subset V$. Let $\mathcal{B} = \{H_V : V \in \mathcal{P}\}$. Since G is a P-space, $\mathcal{B}$ is a base of G at e. □

Let G be a rectifiable space. We say that G is a rectifiable space with no small rectifiable subspace or, for brevity, an NSS-rectifiable space if there exists a neighborhood U of e such that every rectifiable subspace H of G contained in U is trivial, that is, $H = \{e\}$.

Theorem 3.7. *If G is a NSS-rectifiable space, then the following two conditions are equivalent:*
(1) *there exists a non-empty compact G_δ-set C in G;*
(2) *the right neutral element e of G is a G_δ-point in G.*

Proof. It is only need to show that the first condition implies the second one. Since the space G is homogeneous, we can assume that $e \in C = \bigcap_{n\in N} U_n$. Since G is a NSS-rectifiable space, there exists a neighborhood U of e such that every rectifiable subspace H of G contained in U is trivial. Let $C' = \bigcap_{n\in N}(U_n \cap U)$. It is clear that $C' \subset C \bigcap U$. Buy the regularity of G, there exists an open neighborhood V_n of e in G such that $\overline{V_{n+1}} \subset V_n \cap U_n \cap U$, for each $n \in N$. Let $C'' = \bigcap_{n\in N} V_n$. It is easy to see that $C'' = \bigcap_{n\in N} \overline{V_n} \subset C \cap U$. Thus C'' is a a non-empty compact G_δ-set contained in U. There is an open neighborhood $W_1^1 \subset V_1$. Assume that open neighborhoods W_i^1 of e in G are defined for each $i = 1, 2, \cdots, n$. Then there is an open neighborhood W_{n+1}^1 of e in G such that $p(W_{n+1}^1, W_{n+1}^1) \subset W_n^1$, $q(W_{n+1}^1, W_{n+1}^1) \subset W_n$ and $\overline{W_{n+1}}^1 \subset W_n^1$. Put $H_1 = \bigcap_{n\in N} W_n^1$. It is clear that H_1 is a closed rectifiable subspace contained in V_1. Similarly, we can define closed rectifiable subspace H_n contained in V_n for each $n = 2, 3, \cdots$. Put $H = \bigcap_{n\in N} H_n$. It is clear that H is a G_δ-set and is a rectifiable subspace contained in C''. Since $C'' \subset U$ and G is NSS, $H = \{e\}$. This completes the proof. □

Theorem 3.8. *Every locally compact NSS rectifiable space G is first-countable.*

Proof. Clearly, each locally compact regular space contains a non-empty compact G_δ-set. Therefore, according to Theorem 3.7, the right neutral element e of G is a G_δ-point in G. Since G is locally compact and Hausdorff, G is first-countable at e. Hence, by homogeneity, the space G is first-countable. □

In [5], the authors proved that each topological group is a topological gyrogroup and each topological gyrogroup is a rectifiable space. It is easy to see that the following corollary is true.

Corollary 3.9. ([4]) *Every locally compact NSS-gyrogroup $(G, \tau, \oplus)$ is first-countable.*

References

[1] A.V. Arhangel'skiĭ, A. Bella, The diagonal of a first countable paratopological group, submetrizablity, and related results, Appl. Gen. Topol. 8 (2007) 2017–212.
[2] A.V. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press, World Sci. 2008.
[3] W. Atiponrat, Topological gyrogroups: generalization of topological groups, Topol. Appl. 224 (2017) 73–82.
[4] M. Bao, F.C. Lin, Feathered Gyrogroups and Gyrogroups with countable Pseudocompact, Filomat 33 (2019) 5113–5124.
[5] Z. Cai, S. Lin, W. He, A note on paratopological loops, Bull. Malaysian Math. Sci. Soc 42 (2019) 2535–2547.
[6] M.M. Choban, The structure of locally compact algebras, Serdica 18 (1992) 129–137.
[7] R. Engelking, General Topology, (revised and completed edition), Heldermann Verlag, Berlin, 1989.
[8] A.S. Gul'ko, Rectifiable spaces, Topology Appl. 68 (1996) 107–112.
[9] R. Hodel, Cardinal functions I, in: K. Kunen and J. E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., Amsterdam, 1984.
[10] I. Juhász, Cardinal functions II, in: K. Kunen and J. E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., Amsterdam, 1984.
[11] F. Lin, Pseudocompact rectifiable spaces, Topology Appl. 164 (2014) 215–228.
[12] F. Lin, C. Liu, S. Lin, A note on rectifiable spaces, Topology Appl. 159 (2012) 2090–2101.
[13] F. Lin, R. Shen, On rectfiable spaces and paratopological groups, Topology Appl. 158 (2011) 597–610.
[14] F. Lin, J. Zhang, K. Zhang, Locally σ-compact rectifiable spaces, Topology Appl. 193 (2015) 182–191.
[15] Q.-T.Luong, V.T. Ong, Some properties of rectifiable spaces, Fasciculi Math. 60 (2018) 181–190.
[16] O.V. Ravsky, Paratopological groups I, Mat. Stud. 16 (2001) 37–48.

[17] O.V. Ravsky, Paratopological groups II, Mat. Stud. 17 (2002) 93–101.
[18] J. Zhang, Cardinal invariants in rectifiable spaces, Journal of Nanjing Normal University (Natural Science Edition) 39 (2016) 1–4.
[19] J. Zhang, Cardinal invariants in quasitopological groups, Houston J. Math., Topology Appl. 265 (2019) 1–9.
[20] V.V. Uspenskii, Topological groups and Dugundji compacta, Mat. Sb., 1989, 180: 1092–1118(Russian), English transl. in: Math. USSR Sb. 67 (1990) 555–580.